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On Linear Recurrence Equations Arising from Compositions of Positive Integers

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Abstract

For an arithmetic function f_0, we define a new arithmetic function f_1, generalizing the linear recurrence for the numbers of compositions of positive integers. Using f_1 in the same way, we then define f_2, and so on.

We establish some patterns related to the sequence $f_1, f_2, \ldots$. Our investigations depend on the following result: if f_0 satisfies a linear recurrence equation of order k, then each function f_m will also satisfy a linear recurrence equation of the same order.

In several results, we derive a recurrence equation for $f_m(n)$, in terms of m and n. For each result, we give a combinatorial meaning for $f_m(n)$ in terms of the number of restricted words over a finite alphabet.

We also find new combinatorial interpretations of the Fibonacci polynomials, as well as the Chebyshev polynomials of the second kind.

1 Introduction

Researchers usually investigate compositions of positive integers through their generating functions, connecting them to the so-called invert transforms.

In this paper, we take a different approach. Investigating some generalizations of the linear recurrence equations for the numbers of compositions, we proceed to show that certain linear recurrences may be considered as recurrences for the numbers of some kinds of compositions.

In conclusion, we provide a number of results, which illustrate some consequences of our findings. We are interested in two particular problems: obtaining a recurrence equation for $f_m(n)$ when f_0 is given, and investigating relationships of the sequence $f_1, f_2, \ldots$ with the number of restricted words over a finite alphabet.

In our interpretation of $f_m(n)$ in terms of restricted words, the number of letters in the alphabet is determined by the parameter m, while the length of a word is determined by the parameter n.

In particular, if f_0 takes values in $\{0,1\}$, then $f_1(n)$ counts the number of some compositions of n, so that f_1 always produces a relationship between the compositions and the binary words.

In fact, our sequence of functions might be understood as a way to transfer some properties from the binary words to words over a finite alphabet.

We derive connections our sequence of our sequence with the Fibonacci polynomials and the Chebyshev polynomials of the second kind. Fibonacci numbers are generalized in several ways.

We finish the paper by giving the combinatorial meaning for a class of sequences when f_0 satisfies a particular linear recurrence equation of second order with constant coefficients.

We start with the definition of our sequence.

Definition 1. We define a pair (f_0, f_1) of arithmetic functions as follows: Let f_0 be an arbitrary arithmetic function. A function f_1 is recursively defined through the convolution

$$f_1(n) = \sum_{i=1}^{n} f_0(i) f_1(n-i),\ (n \geq 1), \tag{1}$$

where $f_1(0) = 1$.

Inductively, for each $m > 0$, we define the pair (f_{m-1}, f_m).

Corollary 2. *For each $m \geq 1$, we have*

$$f_m(1) = f_0(1),\ f_m(2) = m f_0(1)^2 + f_0(2).$$

Proof. We use induction with respect to m. From Equation (1), we conclude that the claim holds for $m = 1$. Suppose that the claim is true for m. Then $f_{m+1}(1) = f_m(1)$, and so $f_{m+1}(1) = f_0(1)$. In the same way, we obtain $f_{m+1}(2) = f_m(1)^2 + f_m(2)$. Applying the induction hypothesis gives

$$f_{m+1}(2) = f_0(1)^2 + m f_0(1)^2 + f_0(2) = (m+1) f_0(1)^2 + f_0(2).$$

□

Remark 3. If $f_0(i) = 1, (i \geq 1)$ then Equation 1 becomes the recurrence for the numbers of compositions of positive integers.

Remark 4. Note that f_m is a particular case of 1-determinants, defined in Janjić [2].

Using the notion of the invert transform, we easily find the generating function for the sequence $f_1(1), f_1(2), \ldots$.

Proposition 5. *The sequence* $\{f_1(n)\}_{n=1}^{\infty}$ *is the invert transform of the sequence* $\{f_0(n)\}_{n=1}^{\infty}$.

