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Exact and Asymptotic Evaluation of the Number of Distinct Primitive Cuboids

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Abstract

We express the number of distinct primitive cuboids with given odd diagonal in terms of the twisted Euler function with alternating Dirichlet character of period four, and two counting formulas for binary sums of squares. Based on the asymptotic behaviour of the sums of these formulas, we derive an approximation formula for the cumulative number of primitive cuboids.

1 Introduction

A primitive cuboid is a rectangular parallelepiped with natural edges and inner diagonal that have no common factor. It is best described by a solution in non-zero natural numbers x, y, z, t of the Diophantine equation $x^2 + y^2 + z^2 = t^2$ satisfying $\gcd(x, y, z, t) = 1$, where x, y, z are the edges and t is the inner diagonal.

With the exception of Shanks [23], [24, Thm. 86, p. 293], which counts the number of primitive cuboids with prime diagonal, no general counting formula seems to be known. Generalizing the proof of Shanks and using results on the twisted Euler function, an exact and an asymptotic formula for the number of distinct primitive cuboids is derived. A more detailed account of the content follows.

[1]Dedicated to the 75th birthday of N. J. A. Sloane on October 10, 2014.

Section 2 begins with a brief historical survey. The existence of solutions to the primitive cuboid equation was settled by Hurwitz [11] who also gave a formula for the total number of representations of a square as a sum of three squares counting zeros, permutations and sign changes. The total number of primitive representations was first evaluated by Gauss [6]. In modern number theory, it belongs to a variety of similar formulas with dimensions going up to twelve as in Cooper and Hirschhorn [2]. Gauss's formula is reinterpreted in terms of the twisted Euler function associated to the Dirichlet character of period four. Lemma 4 states the required exact counting formula. Section 3 derives an asymptotic formula for the corresponding cumulative number of primitive cuboids and illustrates its very good approximation.

2 Primitive cuboids and the twisted Euler function

It is well-known that the solutions of the Diophantine equation, also called cuboids and Pythagorean quadruples,

$$x^2 + y^2 + z^2 = t^2 \tag{1}$$

in non-zero natural numbers x, y, z, t can be obtained from the identity of Lebesgue [17]

$$x = p^2 + q^2 - r^2 - s^2,\ y = 2(pr + qs),\ z = 2(ps - qr),\ t = p^2 + q^2 + r^2 + s^2.$$

Since every integer is a sum of four squares by the theorem of Lagrange [16], it has been stated [3, Chap. VII, p. 265], and [20, p. 194], that every square can be written as a sum of three squares. However, when restricted to non-zero squares, all numbers of the form 2^k and $2^k \cdot 5$ are exceptions, a result due to Hurwitz ([11], [3, p. 271], [25, p. 101]).

Theorem 1. *The three squares equation $x^2 + y^2 + z^2 = t^2$, $0 < x \leq y \leq z$, has a solution if, and only if, the positive integer t is not of the form 2^k or $2^k \cdot 5$, $k \in \mathbb{N}$.*

Proof. Since a proof by Hurwitz is not available, one must rely on Gordon and Fraser [9] or Fraedrich [5]. □

Hurwitz [11] also stated (without proof) a formula for the number of representations $r_3(t^2)$, where in general $r_k(m)$ denotes the total number of representations of m as a sum of $k \geq 2$ squares such that $x_1^2 + x_2^2 + \cdots + x_k^2 = m$ counting zeros, permutations and sign changes. Dickson [3, p. 271] reproduces this formula. A modern version of it with an elementary proof is due to [15].

Theorem 2. *Given the unique decomposition $t = \prod_{i=1}^m p_i^{s_i}$ in prime numbers p_i and power exponents s_i, one has $r_3(t^2) = 6 \cdot \prod_{i=1}^m \left(\sigma(p_i^{s_i}) - (-1)^{(p_i-1)/2}\sigma(p_i^{s_i-1})\right)$, where $\sigma(m)$ counts the sum of all positive divisors of m.*

