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# Generalizing Tuenter's Binomial Sums

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**Abstract**

Tuenter considered centered binomial sums of the form

$$S_r(n) = \sum_{k=0}^{2n} \binom{2n}{k} |n-k|^r,$$

where $r$ and $n$ are non-negative integers. We consider sums of the form

$$U_r(n) = \sum_{k=0}^{n} \binom{n}{k} |n/2-k|^r,$$

which are a generalization of Tuenter's sums and may be interpreted as moments of a symmetric Bernoulli random walk with $n$ steps. The form of $U_r(n)$ depends on the parities of both $r$ and $n$. In fact, $U_r(n)$ is the product of a polynomial (depending on the parities of $r$ and $n$) times a power of two or a binomial coefficient. In all cases the polynomials can be expressed in terms of Dumont-Foata polynomials. We give recurrence relations, generating functions and explicit formulas for the functions $U_r(n)$ and/or the associated polynomials.

# 1 Introduction

We consider centered binomial sums of the form

$$U_r(n) = \sum_{k=0}^{n} \binom{n}{k} \left|\frac{n}{2} - k\right|^r, \tag{1}$$

where $r, n \geq 0$. These generalize the binomial sums

$$S_r(n) = \sum_{k=0}^{2n} \binom{2n}{k} |n-k|^r, \tag{2}$$

previously considered by Tuenter [7] and other authors, since $S_r(n) = U_r(2n)$, but $U_r(n)$ is well-defined for both even and odd values of $n$. The generalization arises naturally in the study of certain two-fold centered binomial sums of the form $\sum_j \sum_k \binom{2n}{n+j}\binom{2n}{n+k} P(j,k)$, see Brent et al. [2, Lemma 6.4].

In definitions such as (1) and (2) we always interpret $0^0$ as 1. Thus $U_0(n) = 2^n$ for all $n \geq 0$. We define $U_r(m) = 0$ if $m < 0$.

For $r > 0$ we can avoid the absolute value function in (1) by writing

$$U_r(n) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \left(\frac{n}{2} - k\right)^r.$$

Tuenter [7] showed in a direct manner that, for $r \geq 0$ and $n > 0$, $S_r(n)$ satisfies the recurrence

$$S_{r+2}(n) = n^2 S_r(n) - 2n(2n-1) S_r(n-1). \tag{3}$$

Observe that this recurrence splits into two separate recurrences, one involving odd values of $r$ and the other involving even values of $r$. Also, $S_0(n) = 2^{2n}$ and $S_1(n) = n\binom{2n}{n}$. It follows that, for $r, n \geq 0$,

$$S_{2r}(n) = Q_r(n) 2^{2n-r} \quad \text{and} \quad S_{2r+1}(n) = P_r(n) n \binom{2n}{n}, \tag{4}$$

where $P_r(n)$ and $Q_r(n)$ are polynomials of degree $r$ with integer coefficients, satisfying the recurrences

$$P_{r+1}(n) = n^2 P_r(n) - n(n-1) P_r(n-1), \tag{5}$$

$$Q_{r+1}(n) = 2n^2 Q_r(n) - n(2n-1) Q_r(n-1) \tag{6}$$

for $r \geq 0$, with initial conditions $P_0(n) = Q_0(n) = 1$.

The Dumont-Foata polynomials $F_r(x,y,z)$ are 3-variable polynomials satisfying the recurrence relation

$$F_{r+1}(x,y,z) = (x+z)(y+z) F_r(x,y,z+1) - z^2 F_r(x,y,z) \tag{7}$$

for $r \geq 1$, with $F_1(x,y,z) = 1$. Dumont and Foata [4] gave a combinatorial interpretation for the coefficients of $F_r(x,y,z)$ and showed that $F_r(x,y,z)$ is symmetric in the three variables $x, y, z$.

Tuenter [7] showed that $P_r(n)$ and $Q_r(n)$ may be expressed in terms of Dumont-Foata polynomials. In fact, for $r \ge 1$,

$$P_r(n) = (-1)^{r-1} n F_r(1, 1, -n) \;\text{ and }\; Q_r(n) = (-2)^{r-1} n F_r(\tfrac{1}{2}, 1, -n). \tag{8}$$

Thus, we can obtain explicit formulas and generating functions for the polynomials $P_r(n)$ and $Q_r(n)$ as special cases of the results of Carlitz [3] on Dumont-Foata polynomials.