Proof. It is easy to see that Equation (1) is equivalent to

$$\left[1+\sum_{i=1}^{\infty} f_1(i)x^i\right] \cdot \left[1-\sum_{i=1}^{\infty} f_0(i)x^i\right] = 1.$$

Hence, the generating function of the sequence $f_1(1), f_1(2), \ldots$ is

$$\frac{1}{1-\sum_{i=1}^{\infty} f_0(i)x^i}.$$

□

In general, an arithmetic function maps the set of positive (or nonnegative) integers into the set of complex numbers. The results of the next section are valid for such functions.

2 A result on linear recurrences

We proceed to prove that, if f_0 satisfies a linear recurrence equation of order k, then f_1 also satisfies a linear recurrence of the same order. To stress the dependence of f_0 and f_1 on k, we write $f_i(n;k)$ instead of $f_i(n)$ for $i = 1, 2$.

Theorem 6. *Let* f_0 *be an arithmetic function, and let* k *be a positive integer. If there exist constant numbers (complex)* $a_0(1), a_0(2), \ldots, a_0(k)$ *such that*

$$f_0(n+k;k) = \sum_{i=1}^{k} a_0(i) f_0(n+k-i;k), (n \geq 1), \tag{2}$$

where $f_0(1;k), f_0(2;k), \ldots, f_0(k;k)$ *are arbitrary numbers (complex), then we have*

$$f_1(i;k) = \sum_{j=1}^{i} f_0(j;k) f_1(i-j;k), (i = 1, 2, \ldots, k), \text{ and} \tag{3}$$

$$f_1(n+k;k) = \sum_{i=1}^{k} a_1(i) f_1(n+k-i;k), (n \geq 1), \tag{4}$$

where

$$a_1(1) = a_0(1) + f_0(1;k),$$

$$a_1(i) = a_0(i) + f_0(i;k) - \sum_{j=1}^{i-1} a_0(j) f_0(i-j;k), (2 \leq i \leq k). \tag{5}$$

Proof. We first consider the case $k=1$. If $f_0(n+1;1) = a_0(1)f_0(n;1), (n \geq 1)$, then

$$f_1(n+1;1) = \sum_{i=1}^{n+1} f_0(i;1)f_1(n+1-i;1) = f_0(1;1)f_1(n;1) + \sum_{i=2}^{n+1} f_0(i;1)f_1(n+1-i;1),$$

which implies $f_1(n+1;1) = [f_0(1;1) + a_0(1)]f_1(n;1)$. Hence, Theorem 6 is true for $k=1$.

Assume that $k > 1$. For $1 \leq i \leq k$, we have

$$f_1(n+k-i;k) = \sum_{j=1}^{n+k-i} f_0(j;k)f_1(n+k-i-j;k).$$

We denote $f_1(n+k-i;k) = X_i + Y_i, (i = 1, \ldots, n)$, where

$$X_i = \sum_{j=1}^{k-i} f_0(j;k)f_1(n+k-i-j;k),\ Y_i = \sum_{j=k-i+1}^{n+k-i} f_0(j;k)f_1(n+k-i-j;k), \tag{6}$$

with $X_k = 0$. It follows that

$$\sum_{i=1}^{k} a_0(i)X_i = \sum_{i=2}^{k}\left[\sum_{j=1}^{i-1} a_0(j)f_0(i-j;k)\right] f_1(n+k-i;k). \tag{7}$$

Also, after some calculations, we obtain

$$\sum_{i=1}^{k} a_0(i)Y_i = f_1(n+k;k) - \sum_{i=1}^{k} f_0(i;k)f_1(n+k-i;k). \tag{8}$$

From equations (7) and (8), we obtain

$$\sum_{i=1}^{k} a_i f_1(n+k-i;k) = \sum_{i=1}^{k} a_i(X_i+Y_i) =$$

$$\sum_{i=2}^{k}\left[\sum_{j=1}^{i-1} a_0(j)f_0(i-j;k)\right] f_1(n+k-i;k) + f_1(n+k;k) - \sum_{j=1}^{k} f_0(j;k)f_1(n+k-j;k).$$

