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A Generating Function for the Diagonal $T_{2n,n}$ in Triangles

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Abstract

We present techniques for obtaining a generating function for the diagonal $T_{2n,n}$ of the triangle formed from the coefficients of a generating function $G(x)$ raised to the power k. We obtain some relations between central coefficients and coefficients of the diagonal $T_{2n,n}$, and we also give some examples.

1 Introduction

A triangle is a classic object of research in combinatorics. For instance, the Pascal triangle, the Bernoulli-Euler triangle, the Catalan triangle, and the Motzkin triangle are discussed in many papers and books [3, 10, 7].

Let $G(x)$ be an ordinary power series without a constant term, i.e., $G(x) = \sum\limits_{n>0} g_n\, x^n$, where $g_0 = 0$ and $g_1 \neq 0$. In this paper we deal with the triangle $T_{n,k}$ defined as follows:

$$[G(x)]^k = \sum_{n\geq k} T_{n,k} x^n.$$

Here we assume that $G(x)^0 = T_{0,0} = 1$.

Then the generating function $G(x)$ raised to the power k gives the following triangle $T_{n,k}$

$$\begin{array}{cccccccccc}
 & & & & T_{1,1} & & & & & \\
 & & & T_{2,1} & & T_{2,2} & & & & \\
 & & T_{3,1} & & T_{3,2} & & T_{3,3} & & & \\
 & T_{4,1} & & T_{4,2} & & T_{4,3} & & T_{4,4} & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
T_{n,1} & T_{n,2} & & \dots & & \dots & & T_{n,n-1} & & T_{n,n}
\end{array}$$

The following notation will be used throughout this paper. The authors [4, 5] introduced the notion of the *composita* of a given ordinary generating function $G(x) = \sum_{n>0} g(n)x^n$.

Definition 1. The *composita* is the function of two variables defined by

$$G^{\Delta}(n,k) = \sum_{\pi_k \in C_n} g(\lambda_1)g(\lambda_2)\cdots g(\lambda_k), \tag{1}$$

where n, k, λ_i are integers that are greater than 0, C_n is the set of all compositions of n, and π_k is the composition into k parts exactly ($\sum_{i=1}^{k} \lambda_i = n$).

The generating function of the composita is equal to

$$[G(x)]^k = \sum_{n\geqslant k} G^{\Delta}(n,k)x^n = \sum_{n\geq k} T_{n,k}x^n. \tag{2}$$

This notation coincides with the concept of Riordan array $(1, G(x))$ or $\left(\frac{G(x)}{x}, G(x)\right)$, which was given by Shapiro, Getu, Woan, and Woodson [8].

Recently, in [6], we have shown how to find a generating function of the central elements of such triangles

$$C(x) = \sum_{n>0} T_{2n-1,n} x^{n-1} = F'(x), \tag{3}$$

where $F(x)$ is the solution of the equation

$$F(x) = xS(F(x)) \tag{4}$$

and

$$x\, S(x) = G(x).$$

For solving (4), one uses the Lagrange inversion formula (LIF), which was proved by Stanley [10]. In [6], we applied the LIF for the generating functions raised to the k power:

$$[G(x)]^k = \sum_{n\geq k} G^{\Delta}(n,k)\, x^n$$

and

$$[F(x)]^k = \sum_{n\geq k} F^{\Delta}(n,k)\, x^n.$$

We obtained the following relation between two triangles:

$$F^{\Delta}(n,k) = \frac{k}{n} T_{2n-k,n}.$$

In this paper we present a method for obtaining the generating function for the diagonal $T_{2n,n}$ of a triangle $T_{n,k}$. The triangle is given by the following expression

$$[G(x)]^k = \sum_{n\geq k} T_{n,k}\, x^n.$$

2 Main results

The main result of this paper is given in the following theorem.

Theorem 2. *Suppose we have the generating function $G(x) = \sum\limits_{n>0} g_n\, x^n$ that forms a triangle $T_{n,k}$:*

$$[G(x)]^k = \sum_{n\geq k} T_{n,k} x^n.$$

Then the generating function $A(x) = \sum_{n\geq 0} T_{2n,n} x^n$ for the diagonal $T_{2n,n}$ of the triangle is defined by

$$A(x) = \frac{x\, F'(x)}{F(x)}, \tag{5}$$

where $F(x) = x\, S(F(x))$ with $S(x) = \frac{G(x)}{x}$.