What about the number of primitive representations $R_3(t^2)$, where similarly to the preceding $R_k(m)$ is the number of primitive representations of m as a sum of $k \geq 2$ squares counting zeros, permutations and sign changes? In general, given a formula for $r_k(m)$, it is possible to derive a formula for $R_k(m)$ from it through Möbius inversion of the basic relationship [10, Thm. 1, Section 1.1]:

$$r_k(m) = \sum_{d|m^2} R_k\left(\frac{m}{d^2}\right).$$

Cooper and Hirschhorn [2] exploit this technique and obtain a wide variety of formulas for $R_k(m)$ including the range $2 \le k \le 8$ for any m, and the range $9 \le k \le 12$ for certain values of m. In particular, one has [2, Thm. 2, Eq. (1.19)]

$$R_3(t^2) = 6 \cdot \prod_{i=1}^{m} p_i^{s_i-1}\left(p_i - (-1)^{(p_i-1)/2}\right). \tag{2}$$

Recall that the original evaluation of $R_3(m)$ is due to Gauss [6], [3, Chap. VII, p. 262]. Restricting the attention to primitive representations with non-zero entries and $\gcd(x, y, z, t) = 1$, Theorem 1 implies that there exist primitive cuboids with odd diagonals for all $t \neq 5$. A formula for the number of distinct primitive cuboids with odd diagonal t does not seem to exist in the literature, with the exception of a prime number $t = p$, a result due to Shanks [23], [24, Thm. 86, p. 293]. Adopting a terminology similar to the above, let us denote by $R_k^d(m)$ the number of distinct primitive representations of m as a sum of $k \ge 2$ non-zero squares such that $x_1^2 + x_2^2 + \cdots + x_k^2 = m$ with $\prod_{j=1}^{k} x_j \neq 0$. Then the number of distinct primitive cuboids with odd diagonal is described by the arithmetic function $R_3^d(t^2)$.

Theorem 3 (Shanks)**.** *For a prime of the form $p = 8n \pm 1$ or $p = 8n \pm 5$, one has $R_3^d(p^2) = n$.*

We present an alternative but more general formula for arbitrary odd diagonal $t \ge 3$ based on analytic number theory. This viewpoint is best suited to derive an asymptotic formula for the cumulative number of primitive cuboids with odd diagonal $t \ge 3$ less than or equal to x, as done in Section 3. Starting point is the observation that the finite product in (2) identifies with the *twisted Euler (totient) function*

$$\varphi(t, \chi) = t \cdot \prod_{p|t} (1 - \chi(p)/p), \tag{3}$$

where the subscript $p|t$ stands for the primes p_i that divides $t = \prod_{i=1}^{m} p_i^{s_i}$, and $\chi(\cdot)$ is the alternating multiplicative *Dirichlet character* of period four defined by

$$\chi(p) = \begin{cases} 0, & \text{if } p = 2; \\ 1, & \text{if } p \equiv 1 \pmod 4); \\ -1, & \text{otherwise.} \end{cases} \tag{4}$$

Inserted into (2) the equation gives $R_3(t^2) = 6 \cdot \varphi(t, \chi)$. This formula yields the number of primitive solutions of (1) counting permutations, sign changes, and zeros. The distinct solutions are of three different forms, namely (x, y, z), (x, y, y) and $(x, y, 0)$, where $0 < x < y < z$. Counting permutations and sign changes the number of resulting representations for each of these forms are, respectively, 48 for the form (x, y, z) and 24 for the forms (x, y, y) and $(x, y, 0)$. Now the number of primitive (respectively, distinct primitive) representations of t^2 as a sum of two squares is equal to, respectively, [2, Thm. 1, Eq. (1.6)]

$$R_2(t^2) = \begin{cases} 4 \cdot 2^m, & \text{if } p_i \equiv 1 \pmod 4,\ i = 1, \ldots, m; \\ 0, & \text{otherwise;} \end{cases}$$

$$R_2^d(t^2) = \begin{cases} 2^{m-1}, & \text{if } p_i \equiv 1 \pmod 4,\ i = 1, \ldots, m; \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, if one denotes the number of primitive representations of the form $x^2 + 2y^2 = t^2$ by $R_2(t^2; 2)$ and the corresponding number of distinct ones by $R_2^d(t^2; 2)$, one has

$$R_2(t^2; 2) = \begin{cases} 4 \cdot 2^{m-1}, & \text{if } p_i \equiv 1, 3 \pmod 8,\ i = 1, \ldots, m; \\ 0, & \text{otherwise;} \end{cases}$$

$$R_2^d(t^2; 2) = \begin{cases} 2^{m-1}, & \text{if } p_i \equiv 1, 3 \pmod 8,\ i = 1, \ldots, m; \\ 0, & \text{otherwise.} \end{cases}$$

The total number of representations of these different forms must by (2) and (3) satisfy the basic relationship

$$48 \cdot \left(R_3^d(t^2) - R_2^d(t^2; 2)\right) + 24 \cdot R_2^d(t^2; 2) + 24 \cdot R_2^d(t^2) = R_3(t^2) = 6 \cdot \varphi(t, \chi).$$

The resulting counting formula is summarized as follows.