We finally have

$$f_1(n+k;k) = \sum_{i=1}^{k}[a_i + f_0(i;k)]f_1(n+k-i;k) - \sum_{i=2}^{k}\left[\sum_{j=1}^{i-1} a_i f_0(i-j,k)\right] f_1(n+k-i;k),$$

which is Equation (4), under the conditions (3). □

Remark 7. In the conditions of Theorem 6, we may always find an explicit formula for $f_1(n;k)$, when $k = 1,2,3,4$ by solving the characteristic equations.

The simplest case of Theorem 6 is

Corollary 8. *If* $f_0(n+1) = af_0(n), (n \geq 1)$*, for some* a*, then for* $m \geq 1$ *we have*

$$f_m(n+1) = [mf_0(1) + a]f_m(n), (n \geq 1).$$

The explicit formula for f_m *is*

$$f_m(n) = f_0(1)[mf_0(1) + a]^{n-1}, (n \geq 1).$$

Proof. For $m = 1$, according to Theorem 6, we have $a_1(1) = a + f_0(1)$, and the claim is true for $m = 1$. The rest follows by induction. □

We finished this section by stating the recurrence equation for $f_m, (m \geq 1)$, assuming that f_0 satisfies a linear recurrence of order 2.

Corollary 9. *If* $f_0(1), f_0(2)$ *are arbitrary, and*

$$f_0(n+2) = a_0(1)f_0(n+1) + a_0(2)f_0(n),$$

then

$$f_m(1) = f_0(1),\ f_m(2) = mf_0(1)^2 + f_0(2), \tag{9}$$

and

$$f_m(n+2) = a_m(1)f_m(n+1) + a_m(2)f_m(n), \tag{10}$$

where $a_m(1) = a_0(1) + mf_0(1), a_m(2) = a_0(2) - ma_0(1)f_0(1) + mf_0(2)$*.*

Proof. This is a particular case of Theorem 6. □

We stress that in all that follows the values of arithmetical functions $f_m, (m \geq 0)$ will be nonnegative integers.

3 Compositions and words

The function f_1 generalizes the notions of the numbers of compositions of positive integers for several kinds of compositions. We state some of them.

Corollary 10.

1. *If* $f_0(i)$ *is either* 1 *or* 0 *and* $Q = \{i : f_0(i) = 1\}$*, then* $f_1(n)$ *equals the number of compositions of* n*, the parts of which belong to* Q*.*
2. *If* $f_0(i) = 1$ *for all* i*, then* $f_1(n)(= 2^{n-1})$ *equals the number of all compositions of* n*.*
3. *If* $f_0(i) \geq 0$ *for all* i*, then* $f_1(n)$ *equals the number of the colored compositions of* n*, in which part* i *may appear in* $f_0(i)$ *different colors.*

Proof. All claims are easy to prove. □

Remark 11. Note that Claim 3 above is, in fact, the combinatorial interpretation of the invert transform in Bernstein and Sloane [1].

The Catalan numbers C_i, $(i = 0, 1, \ldots)$ may also be considered as a kind of colored compositions.

Proposition 12. *If $f_0(i) = C_{i-1}$ for all $i \geq 1$, then $f_1(n) = C_n$.*

Proof. The claim follows from Segner's formula for the Catalan numbers; see Koshy [5, Formula (5.6), p. 117]. □

Now we show that the function f_1 also counts the number of the k-matrix compositions, considered by Munarini at al. [6].