Proof. Suppose we have the following Laurent series

$$\Phi(z) = \varphi\, z + \varphi_0 + \frac{\varphi_1}{z} + \cdots + \frac{\varphi_n}{z^n} + \cdots$$

Then, raising this generating function to the power k, we get

$$[\Phi(z)]^k = \Phi_k(z) + E_k(z),$$

where $\Phi_k(z)$ contains the nonnegative powers of z and $E_k(z)$ contains the remaining powers of z. According to Suetin [11], $\Phi_k(z)$ is the Faber polynomial.

Let us consider the generating function $G(z)$ in terms of $\Phi(z)$. That is,

$$[G(z)]^k = [z^2\,\Phi(1/z)]^k = \sum_{n\ge k} T_{n,k}\,z^n.$$

Then we have

$$[\Phi(z)]^k = z^{2k}\sum_{n\ge k} T_{n,k}\,z^{-n}.$$

After transformation, the Faber polynomial is equal to

$$\Phi_n(z) = \sum_{k=0}^{n} T_{2n-k,n}\,z^k, \tag{6}$$

For the case $z=0$, we have

$$\Phi_n(0) = T_{2n,n}. \tag{7}$$

According to Curtiss [2] and Suetin [11], the generating function for the Faber polynomials is equal to

$$\frac{t\phi'(t)}{\phi(t)-z} = \sum_{n\ge 0}\Phi_n(z)t^{-n},$$

where $\phi(t)$ is the compositional inverse of $\Phi(t)$.

Then the generating function for the case $z=0$ is equal to

$$\frac{t\phi'(t)}{\phi(t)} = \sum_{n\ge 0}\Phi_n(0)t^{-n}.$$

Next we set $t=\frac{1}{x}$. Taking into account that

$$(\phi(1/x))' = \phi'(1/x)\,(1/x)' = \frac{-\phi'(1/x)}{x^2}$$

or

$$\phi'(1/x) = -x^2\,(\phi(1/x))'$$

we get the generating function for $\Phi_n(0)$

$$A(x) = -\frac{x(\phi(1/x))'}{\phi(1/x)} = \sum_{n\ge 0}\Phi_n(0)x^n. \tag{8}$$

Since $\phi(t)$ is the compositional inverse of $\Phi(t)$, the following identity holds:

$$\Phi(\phi(t)) = t.$$

If we substitute $1/x$ for t, then we obtain the following relation:

$$\Phi(\phi(1/x)) = 1/x.$$

Since

$$\Phi(x) = x^2 G(1/x) = xS(1/x),$$

we get

$$\phi(1/x)S(1/\phi(1/x)) = \frac{1}{x}.$$

Then

$$\frac{1}{\phi(1/x)} = xS(1/\phi(1/x)).$$

According to (4), we have

$$\frac{1}{\phi(1/x)} = F(x)$$

Therefore, according to (7) and (8), the generating function for the diagonal $T_{2n,n}$ is equal to

$$A(x) = -\frac{x(\frac{1}{F(x)})'}{\frac{1}{F(x)}} = \frac{x\,F'(x)}{F(x)} = \sum_{n\geq 0} T_{2n,n}\,x^n.$$

The theorem is thus proved. $\square$

As applications of Theorem 2, we give the following examples.

Example 3. Let us consider the Pascal triangle. This triangle can be defined by the generating function

$$[G(x)]^k = \left(\frac{x}{1-x}\right)^k = \sum_{n\geq k} \binom{n-1}{n-k} x^n$$

The solution of the equation

$$F(x) = \frac{x}{1-F(x)}$$

is the generating function

$$F(x) = \frac{1-\sqrt{1-4x}}{2}.$$

Therefore, the generating function for the diagonal $T_{2n,n}$ with the general term $\binom{2n-1}{n}$ is

$$A(x) = \frac{x\,F'(x)}{F(x)} = \frac{2x}{\left(1-\sqrt{1-4x}\right)\,\sqrt{1-4x}}.$$

