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On Certain Sums of Stirling Numbers with Binomial Coefficients

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Abstract

We study two sums involving the Stirling numbers and binomial coefficients. We find their closed forms, and discuss the connection between these sums.

Dedicated to the memory of our mentors,
Professors Leonard Carlitz and Albert Nijenhuis

1 Introduction

Stirling numbers of the first and second kind, denoted by $s(n,k)$ and $S(n,k)$ respectively, in Riordan's [8] popular notation, have long fascinated mathematicians. They were named for James Stirling [13] who used them in 1730. In 1852 Schläfli [10] studied relations between $s(n,k)$ and $S(n,k)$. Then in 1960 Gould [3] extended Schläfli's work by discovering the pair of dual relations

$$\begin{aligned}(-1)^n S(m,m-n) &= \sum_{k=0}^{n}\binom{n+m}{n-k}\binom{n-m}{n+k}s(n+k,k),\\ (-1)^n s(m,m-n) &= \sum_{k=0}^{n}\binom{n+m}{n-k}\binom{n-m}{n+k}S(n+k,k).\end{aligned}$$

Prompted by a recent problem [7] in the *Amer. Math. Monthly* that asked the readers to find a closed form expression for $\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}s(n+k,k)$, Gould tried to relate this sum to the first of his dual sums. By choosing $m=n+1$ and noting that $\binom{-1}{n-k}=(-1)^{n+k}$, the first of Gould's relations yields

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}s(n+k,k)=S(n+1,1)=1,$$

which is not quite the proposed result but suggested that we study a wider range of sums.

Motivated by this and other experiments using Maple, we study the following sums:

$$\begin{aligned}f_m(n) &= \sum_{k=0}^{n+m}(-1)^k\binom{2n+m}{n+k}s(n+k,k),\\ F_m(n) &= \sum_{k=0}^{n+m}(-1)^k\binom{2n+m}{n+k}S(n+k,k),\\ g_m(n) &= \sum_{k=0}^{n}(-1)^k\binom{2n+m}{n-k}s(n+k,k),\\ G_m(n) &= \sum_{k=0}^{n}(-1)^k\binom{2n+m}{n-k}S(n+k,k).\end{aligned}$$

For $m\geq 0$, the closed forms for $f_m(n)$ and $F_m(n)$ are easy to obtain, but the sums $g_m(n)$ and $G_m(n)$ are more complicated. We also study the case of $m<0$. We shall derive formulas for these sums, and discuss their connections. To simplify the notation, define

$$\widehat{f}_m(n)=f_{-m}(n),\quad \widehat{F}_m(n)=F_{-m}(n),\quad \widehat{g}_m(n)=g_{-m}(n),\quad\text{and}\quad \widehat{G}_m(n)=G_{-m}(n).$$

Consequently, throughout this paper, unless otherwise stated, m will denote a nonnegative integer, and n a positive integer.

2 Closed forms for $f_m(n)$ and $F_m(n)$

We start with $f_0(n)$ and determine its value using a combinatorial argument.

Theorem 1. *For any positive integer,*

$$f_0(n) = \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} s(n,k) = (2n-1)!!,$$

where $(2n-1)!!$ *denotes the double factorial* $(2n-1)(2n-3)\cdots 3\cdot 1$.

Proof. Recall that the unsigned Stirling number of the first kind $c(n,k) = (-1)^{n-k}s(n,k)$ counts the number of n-permutations with k disjoint cycles. Then

$$\begin{aligned} f_0(n) &= \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} s(n+k,k) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{n-k} c(n+k,k) \\ &= \sum_{j=0}^{n} (-1)^j \binom{2n}{j} c(2n-j, n-j). \end{aligned}$$

Let S be the set of $2n$-permutations with n cycles, and A_i be the subset of permutations of S with i as a fixed point. Then

$$\left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}\right| = c(2n-j, n-j).$$

It follows from the principle of inclusion-exclusion that $f_0(n)$ is precisely the number of $2n$-permutations without fixed points. Notice that if a permutation in S has no fixed point, it must be a permutation with exactly n transpositions (that is, 2-cycles). After lining up $2n$ objects, we can take the elements two at a time to form n transpositions. Since the order within each transposition does not matter, it is just a matter of calculating the order among the transpositions; hence, there are

$$\frac{(2n)!}{n!\,2^n} = (2n-1)(2n-3)\cdots 3\cdot 1 = (2n-1)!!$$

permutations of $2n$ with exactly n transpositions. Thus, $f_0(n) = (2n-1)!!$. $\square$

The proof suggests we should examine the combinatorial interpretation of $f_m(n)$. Let $c^*(n,k)$ denote the number of n-permutations with k cycles and no fixed points. It is called the *unsigned associated Stirling number of the first kind* ([2, p. 256] and [8, p. 73]). In a similar fashion, we can define $S^*(n,k)$ as the number of partitions of an n-set into k subsets with no singleton subset as any part. The number $S^*(n,k)$ is the *associated Stirling number of the second kind* ([2, p. 221] and [8, p. 77]). See [6] for a more thorough discussion of the associated Stirling numbers.