A k-matrix composition of n is a matrix with k rows, in which the entries are nonnegative integers, there are no columns consisting of zeros only, and the sum of all entries equals n. Let $\mathrm{mc}(n;k)$ denote the number of such compositions. We have

Proposition 13. *If $f_0(i) = \binom{k+i-1}{k-1}$, $(i = 1, 2, \ldots)$, then $\mathrm{mc}(n;k) = f_1(n)$.*

Proof. For given $i, (1 \leq i \leq k)$, the equation $x_1 + x_2 + \cdots + x_k = i$ has $\binom{k+i-1}{k-1}$ nonnegative solutions. This means that there are $\binom{k+i-1}{k-1} \cdot \mathrm{mc}(n-i;k)$ k-matrix compositions of n, ending with a column having the sum of all elements equal i. Taking $\mathrm{mc}(0;k) = 1$, we obtain

$$\mathrm{mc}(n;k) = \sum_{i=1}^{n} \binom{i+k-1}{k-1} \mathrm{mc}(n-i;k).$$

Comparing this equation with Equation (1), we conclude that $\mathrm{mc}(n;k) = f_1(n)$. □

Remark 14. The sequences in Sloane [7], the members of which count k-compositions are: A003480, A145839, A145840, A145841, A161434.

We next note that, for a suitably chosen f_0, the function f_1 counts the number of particular partitions of positive integers. Namely, for the set $Q = \{q_1, q_2, \ldots, q_n\}$ of positive integers, we let $p(Q,n)$ denote the number of partitions of n, the parts of which belong to Q.

Janjić and Petković, in [3], consider the following function:

$$a_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{n+k+1}{i-j} \binom{j, n}{k, Q}, (i \geq 0),$$

where numbers $\binom{j,n}{k,Q}$ are defined in the following way:

1. $k = \sum_{t=1}^{n} (q_t - 1)$.
2. Consider a set X consisting of n blocks $X_1, X_2, \ldots, X_n$, such that $|X_i| = q_i$ for all i and an additional block Y, such that $|Y| = j$.

The number $\binom{j,n}{k,Q}$ is the number of $(n+k)-$subsets of X intersecting each block X_i, $(i = 1, 2, \ldots, n)$.

From Janjić [4, Theorem 2], we obtain

Proposition 15. *If $f_0(i) = -a_i$ for all i, then*

$$f_1(n) = p(Q, n),\ (n = 1, 2, \ldots, n).$$

Remark 16. Some sequences in Sloane [7] generated by f_1 in Proposition 15 are: A001401, A001045, A008616, A008676, A109707, A025795, A008677, A025839, A029144, A029280.

Corollary 17. *If $a = 1$, $m = 1$, and $f_0(1) = p$, where p is a positive integer, then $f_1(n) = p(1+p)^{n-1}$, which is the number of the colored compositions of n, such that each part may appear in p different colors.*

Remark 18. Some sequences in Sloane [7] generated by f_1 from Corollary 17 are: A000079, A000244, A025192, A020699, A093138.

Assume that $f_0(i) = 1$, $(i \geq 1)$. Then, by Corollary 17 we have

$$f_m(n) = (m+1)^{n-1}.$$

In this particular case, we obtain the following relationship of our sequence and numbers of words over an alphabet with $m+1$ letters.

Corollary 19. *The number $f_m(n)$ equals the number of words of length $n-1$ over an alphabet of $m+1$ letters.*

We consider the case when values of f_0 are either 0 or 1. Take $g_0(i) = 1, (i = 1, 2, \ldots)$. Then, for all i, we have $f_0(i) \leq g_0(i)$. Since each term in Equation (1) is nonnegative, we conclude that $f_m(n) \leq g_m(n), (n \geq 0)$ for each m, and since $g_m(n) = (m+1)^{n+1}$, we obtain

Proposition 20. *If the values of f_0 are either 0 or 1, then f_m counts some restricted words over the alphabet $\{0, 1, \ldots, m\}$.*

Our further investigation may be summarized by the following two problems.