We show that all the above results for $S_r(n)$ can be generalized to cover $U_r(n)$. In particular, Theorem 1 shows that $U_r(n)$ satisfies a recurrence (9) similar to the recurrence (3) satisfied by $S_r(n)$. Also, $U_r(n)$ is the product of a polynomial (depending on the parity of $r$) times a power of two or a binomial coefficient, and Theorem 2 shows that these polynomials satisfy three-term recurrence relations analogous to (5)–(6). Using the recurrences, the polynomials can be expressed in terms of Dumont-Foata polynomials, so the results of Carlitz allow us to obtain explicit formulas (in §3) and exponential generating functions (in §4).

# 2 Recurrence relations

Theorems 1–2 give recurrence relations for $U_r(n)$ and associated polynomials. The recurrence (9) in Theorem 1 implies the recurrence (3) satisfied by $S_r(n)$.

**Theorem 1.** *For all $r, n \ge 0$, $U_r(n)$ satisfies the recurrence*

$$4U_{r+2}(n) = n^2 U_r(n) - 4n(n-1)U_r(n-2), \tag{9}$$

*and may be computed from the recurrence using the initial conditions*

$$U_0(n) = 2^n,\; U_1(2n) = n\binom{2n}{n},\; U_1(2n+1) = (2n+1)\binom{2n}{n} \text{ for all } n \ge 0.$$

*Proof.* We have

$$4U_{r+2}(n) = \sum_{k=0}^{n} 4\binom{n}{k}\left|\frac{n}{2} - k\right|^{r+2} = \sum_{k=0}^{n}\binom{n}{k}\left|\frac{n}{2} - k\right|^{r}(n-2k)^2, \tag{10}$$

$$n^2 U_r(n) = \sum_{k=0}^{n}\binom{n}{k}\left|\frac{n}{2} - k\right|^{r} n^2, \tag{11}$$

and

$$\begin{aligned} 4n(n-1)U_r(n-2) &= \sum_{k=0}^{n-2} 4n(n-1)\binom{n-2}{k}\left|\frac{n-2}{2} - k\right|^{r} \\ &= \sum_{k=1}^{n-1} 4n(n-1)\binom{n-2}{k-1}\left|\frac{n}{2} - k\right|^{r} \\ &= \sum_{k=1}^{n-1} 4k(n-k)\binom{n}{k}\left|\frac{n}{2} - k\right|^{r}. \end{aligned} \tag{12}$$

Since $(n-2k)^2 - n^2 - 4k(n-k) = 0$, the recurrence (9) follows from (10)–(12).

For the initial conditions, it is easily verified that $U_0(n) = 2^n$. The solution by Hillman [5] to the Putnam problem 35-A4 gives $U_1(n) = n\binom{n-1}{\lfloor n/2 \rfloor}$, and taking account of the parity of $n$ gives the remaining conditions. □

It is now clear that the structure of $U_r(n)$ depends upon the parities of both $r$ and $n$, and one can elucidate the recursion (9) by the following substitutions:

$$U_{2r+1}(2n-1) = 2^{-(2r+1)} n \overline{P}_r(n) \binom{2n}{n}, \tag{13}$$

$$U_{2r}(2n+1) = 2^{2n+1-2r} \overline{Q}_r(n). \tag{14}$$

It is easily verified that $\overline{P}_r(n)$ and $\overline{Q}_r(n)$ are polynomials in $n$ of degree $r$. By substitution into (9), we obtain the following theorem.

**Theorem 2.** *For $r \geq 0$, the polynomials $\overline{P}_r(n)$ and $\overline{Q}_r(n)$ defined by equations (13)–(14) satisfy the following recurrence relations:*

$$\overline{P}_{r+1}(n) = (2n-1)^2 \, \overline{P}_r(n) - 4(n-1)^2 \, \overline{P}_r(n-1), \tag{15}$$

$$\overline{Q}_{r+1}(n) = (2n+1)^2 \, \overline{Q}_r(n) - 2n(2n+1)\overline{Q}_r(n-1), \tag{16}$$

*with initial conditions $\overline{P}_0(n) = \overline{Q}_0(n) = 1$.*

# 3 Explicit formulas

Using the recurrence (7) and matching the initial conditions, we can verify that the polynomials $\overline{P}_r(n)$ and $\overline{Q}_r(n)$ can be expressed in terms of Dumont-Foata polynomials, as follows:

$$\overline{P}_r(n) = (-4)^r F_{r+1}\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} - n\right), \tag{17}$$

$$\overline{Q}_r(n) = (-1)^{r-1} 2^{2r-1} \left(n + \tfrac{1}{2}\right) F_r\left(\tfrac{1}{2}, 1, -n - \tfrac{1}{2}\right). \tag{18}$$