Lemma 2. *The identity $f_m(n) = (-1)^m c^*(2n+m, n+m)$ holds for any integers m and n such that $n+m \geq 1$.*

Proof. Let S be the set of $(2n+m)$-permutations with $n+m$ cycles, and A_i be the subset of permutations of S with i as a fixed point. Then

$$\left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}\right| = c(2n+m-j, n+m-j).$$

Therefore, according to the principle of inclusion-exclusion,

$$\begin{aligned} c^*(2n+m, n+m) &= \sum_{j=0}^{n+m} (-1)^j \binom{2n+m}{j} c(2n+m-j, n+m-j) \\ &= \sum_{k=n+m}^{0} (-1)^{n+m-k} \binom{2n+m}{n+m-k} c(n+k, k) \\ &= (-1)^m \sum_{k=0}^{n+m} (-1)^k \binom{2n+m}{n+k} s(n+k, k). \end{aligned}$$

Therefore, $c^*(2n+m, n+m) = (-1)^m f_m(n)$. □

By using an almost identical argument, we obtain a similar result for the associated Stirling numbers of the second kind.

Lemma 3. *The identity $F_m(n) = (-1)^{n+m} S^*(2n+m, n+m)$ holds for any integers m and n such that $n+m \geq 1$.*

These combinatorial interpretations allow us to determine the exact values of $f_m(n)$ and $F_m(n)$ for $m \geq 0$. Again, due to their similarity, we shall only prove the first result.

Theorem 4. *For any integer $n \geq 1$, $f_0(n) = (2n-1)!!$, and $f_m(n) = 0$ if $m > 0$.*

Proof. Lemma 2 states that $f_m(n) = (-1)^m c^*(2n+m, n+m)$. If $m > 0$, it is clear that $2n+m < 2(n+m)$, hence $c^*(2n+m, n+m) = 0$. When $m = 0$, we have $f_0(n) = c^*(2n, n)$, which counts the number of permutations with exactly n transpositions. Hence, $f_0(n) = (2n-1)!!$. □

Theorem 5. *For any integer $n \geq 1$, $F_0(n) = (-1)^n (2n-1)!!$, and $F_m(n) = 0$ if $m > 0$.*

The same conclusions can be drawn from generating functions.

Lemma 6. *Let T be a nonempty set of positive integers. Define $c_T(n,k)$ as the number of n-permutations with k disjoint cycles whose lengths belong to T. Then*

$$c_T(x,y) := \sum_{n,k \geq 0} c_T(n,k) \frac{x^n}{n!} y^k = \exp\left(y \sum_{j \in T} \frac{x^j}{j} \right).$$

Proof. The result follows easily from the exponential formula [15]. Alternatively, we can prove it directly as follows. For any n-permutation, let n_j denotes the number of j-cycles. It is a routine exercise to show that

$$c_T(n,k) = \sum\nolimits_T \frac{n!}{\prod_{j\in T} n_j!\, j^{n_j}},$$

where the summation $\sum_T$ is taken over all $n_j \geq 0$, where $j \in T$, such that $\sum_{j\in T} n_j = k$, and $\sum_{j\in T} jn_j = n$. Then

$$\begin{aligned}\sum_{n,k\geq 0} c_T(n,k)\,\frac{x^n}{n!}\,y^k &= \sum_{n,k\geq 0}\sum\nolimits_T \prod_{j\in T} \frac{x^{jn_j}y^{n_j}}{n_j!\,j^{n_j}} \\ &= \sum_{n,k\geq 0}\sum\nolimits_T \prod_{j\in T} \frac{1}{n_j!}\left(\frac{x^j y}{j}\right)^{n_j}.\end{aligned}$$

Noting that this is in the form of a convolution, we determine that

$$c_T(x,y) = \prod_{j\in T}\sum_{n_j\geq 0}\frac{1}{n_j!}\left(\frac{x^j y}{j}\right)^{n_j} = \prod_{j\in T}\exp\left(\frac{x^j y}{j}\right) = \exp\left(y\sum_{j\in T}\frac{x^j}{j}\right).$$