Problem 21. *Suppose that f_0 takes values either 0 or 1.*

1. *Find either a recurrence or an explicit formula for $f_m(n)$ in terms of m and n.*
2. *Describe the set of restricted words over the alphabet $\{0, 1, \ldots, m\}$ counted by $f_m(n)$.*

Problem 22. *Solve Problem 21 when f_0 may have values other than 0 and 1.*

4 Some results concerning Problem 21

Proposition 23. *Assume that $f_0(1) = 0$, and $f_0(i) = 1, (i > 1)$. Then we have*

$$f_m(1) = 0, f_m(2) = 1,$$

and

$$f_m(n+2) = f_m(n+1) + mf_m(n).$$

Proof. The function f_0 clearly satisfies the following linear recurrence of order 2.

$$f_0(1) = 0, f_0(2) = 1, \text{ and } f_0(n+2) = a_0(1)f_0(n+1) + a_0(2)f_0(n), (n \geq 1),$$

where $a_0(1) = 1, a_0(2) = 0$.

Now the assertion is an immediate consequence of Corollary 9. □

Corollary 24. *For $n \geq 0$, the number $f_m(n+2)$ from Proposition 23 counts the number of words of length $n-1$, over the alphabet $\{0, 1, \ldots, m\}$, where no two consecutive letters are nonzero.*

Proof. We let $g(n)$ denote the number of required words of length n. We have $g(0) = 1 = f_m(3)$, since only an empty word has length 0. Further, we have $g(1) = m + 1 = f_m(4)$.

Consider words of length $n+1, (n > 0)$. There are $g(n)$ such words beginning with 0. If a word begins with $j \neq 0$, then the next letter must be 0, which yields that there are $g(n-1)$ such words. Since j may take m different values, we see that $g(n+1) = g(n) + mg(n-1)$. It follows that $g(n-1) = f_m(n+2), (n \geq 0)$. □

Remark 25. In the case $m = 1$, Corollary 24 gives the following well-known property of the Fibonacci numbers: For $n \geq 1$, Fibonacci number F_{n+1} counts 11-avoiding binary words of length $n-1$. The case $m = 2$ gives the analogous property of the Jacosthal numbers.

Remark 26. Some sequences in Sloane [7] generated by f_m from Corollary 24 for different m's are: A000045, A001045, A006130, A006131, A015440, A015441, A015442, A015443, A015445, A015446, A015447, A053404.

Proposition 27. *We define*

$$f_0(i) = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even.} \end{cases}$$

Then

$$f_m(1) = 1, f_m(2) = m,$$

and

$$f_m(n+2) = mf_m(n+1) + f_m(n). \tag{11}$$

Proof. The function f_0 satisfies the following recurrence:

$$f_0(1) = 1, f_0(2) = 0,$$

$$f_0(n+2) = a_0(1)f_0(n+1) + a_0(2)f_0(n),$$

where $a_0(1) = 0, a_0(2) = 1$. The claim is now a simple consequence of Corollary 9. $\square$

Corollary 28. *If $f_0(i)$ is defined as in Proposition 27, then the number $f_m(n+1)$ equals the number n-length over the alphabet $\{0, 1, \ldots, m\}$ avoiding runs of zeros of odd lengths.*

Proof. We let $d(n)$ denote the number of required words of length n. Then, obviously, $d(0) = 1, d(1) = m$. For $n > 1$, there are $md(n-1)$ words of length n beginning with a nonzero letter. If a word begins with 0, then it must begin with at least two zeros. Hence there are $d(n-2)$ such words.

We conclude that $d(n) = f_m(n+1), (n \geq 0)$. $\square$

As an immediate consequence of Proposition 27, we obtain the following connection of our functions with Fibonacci polynomials $F_n(x)$.

Corollary 29. *In terms of Proposition 27, we have*

$$f_m(n) = F_n(m).$$

Proposition 27 also generalizes the Fibonacci number through the golden ratio.