Also, replacing $n$ by $n + \frac{1}{2}$ in the second half of (8) shows that

$$\overline{Q}_r(n) = 2^r Q_r\left(n + \tfrac{1}{2}\right). \tag{19}$$

Carlitz [3, eqns. (1.13) and (1.16)] gives an explicit formula for the Dumont-Foata polynomials. We follow Carlitz and let $(z)_k$ denote the Pochhammer symbol or "rising factorial". Note that, although $F_{r+1}(x, y, z)$ is symmetric in $x, y$ and $z$, the representation given in Proposition 3 is not symmetric. Thus, it is sometimes necessary to permute variables before applying Proposition 3.

**Proposition 3** (Carlitz)**.** *For $r \ge 0$ and $z > 0$,*

$$F_{r+1}(x,y,z) = \sum_{k=0}^{r} (-1)^{r-k} (x+z)_k (y+z)_k \, A_{r,k}(z),$$

*where*

$$A_{r,k}(z) = \frac{2}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(z+j)^{2r+1}}{(2z+j)_{k+1}} \, .$$

Application of Proposition 3 gives, after some simplification, the following formulas for $U_r(n)$, valid for $r \ge 1$ and $n \ge 0$:

$$U_{2r}(n) = 2^{n+1} \sum_{1 \le j \le k \le r} (-1)^j \frac{\left(-\frac{n}{2}\right)_k \left(\frac{1}{2}\right)_k}{(k-j)!(k+j)!} \, j^{2r}, \tag{20}$$

$$U_{2r+1}(2n) = 2n \binom{2n}{n} \sum_{1 \le j \le k \le r} (-1)^j \frac{(-n)_k}{(k-j)! \, (k+1)_j} \, j^{2r}, \tag{21}$$

$$U_{2r-1}(2n-1) = \binom{2n}{n} \sum_{1 \le j \le k \le r} (-1)^j \frac{(-n)_k}{(k-j)! \, (k)_j} \, \left(j - \tfrac{1}{2}\right)^{2r-1} . \tag{22}$$

# 4 Generating functions

The exponential generating function (egf)

$$\sum_{r \ge 0} U_{2r}(n) \frac{z^{2r}}{(2r)!} = 2^n \cosh^n(z/2). \tag{23}$$

generalizes the egf

$$\sum_{r \ge 0} S_{2r}(n) \frac{z^{2r}}{(2r)!} = 2^{2n} \cosh^{2n}(z/2) \tag{24}$$

given by Tuenter [7, §5], since replacing $n$ by $2n$ in (23) gives (24). The proof of (23) is straightforward, and does not require the results of Carlitz.

We can obtain other egfs from the results of Carlitz. First, we note that Carlitz [3, eqn. (4.2)] gives the egf

$$\sum_{r \ge 1} (-1)^r F_r(x,y,1) \frac{z^{2r}}{(2r)!} = \frac{1}{xy} \sum_{k \ge 1} (-1)^k \frac{(x)_k (y)_k}{(2k)!} \left(2 \sinh \frac{z}{2}\right)^{2k} . \tag{25}$$

In view of (4) and (8), this allows us to obtain an egf for $U_{2r+1}(2n)$:

$$\sum_{r \ge 0} U_{2r+1}(2n) \frac{z^{2r}}{(2r)!} = n \binom{2n}{n} \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \binom{2k}{k}^{-1} \sinh^{2k}\left(\frac{z}{2}\right) \cdot \tag{26}$$

In order to calculate $U_{2r+1}(2n)$ from (26), it is only necessary to sum the terms on the right-hand side for $k \le \min(r, n)$.

The remaining case $U_{2r+1}(2n-1)$ is more difficult because (25) does not apply. We can use the egf

$$\begin{aligned}\sum_{r\ge 0}(-1)^r F_{r+1}(x,y,z)\frac{u^{2r+1}}{(2r+1)!}\\ &= 2\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^j(x+z)_k(y+z)_k(2z)_j}{j!(k-j)!(2z)_{k+1}(2z+k+1)_j}\sinh((z+j)u), \qquad (27)\end{aligned}$$

which follows from the discussion in Carlitz [3, pp. 221–222]. Using (13), (17) and (27), after some simplification followed by a change of variables ($u \mapsto z$), we obtain the egf:

$$\begin{aligned}\sum_{r\ge 0}U_{2r+1}(2n-1)\frac{z^{2r+1}}{(2r+1)!}\\ &= n\binom{2n}{n}\sum_{0\le j\le k<n}\frac{(-1)^{k-j}\binom{n-1}{k}\binom{2k}{k-j}}{\binom{2k}{k}}\,\frac{\sinh\left(\left(j+\frac{1}{2}\right)z\right)}{j+k+1}\,, \qquad (28)\end{aligned}$$

which is valid for $n \ge 1$.