□

Lemma 7. *Let T be a nonempty set of positive integers. Define $S_T(n,k)$ as the number of ways to partition an n-set into k subsets with cardinalities belonging to T. Then*

$$S_T(x,y) := \sum_{n,k\geq 0} S_T(n,k)\,\frac{x^n}{n!}\,y^k = \exp\left(y\sum_{j\in T}\frac{x^j}{j!}\right).$$

Proof. The proof is identical to that of Lemma 6, except that

$$S_T(n,k) = \sum\nolimits_T \frac{n!}{\prod_{j\in T} n_j!\,(j!)^{n_j}}.$$

□

For our purpose, we need $T = \mathbb{N} - \{1\}$. We find

$$c^*(x,y) := \sum_{n,k\geq 0} c^*(n,k)\,\frac{x^n}{n!}\,y^k = \exp\left(y\sum_{j\geq 2}\frac{x^j}{j}\right) = (1-x)^{-y}e^{-xy}, \tag{1}$$

and

$$S^*(x,y) := \sum_{n,k\geq 0} S^*(n,k)\,\frac{x^n}{n!}\,y^k = \exp\left(y\sum_{j\geq 2}\frac{x^j}{j!}\right) = e^{y(e^x-1-x)}. \tag{2}$$

From the generating function $c^*(x,y)$, it is clear that the coefficient of $x^r y^t$ is zero if $r < 2t$. Hence, $f_m(n) = (-1)^m c^*(2n+m, n+m) = 0$ if $m > 0$. For $m = 0$, the coefficient of $\frac{x^{2n}}{(2n)!}\,y^n$ is $\frac{(2n)!}{n!\,2^n} = (2n-1)!!$. The argument for $F_m(n)$ is similar.

3 Formulas for $\widehat{f}_m(n)$ and $\widehat{F}_m(n)$

Lemma 2 shows that $f_m(n) = (-1)^m c^*(2n+m, n+m)$. Its combinatorial interpretation implies that $f_m(n)$ is nonzero if $1-n \le m \le 0$. We obtain the following result.

Theorem 8. *For any integer m that satisfies $0 < m \le n-1$,*

$$\widehat{f}_m(n) = \sum \frac{(-1)^m (2n-m)!}{\prod_{i\ge 2} n_i!\, i^{n_i}},$$

where the summation is taken over all integers $n_2, n_3, n_4, \ldots \ge 0$ such that $\sum_{i\ge 2} n_i = n-m$, and $\sum_{i\ge 2} i n_i = 2n-m$.

We shall present an analytic proof as well as a combinatorial proof.

Proof. Since $f_m(n) = (-1)^m c^*(2n+m, n+m)$, we gather from the generating function $c^*(x,y)$ that $f_m(n)$ is $(-1)^m(2n+m)!$ times the coefficient of $x^{2n+m}y^{n+m}$ in the power series expansion of

$$\exp\left[y\left(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)\right] = \sum_{k=0}^{\infty} \frac{y^k}{k!}\left(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)^k.$$

We conclude that $f_m(n)$ is $(-1)^m(2n+m)!/(n+m)!$ times the coefficient of x^{2n+m} in the power series expansion of

$$\left(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)^{n+m}.$$

For $m<0$, replace m with $-m$. Then $\widehat{f}_m(n)$ is $(-1)^m(2n-m)!/(n-m)!$ times the coefficient of x^{2n-m} in the expansion of the polynomial

$$\left(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^2}{4}+\cdots\right)^{n-m}.$$

Applying the multinomial theorem yields the result immediately. □

Here is an alternate proof.

Proof. Since $\widehat{f}_m(n) = f_{-m}(n) = (-1)^m c^*(2n-m, n-m)$, it suffices to find a formula for the number of permutations of $2n-m$ with exactly $n-m$ cycles none of which is a 1-cycle. Assume there are n_i cycles of length i, then $n_2, n_3, n_4, \ldots \ge 0$, and

$$\begin{aligned} n_2+n_3+n_4+\cdots &= n-m, \\ 2n_2+3n_3+4n_4+\cdots &= 2n-m, \end{aligned}$$

and there are

$$\frac{(2n-m)!}{\prod_{i\ge 2} n_i!\, i^{n_i}}$$

such permutations. This, when combined with the addition principle, completes the proof. □

An almost identical argument leads to the next result.