Corollary 30. *If α and β are solutions of the characteristic equation of Equation (11), then the explicit formula for f_m in Proposition 27 is*

$$f_m(n) = C_1(\Phi_m)^n + C_2(-\Phi_m^{-1})^n,$$

where

$$C_1 = \frac{f_m(2) - \beta f_m(1)}{\alpha(\alpha - \beta)}, C_2 = -\frac{f_m(2) - \alpha f_m(1)}{\beta(\alpha - \beta)}.$$

For Φ_m, we have

1. *Φ_1 is the golden ratio, that is, $\Phi_1 = \frac{1+\sqrt{5}}{2}$.*

2. *Φ_2 is the silver ratio, that is, $\Phi_2 = 1 + \sqrt{2}$.*

3. *Φ_3 is the bronze ratio, that is, $\Phi_3 = \frac{3+\sqrt{13}}{2}$.*

4. *Generally, $\Phi_m = \frac{m+\sqrt{4+m^2}}{2}$.*

Remark 31. Some sequences from Sloane's [7] generated by recurrence from Corollary 30 are: A000045, A000129, A006190, A001076, A052918, A005668, A054413, A041025, A041041.

Proposition 32. *Let k be a positive integer. We define*

$$f_0(i;k) = \begin{cases} 1, & \text{if } 1 \le i \le k; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $m \ge 1$, we have

$$f_m(i;k) = (m+1)^{i-1}, (i = 1, 2, \ldots, k), \tag{12}$$

and

$$f_m(n+k;k) = m \sum_{i=1}^{k} f_1(n+k-i;k). \tag{13}$$

Proof. In this case, we have $f_0(i;k) = 1, a_0(i) = 0, (i = 1, 2, \ldots, k)$. Equation (3) now gives $a_1(i) = 1, (1 \le i \le k)$, which yields that Equation (13) holds for $m = 1$.

Assume that Equation (13) holds for m. To prove that $f_{m+1}(i;k) = (m+2)^{i-1}, (1 \le i \le k)$, it is enough to use induction and the identity:

$$(m+2)^{i-1} = (m+1)^{i-1} + \sum_{j=1}^{i-1} (m+1)^{j-1}(m+2)^{i-1-j}.$$

This identity is easy to prove.

We have $a_m(i) = m, (i = 1, \ldots, k)$. Using equation (5), we obtain $a_{m+1}(1) = m + f_m(1;k) = m+1$. For $2 \le i \le k$, using induction and the identity

$$m+1 = m + (m+1)^{i-1} - m \sum_{j=1}^{i-1} (m+1)^{i-1-j},$$

which is also easy to prove, we conclude that the claim holds for $m+1$. $\square$

It is known that F_{n+2} equals the number of 00-avoiding binary words of length n. Example 32 generalizes this property in the following way:

Corollary 33. *The number $f_m(n+1;k)$ from Proposition 32 equals the number of words of length n over the alphabet $\{0, 1, \ldots, m\}$, avoiding runs of k zeros.*

Proof. Let $g(n;k)$ denote the number of required words of length n. We have $g(0;k) = 1$, since only the empty word has length 0. Next, we have $g(i;k) = (m+1)^i, (1 \le i \le k-1)$, since no word of length $< k$ has a run of k zeros. Suppose that $n \ge k$. We calculate $g(n+k-1;k), (n \ge 1)$. There are $mg(n+k-2;k)$ words of length $n+k-1$, beginning with a nonzero letter. The remaining words begin with 0. It is clear that there are $mg(n+k-3;k)$ words beginning with an isolated zero. In the same way, we have $mg(n+k-4;k)$ words beginning with the run of two isolates zeros, and so on.

Finally, there are $mg(n+1;k)$ words beginning with $k-1$ successive zeros. Hence,

$$g(n+k-1;k) = m\sum_{i=1}^{k} g(n+k-1-i;k), (n \geq 1).$$

We see that $g(n;k)$ and $f_m(n+1;k)$ satisfy the same recurrence equation. $\square$

Remark 34. In particular, the function $f_1(n+1;2)$ counts binary words with no adjacent zeros. It follows that $f_1(n+1;2) = F_{n+2}$. The case $k=3$ produces analogous property of Tribonacci numbers, and so on.