Example 4. Let us find the generating function $A(x) = \sum_{n\geq 0} T_{2n,n} x^n$ for the triangle defined by the following generating function

$$G(x) = x + x^2 + x^3.$$

Solving the equation

$$F(x) = x(1 + F(x) + F(x)^2),$$

we get the generating function for the Motzkin numbers (see the sequence A001006 in [9])

$$F(x) = \frac{-\sqrt{-3x^2-2x+1}-x+1}{2x}.$$

Then we have

$$\frac{x\,F'(x)}{F(x)} = \frac{\sqrt{-3x^2-2x+1}+x-1}{(x-1)\,\sqrt{-3x^2-2\,x+1}-3x^2-2x+1}.$$

After transformation, we obtain

$$A(x) = \frac{1}{\sqrt{-3x^2-2x+1}}.$$

Example 5. Let us find the generating function $A(x) = \sum_{n\geq 0} T_{2n,n}x^n$ for the triangle defined by the following expression

$$[G(x)]^k = \left[\frac{1-\sqrt{1-4x}}{2}\right]^k = \sum_{n\geq k} \frac{k\binom{2n-k-1}{n-k}}{n}x^n.$$

The solution of the functional equation (4) for this case is the following generating function (see sequence A001764 in [9])

$$F(x) = \frac{2}{\sqrt{3x}}\sin\left(\frac{1}{3}\arcsin\left(\frac{\sqrt{27x}}{2}\right)\right).$$

Therefore, the desired generating function has the form

$$A(x) = \frac{xF'(x)}{F(x)} = 1 + \sum_{n>0}\frac{\binom{3n-1}{n}}{2}x^n = \frac{\sqrt{3x}}{2\sqrt{4-27x}}\cot\left(\frac{1}{3}\arcsin\left(\frac{\sqrt{27x}}{2}\right)\right) + \frac{1}{2}.$$

Example 6. Let us consider the triangle defined by the expression

$$[G(x)]^m = [x^2\cot(x)]^m = \sum_{n\geq m} T_{n,m}x^n,$$

where

$$T_{n,m} = (-1)^{\frac{n-m}{2}}\sum_{l=0}^{m}\frac{2^{n-2m+l}}{(n-2m+l)!}\binom{m}{l}\sum_{k=0}^{n-2m+l}\frac{s\,(l+k,l)\;S\,(n-2m+l,k)}{\binom{k+l}{l}}.$$

Here $s(n,k)$ and $S(n,k)$ stand for the Stirling numbers of the first and second kinds, respectively [1, 3].

This triangle forms the sequence A199542 in [9]. Then we have

$$T_{2n,n} = (-1)^{\frac{n}{2}} \sum_{l=0}^{n} 2^l \left(\sum_{k=0}^{l} \frac{k!\, S\,(l,k)\ s\,(l+k,l)}{(l+k)!} \right) \binom{n}{l}.$$

For the equation $F(x) = x\, F(x) \cot(F(x))$, the solution is the generating function $\arctan(x)$.

Hence,

$$\frac{xF'(x)}{F(x)} = \frac{x}{(1+x^2)\arctan(x)} = 1 - \frac{2x^2}{3} + \frac{26x^4}{45} - \frac{502x^6}{945} + \frac{7102x^8}{14175} + \cdots$$

Therefore, we obtain

$$A(x) = \frac{x}{(1+x^2)\arctan(x)} = \sum_{n\geq 0} (-1)^{\frac{n}{2}} \sum_{l=0}^{n} 2^l \left(\sum_{k=0}^{l} \frac{k!\, S\,(l,k)\ s\,(l+k,l)}{(l+k)!} \right) \binom{n}{l} x^n.$$

Next, we derive some interesting identities between coefficients in triangles.

Theorem 7. *Suppose we have the triangle $T_{n,k}$, which is generated by $G(x)^k = \sum_{n\geqslant k} T_{n,k}x^n$. Then the following identity holds for the central coefficients of the triangle*

$$T_{2n-1,n} = \sum_{i=1}^{n} \frac{1}{i} T_{2i-1,i}\, T_{2(n-i),n-i}. \tag{9}$$

Proof. The result follows from Theorem 2 and the expression (3). We point out that

$$F(x) = \sum_{n>0} \frac{1}{n} T_{2n-1,n} x^n$$

and

$$\frac{x\, F'(x)}{F(x)} = \sum_{n\geqslant 0} T_{2n,n} x^n.$$

Since

$$x\, F'(x) = \left(\frac{x\, F'(x)}{F(x)} \right) F(x),$$

by applying the multiplication rule for formal power series, we obtain the desired result. □

Example 8. Using Theorem 7, we obtain the identities for the Stirling numbers.