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## Appendix 1: The polynomials $\overline{P}_r, \overline{Q}_r$ for $r \le 5$

$$\begin{aligned}
\overline{P}_0(n) &= 1,\\
\overline{P}_1(n) &= 4n - 3,\\
\overline{P}_2(n) &= 32n^2 - 56n + 25,\\
\overline{P}_3(n) &= 384n^3 - 1184n^2 + 1228n - 427,\\
\overline{P}_4(n) &= 6144n^4 - 29184n^3 + 52416n^2 - 41840n + 12465,\\
\overline{P}_5(n) &= 122880n^5 - 829440n^4 +\\
&\qquad 2258688n^3 - 3076288n^2 + 2079892n - 555731,\\
\overline{Q}_0(n) &= 1,\\
\overline{Q}_1(n) &= 2n + 1,\\
\overline{Q}_2(n) &= 12n^2 + 8n + 1,\\
\overline{Q}_3(n) &= 120n^3 + 60n^2 + 2n + 1,\\
\overline{Q}_4(n) &= 1680n^4 - 168n^2 + 128n + 1,\\
\overline{Q}_5(n) &= 30240n^5 - 25200n^4 + 5040n^3 + 7320n^2 - 2638n + 1.
\end{aligned}$$

A similar table for Tuenter's polynomials $P_r(n), Q_r(n)$ may be found in Brent [1]. The triangles of coefficients of $-P_r(-n)/n, -Q_r(-n)/n$, and $\overline{Q}_r(-n)$ for $r \ge 1$ are OEIS [6] sequences A036970, A083061, and A160485 respectively. We have contributed the coefficients of $\overline{P}_r(n)$ as sequence A245244. The values $(-1)^r\overline{P}_r(0)$ are sequence A009843 (see Appendix 2 for details).

The bijection (19) between A083061 and A160485 (by a shift of $\pm\frac{1}{2}$ and scaling by a power of 2) was not mentioned in the relevant OEIS entries as at July 14, 2014; we have now contributed comments to this effect.

## Appendix 2: Special values of $P_r, Q_r, \overline{P}_r, \overline{Q}_r$

In Table 1, we give values of the polynomials $P_r, Q_r, \overline{P}_r$, and $\overline{Q}_r$ at $0, 1, \infty$. Here $P(\infty)$ denotes the leading coefficient of the polynomial $P(z)$, $\delta_{i,j}$ is the Kronecker delta, and $\mathcal{S}_r$ is the $r$-th Secant number. The values $(-1)^r\overline{P}_r(0)$ are OEIS sequence A009843, and are given by the egf

$$\sum_{r=0}^{\infty} \overline{P}_r(0)\frac{x^{2r+1}}{(2r+1)!} = \frac{x}{\cosh x}\,. \tag{29}$$

They may be expressed in terms of the Secant numbers $\mathcal{S}_r$, which comprise OEIS sequence A000364. In view of (17), we obtain a special value of the Dumont-Foata polynomials:

$$F_{r+1}\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) = 2^{-2r}(2r+1)\mathcal{S}_r. \tag{30}$$

| $z$ | $P_r(z)$ | $Q_r(z)$ | $\overline{P}_r(z)$ | $\overline{Q}_r(z)$ |
|---|---|---|---|---|
| 0 | $\delta_{0,r}$ | $\delta_{0,r}$ | $(-1)^r(2r+1)\mathcal{S}_r$ | 1 |
| 1 | 1 | $\max(1, 2^{r-1})$ | 1 | $(3^{2r}+3)/4$ |
| $\infty$ | $r!$ | $(2r)!/(2^r r!)$ | $2^{2r}r!$ | $(2r)!/r!$ |

Table 1: Special values of the polynomials

The values $\overline{Q}_r(1)$ comprise OEIS sequence A054879. The values in the last row of Table 1 may also be found in OEIS: they are sequences A000142, A001147, A047053, and A001813.

Tuenter [7] observed that, for $r \geq 1$, the constant terms of $-P_r(n)/n$ are the Genocchi numbers (A001469), and the constant terms of $(-1)^{r-1}Q_r(n)/n$ are the reduced tangent numbers (A002105).

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