Remark 35. Several sequences in Sloane [7] generated by f_m form Corollary 33 are: A000045, A028859, A155020, A125145, A086347, A180033, A180167, A000073, A119826, A000078, A209239, A001591, A001592, A122189, A079262, A104144, A122265, A168082, A168083, A168084, A220469, A220493, A249169.

5 Some results concerning Problem 22

We first derive a recursion for f_m in the case when f_0 is a linear function of i.

Proposition 36. *If $p, q \neq 0$ are arbitrary numbers, and $f_0(i) = q(i-1)+p, (i \geq 1)$, then*

$$f_m(1) = p,\ f_m(2) = q + p + mp^2, \tag{14}$$

$$f_m(n+2) = (mp+2)f_m(n+1) + (mq - mp - 1)f_m(n),\ (n > 0). \tag{15}$$

Proof. It is easy to see that f_0 satisfies the following recurrence:

$$f_0(1) = p, f_0(2) = q + p,$$

$$f_0(n+2) = a_0(1)f_0(n+1) + a_0(2)f_0(n),$$

where

$$a_0(1) = 2, a_0(2) = -1.$$

Therefore we may apply Corollary 9. It follows that

$$f_{m+1}(1) = f_m(1) = p,\ f_{m+1}(2) = [f_m(1)]^2 + f_m(2) =$$
$$p^2 + q + p + mp^2 = q + p + (m+1)p^2.$$

Also, we have

$$a_{m+1}(1) = a_m(1) + f_m(1) = mp + 2 + p = (m+1)p + 2,$$
$$a_{m+1}(2) = a_m(2) + f_m(2) - a_m(1)f_m(1) =$$
$$mq - mp - 1 + q + p + mp^2 - (mp+2)p = (m+1)q - (m+1)p - 1,$$

and the proposition is proved. $\square$

In a particular case of the preceding proposition, we find the combinatorial meaning of $f_m(n)$ in terms of restricted words.

Corollary 37. *1. If $f_0(i) = i, (i \geq 1)$ then, for $m \geq 1$, we have*

$$f_m(1) = 1,\ f_m(2) = m + 2, \tag{16}$$

$$f_m(n+2) = (m+2)f_m(n+1) - f_m(n),\ (n > 0). \tag{17}$$

2. The number $f_m(n)$ equals the number of 01-avoiding words of length $n-1$ over the alphabet $\{0, 1, 2, 3, \ldots, m+1\}$.

3. For $n \geq 1$, we have $f_m(n) = U_{n-1}(\frac{m+2}{2})$, where $U_n(x)$ is the nth Chebyshev polynomial of the second kind, evaluated at $x = \frac{m+2}{2}$.

Proof. 1. This is the case $q = p = 1$ of Proposition 36.

2. We let $g(n)$ denote the number of required words of length $n-1$. It is clear that $g(1) = 1$, and $g(2) = m + 2$.

   At the beginning of each word of length $n+1$ put a letter of $\{0, 1, \ldots, m+1\}$ to obtain $(m+2)g(n+1)$ words of length $n+2$. Exactly $g(n)$ of these words begin with 01. Subtracting this number from $(m+2)g(n+1)$, we conclude that g satisfies Recurrence (17), which means that $g(n) = f_m(n), (n \geq 1)$.

3. Equations (16) and (17) are the recurrence for the numbers $U_{n-1}(\frac{m+2}{2})$.

□

We see that, for $m = 1$, formula (17) is the recurrence for Fibonacci numbers of even indices, that is $f_1(n) = F_{2n}, (n = 1, 2, \ldots)$.

We stress two particular results. The first concerns a relationship between Fibonacci numbers and Chebyshev polynomials of the second kind, and the second describes the restricted words counting by F_{2n}.