Lemma 4. *The number of distinct primitive cuboids with odd diagonal $t \geq 3$ is given by*

$$R_3^d(t^2) = \tfrac{1}{8} \cdot \varphi(t, \chi) + \tfrac{1}{2}\left(R_2^d(t^2; 2) - R_2^d(t^2)\right). \tag{5}$$

Clearly, the exact calculation of all three terms in (5) requires a factorization table of the distinct prime factors for all odd numbers. Since such tables are necessarily limited to the finite computing and storage capacity of computers, a search for alternative computational tools is necessary. It turns out to be more efficient to study the partial sums below x of the twisted Euler function over all natural numbers $2 \leq n \leq x$ (including even ones), which is denoted by

$$\Phi(x, \chi) = \sum_{2 \leq n \leq x} \varphi(n, \chi).$$

Such sums have been recently studied by Kaczorowski [12] and Kaczorowski and Wiertelak [13, 14], where the required result will be stated in Theorem 6. For now, some preliminary formula that accounts separately for sums over odd and even numbers is needed. The elementary analysis applies as well to other multiplicative Dirichlet characters $\chi(\cdot)$ satisfying $\chi(2)=0$. Consider the partial sums of the twisted Euler function over odd and even numbers denoted by

$$\Phi_o(x,\chi)=\sum_{\substack{3\le n\le x\\ n\ \text{odd}}}\varphi(n,\chi),\quad \Phi_e(x,\chi)=\sum_{\substack{2\le n\le x\\ n\ \text{even}}}\varphi(n,\chi),\quad \Phi(x,\chi)=\Phi_o(x,\chi)+\Phi_e(x,\chi). \tag{6}$$

Lemma 5. *Let $\chi(\cdot)$ be a multiplicative character satisfying $\chi(2)=0$. Then the even partial sums in (6) are determined by the formula $\Phi_e(x,\chi)=2\cdot(1+\Phi(2^{-1}\cdot x,\chi))$.*

Proof. Under the assumption $\chi(2)=0$ the following basic identity holds:

$$\Phi(x,\chi)=\sum_{k=1}^{\lfloor \ln x/\ln 2\rfloor}2^k+\sum_{k=0}^{\lfloor \ln x/\ln 2\rfloor-1}2^k\cdot\Phi_o(2^{-k}\cdot x,\chi). \tag{7}$$

Indeed, if $2\le n\le x$ is even, then there exists $1\le k\le\lfloor \ln x/\ln 2\rfloor$ such that $n=2^k\cdot m$ with m odd such that $1\le m\le 2^{-k}\cdot x$, which implies that

$$\Phi_e(x,\chi)=\sum_{k=1}^{\lfloor \ln x/\ln 2\rfloor}2^k+\sum_{k=1}^{\lfloor \ln x/\ln 2\rfloor-1}2^k\cdot\Phi_o(2^{-k}\cdot x,\chi).$$

Changing the index of summation and noting that $\lfloor \ln x/\ln 2\rfloor-1=\lfloor \ln(x/2)/\ln 2\rfloor$, one sees immediately that

$$\Phi_e(x,\chi)=2\cdot\left(1+\sum_{j=1}^{\lfloor \ln(x/2)/\ln 2\rfloor}2^j+\sum_{j=0}^{\lfloor \ln(x/2)/\ln 2\rfloor-1}2^j\cdot\Phi_o(2^{-j}\cdot(x/2),\chi)\right),$$

which coincides with $2\cdot(1+\Phi(2^{-1}\cdot x,\chi))$ by (7). □

3 The cumulative number of primitive cuboids

Based on (5) the total number of primitive cuboids with odd diagonal $3\le t\le x$ is equal to

$$N_3(x)=\tfrac{1}{8}\cdot\Phi_o(x,\chi)+\tfrac{1}{2}\cdot\left(N_2(x;2)-N_2(x)\right), \tag{8}$$

with the following cumulative counting functions

$$N_3(x)=\sum_{\substack{3\le t\le x\\ t\ \text{odd}}}R_3^d(t^2),\ \ N_2(x;2)=\sum_{\substack{3\le t\le x\\ t\ \text{odd}}}R_2^d(t^2;2),\ \ N_2(x)=\sum_{\substack{3\le t\le x\\ t\ \text{odd}}}R_2^d(t^2). \tag{9}$$