The Stirling numbers of the first kind $s(n,k)$ count the number of permutations of n elements with k disjoint cycles. The Stirling numbers of the first kind are defined by the following generating function [1]:

$$\psi_k(x) = \sum_{n\geq k} s(n,k) \frac{x^n}{n!} = \frac{1}{k!} \ln^k(1+x).$$

With the help of (9), we find the following identity for the Stirling numbers of the first kind:

$$s(2n-1,n)=\sum_{i=1}^{n}\frac{\binom{2n-1}{2i-1}}{\binom{n}{i}}\,\frac{s\left(2i-1,i\right)\;s\left(2\left(n-i\right),n-i\right)}{i}.$$

The Stirling numbers of the second kind $S(n,k)$ count the number of ways to partition a set of n elements into k nonempty subsets. The Stirling numbers of the second kind are defined by the following generating function [1]:

$$\Phi_k(x)=\sum_{n\geq k}S(n,k)\frac{x^n}{n!}=\frac{1}{k!}(e^x-1)^k.$$

Using Theorem 7, we derive the following identity

$$S(2n-1,n)=\sum_{i=1}^{n}\frac{\binom{2n-1}{2i-1}}{\binom{n}{i}}\,\frac{S\left(2i-1,i\right)\;S\left(2\left(n-i\right),n-i\right)}{i}.$$

Example 9. Suppose we have the triangle defined by the following expression

$$(x\,e^x)^k=\sum_{n\geqslant k}\frac{k^{n-k}}{(n-k)!})x^n.$$

Then, using (9), we get

$$\frac{n^{n-1}}{(n-1)!}=\sum_{i=1}^{n}\frac{i^{i-2}\;(n-i)^{n-i}}{(i-1)!\;(n-i)!}$$

or after simple manipulation

$$n^{n-1}=\sum_{i=1}^{n}\binom{n-1}{i-1}i^{i-2}\;(n-i)^{n-i}.$$

Example 10. Suppose we have the triangle defined by the following expression

$$\left(\frac{x}{(1-x)^m}\right)^k=\sum_{n\geqslant k}\binom{n+(m-1)k-1}{n-k}x^n.$$

Then, according to (9), we obtain

$$\binom{(m+1)\,n-2}{n-1}=\sum_{i=1}^{n}\frac{1}{i}\binom{im+i-2}{i-1}\binom{(m+1)\,n-im-i-1}{n-i}.$$

If we put $m=2$, we derive the following identity

$$\binom{3n-2}{n-1}=\sum_{i=1}^{n}\frac{\binom{3i-2}{i-1}\binom{3n-3i-1}{n-i}}{i}.$$

Example 11. Suppose we have the triangle defined by the following expression

$$[G(x)]^k = \left(\frac{x^2}{e^x - 1}\right)^k = \sum_{n\geqslant k} T_{n,k} x^n,$$

where

$$T_{n,m} = \frac{m!}{(n-m)!} \sum_{k=0}^{n-m} \frac{k!\, S_1(m+k,m)\ S_2(n-m,k)}{(m+k)!}.$$

The solution of the equation (4) for this case, that is, $F(x) = x\frac{F(x)}{e^{F(x)}-1}$, is the generating function $\ln(1+x)$ (see the sequence A191578 in [9]).

Then, according to Theorem 2, we have

$$\frac{xF'(x)}{F(x)} = \frac{x}{(1+x)\ln(1+x)} = \sum_{n\geqslant 0} T_{2n,n}\, x^n.$$

where

$$T_{2n,n} = \sum_{k=0}^{n} \frac{k!\, S_2(n,k)\ S_1(n+k,n)}{(n+k)!}.$$

This is reflected in the sequence A002208 in [9].

Using Theorem 7, we obtain the following identity

$$\sum_{m=0}^{n-1} \frac{(-1)^m}{n-m} \sum_{k=0}^{m} \frac{k!\, S_2(m,k)\ S_1(m+k,m)}{(m+k)!} = 1.$$

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