Remark 38. 1. For $n \geq 1$, we have

$$F_{2n} = U_{n-1}\left(\frac{3}{2}\right).$$

2. The number F_{2n} equals the number of 01-avoiding words of length $n-1$ over the alphabet $\{0, 1, 2\}$.

Remark 39. Some sequences in Sloane [7] generated by Recurrence (17) are: A001906, A001353, A004254, A001109, A004187, A001090, A018913, A004189, A004190, A004191, A078362, A007655, A078364, A077412, A078366, A049660, A078368, A075843, A092499, A077421, A097778.

We finish our investigation with a result which gives a combinatorial interpretation in terms of restricted words for a class of the functions f_m, when f_0 satisfies a linear recurrence equation of the second order.

Proposition 40. *Let $p > 1$ and q be integers such that $1 \le q \le p$. Define f_0 recursively in the following way:*

$$f_0(1) = 1,\ f_0(2) = p,$$

and

$$f_0(n+2) = (p-1)f_0(n+1) + (p-q)f_0(n),\ (n \ge 1).$$

Then, for $m \ge 0$, we have

$$f_m(1) = 1, f_m(2) = m + p,$$

and

$$f_m(n+2) = (m+p-1)f_m(n+1) + (m+p-q)f_m(n). \tag{18}$$

Proof. We again apply Corollary 9. Clearly, $f_m(1) = 1, f_m(2) = m + p$. Further we have

$$a_{m+1}(1) = f_m(1) + a_m(1) = 1 + m + p - 1 = m + p,$$
$$a_{m+1}(2) = f_m(2) + a_m(2) - f_m(1)a_m(1) =$$
$$m + p + m + p - q - m - p + 1 = m + 1 + p - q.$$

□

Corollary 41. *For $m \ge 0$, the number $f_m(n+1)$ from Proposition 40 equals the number of ii-avoiding words of length n over the alphabet $\{0, 1, \ldots, m+p-1\}$, for $i \in \{0, 1, \ldots, q-1\}$.*

Proof. Let $g(n)$ be the number of required words of length n. We obviously have $g(0) = 1, g(1) = m + p$.

It remains to prove that $g(n)$ satisfies recurrence equation (18). There are $(m+p-q)g(n)$ words of length $n+2$ in which the first two letters are the same. The remaining words begin with two different letters. At the beginning of a word of length $n+1$ which begins with a letter i, put a letter j $(j \ne i)$ to obtain $(m+p-1)$ words of length $n+2$, the second letter of which is i. When i runs over all elements of the alphabet, we obtain all words of length $n+2$ beginning with two different letters. Hence,

$$g(n+2) = (m+p-1)g(n+1) + (m+p-q)g(n), (n \ge 0).$$

We conclude that $g(n) = f_m(n+1), (n \ge 0)$. □

Remark 42. Some sequences in Sloane [7] generated by the preceding corollary are: A000045, A000129, A126473 , A126501, A126528, A122391, A180037, A099842, A003948, A015451, A003949, A015453, A003950, A015454, A003951, A015455, A003952, A003953, A015456, A015457, A003954, A170732, A170733, A170734.

Remark 43. Note that the first two sequences in the above remark are Fibonacci and Pell numbers.

References

[1] M. Bernstein and N. J. A. Sloane, Some canonical sequences of integers, *Linear Algebra Appl.* **226/228**, (1996), 57–72.

[2] M. Janjić, Determinants and recurrence sequences, *J. Integer Seq.* **15** (2012), Article 12.3.5.

[3] M. Janjić and B. Petković, A counting function generalizing binomial coefficients and some other classes of integers, *J. Integer Seq.* **17** (2014), Article 14.3.5.

[4] M. Janjić, A generating function for the numbers of insets, *J. Integer Seq.* **17** (2014), Article 14.9.7.

[5] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, 2009.

[6] E. Munarini, M. Poneti, and S. Rinaldi, Matrix compositions, *FPSAC 2006: Formal Power Series and Algebraic Combinatorics*, San Diego, 2006, pp. 445–456. Available at `http://garsia.math.yorku.ca/fpsac06/papers/49.pdf`.

[7] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

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