In the following, we determine the asymptotic behaviour of these counting functions. To obtain the one for $\Phi_o(x,\chi)$ it suffices by Lemma 5 to find the one for

$$\tilde{\Phi}(x,\chi)=\sum_{1\le n\le x}\varphi(n,\chi)=1+\Phi(x,\chi).$$

Now the twisted Euler function $\varphi(n,\chi)$ is related to the *Dirichlet L-function*, introduced by Dirichlet [4], via its Euler product through the identity

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1-\chi(p)/p)^{-1}, \quad \mathrm{Re}(s) > 1,$$

and the following result about sums of twisted Euler functions.

Theorem 6. *For all $x \geq 1$ one has the asymptotic relationship*

$$\tilde{\Phi}(x,\chi) = \sum_{1\leq n\leq x} \varphi(n,\chi) = \frac{1}{2} L(2,\chi)^{-1} \cdot x^2 + O(x\ln(2x)).$$

Proof. Consult Kaczorowski and Wiertelak [13], and Kaczorowski [12, Thm. 1.1]. □

Applied to our situation, one obtains for the Dirichlet character (4) the asymptotic formula $\Phi(x,\chi) \sim \tilde{\Phi}(x,\chi) \sim (2G)^{-1} \cdot x^2 \quad (x \longrightarrow \infty)$, with *Catalan's constant*

$$G = L(2,\chi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \doteq 0.915965594177.$$

Making use of Lemma 5, one sees that $\Phi_0(x,\chi) = \Phi(x,\chi) - 2\cdot(1+\Phi(2^{-1}\cdot x,\chi))$, which implies the required asymptotic relationship

$$\Phi_0(x,\chi) \sim (4G)^{-1} \cdot x^2 \quad (x \longrightarrow \infty).$$

Catalan's constant is described by the sequence A006752 in Sloane [26]. It has originally been computed to 14 decimals by Catalan [1] and to 24 decimals by Bresse in 1867 making use of a technique from Kummer. It has been computed to 20 and 32 decimals by Glaisher [7, 8] and to $3.1026 \cdot 10^{10}$ decimal digits by Yee and Chan in 2009. Lima [19] and Yee [27] provide more details.

It remains to determine the asymptotic behaviour of the counting functions $N_2(x;2)$, $N_2(x)$ defined in (9). This was achieved a long time ago. Lehmer [18, p. 329] obtains the asymptotic formulas

$$N_2(x;2) \sim \frac{\sqrt{2}\cdot x}{2\pi}, \quad N_2(x) \sim \frac{x}{2\pi}.$$

Inserting the above into the equation (8) one obtains the following result.

Theorem 7. *The cumulative number of primitive cuboids satisfies the asymptotic formula*

$$N_3(x) = \tfrac{1}{8}\cdot\Phi_o(x,\chi) + \tfrac{1}{2}\cdot(N_2(x;2) - N_2(x)) \sim \frac{x^2}{32G} + \frac{(\sqrt{2}-1)\cdot x}{4\pi} \quad (x \longrightarrow \infty). \tag{10}$$

limit x	exact	asymptotic	difference	error (in %)
100	347	344	-3	0.86
200	1364	1371	7	0.51
300	3079	3080	1	0.03
400	5484	5472	-12	0.22
500	8541	8546	5	0.06
600	12299	12302	3	0.02
700	16750	16740	-10	0.06
800	21837	21861	24	0.11
900	27664	27664	0	0.00
1000	34163	34150	-13	0.04

Table 1: Exact and asymptotic counts

It is remarkable that exact and asymptotic counts do not differ very much (at least for lower values of x). We conclude with some related comments. Our main results, namely Lemma 4, (8), and Theorem 7 extend to primitive Pythagorean quadruples the long known similar results for primitive Pythagorean triples. For example, the number of primitive Pythagorean triples with hypotenuse less than or equal to x is approximately equal to $x/2\pi$ [18, p. 28]. The exact count is described by Sloane's OEIS sequence A020882. Roque [21, 22] provides algorithms to generate and count them exhaustively.

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(Concerned with sequences A006752 and A020882.)

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