Theorem 9. *For any integer m that satisfies $0 < m \leq n-1$,*

$$\widehat{F}_m(n) = \sum \frac{(-1)^{n+m}(2n-m)!}{\prod_{i\geq 2} n_i!\,(i!)^{n_i}},$$

where the summation is taken over all integers $n_2, n_3, n_4, \ldots \geq 0$ such that $\sum_{i\geq 2} n_i = n-m$, and $\sum_{i\geq 2} in_i = 2n-m$.

In order to use these two results effectively, take note that the two conditions on the n_i's imply that

$$n_3 + 2n_4 + 3n_5 + 4n_6 + \cdots = m.$$

The possible solutions for $0 \leq m \leq 4$ are summarized below.

m	$2n-m$	n_2	n_3	n_4	n_5	n_6	$\frac{(2n-m)!}{\prod_{i>2} n_i!\, i^{n_i}}$	$\frac{(2n-m)!}{\prod_{i>2} n_i!\,(i!)^{n_i}}$
0	$2n$	n	0	0	0	0	$\frac{(2n)!}{n!\,2^n}$	$\frac{(2n)!}{n!\,2^n}$
1	$2n-1$	$n-2$	1	0	0	0	$\frac{(2n-1)!}{3(n-2)!\,2^{n-2}}$	$\frac{(2n-1)!}{6(n-2)!\,2^{n-2}}$
2	$2n-2$	$n-3$	0	1	0	0	$\frac{(2n-2)!}{4(n-3)!\,2^{n-3}}$	$\frac{(2n-2)!}{24(n-3)!\,2^{n-3}}$
		$n-4$	2	0	0	0	$\frac{(2n-2)!}{18(n-4)!\,2^{n-4}}$	$\frac{(2n-2)!}{72(n-4)!\,2^{n-4}}$
3	$2n-3$	$n-4$	0	0	1	0	$\frac{(2n-3)!}{5(n-4)!\,2^{n-4}}$	$\frac{(2n-3)!}{120(n-4)!\,2^{n-4}}$
		$n-5$	1	1	0	0	$\frac{(2n-3)!}{12(n-5)!\,2^{n-5}}$	$\frac{(2n-3)!}{144(n-5)!\,2^{n-5}}$
		$n-6$	3	0	0	0	$\frac{(2n-3)!}{162(n-6)!\,2^{n-6}}$	$\frac{(2n-3)!}{1296(n-6)!\,2^{n-6}}$
4	$2n-4$	$n-5$	0	0	0	1	$\frac{(2n-4)!}{6(n-5)!\,2^{n-5}}$	$\frac{(2n-4)!}{720(n-5)!\,2^{n-5}}$
		$n-6$	1	0	1	0	$\frac{(2n-4)!}{15(n-6)!\,2^{n-6}}$	$\frac{(2n-4)!}{720(n-6)!\,2^{n-6}}$
		$n-6$	0	2	0	0	$\frac{(2n-4)!}{32(n-6)!\,2^{n-6}}$	$\frac{(2n-4)!}{1152(n-6)!\,2^{n-6}}$
		$n-7$	2	1	0	0	$\frac{(2n-4)!}{72(n-7)!\,2^{n-7}}$	$\frac{(2n-4)!}{1728(n-7)!\,2^{n-7}}$
		$n-8$	4	0	0	0	$\frac{(2n-4)!}{1944(n-8)!\,2^{n-8}}$	$\frac{(2n-4)!}{31104(n-8)!\,2^{n-8}}$

Theorem 8 asserts that

$$\begin{aligned}
\widehat{f}_1(n) &= -\frac{(2n-1)!}{3(n-2)!\,2^{n-2}},\\
\widehat{f}_2(n) &= \frac{(2n-2)!}{4(n-3)!\,2^{n-3}} + \frac{(2n-2)!}{18(n-4)!\,2^{n-4}},\\
\widehat{f}_3(n) &= -\frac{(2n-3)!}{5(n-4)!\,2^{n-4}} - \frac{(2n-3)!}{12(n-5)!\,2^{n-5}} - \frac{(2n-3)!}{162(n-6)!\,2^{n-6}},\\
\widehat{f}_4(n) &= \frac{(2n-4)!}{6(n-5)!\,2^{n-5}} + \frac{47(2n-4)!}{480(n-6)!\,2^{n-6}} + \frac{(2n-4)!}{72(n-7)!\,2^{n-7}} + \frac{(2n-4)!}{1944(n-8)!\,2^{n-8}}.
\end{aligned}$$

The first few absolute values of each of these four sequences are tabulated in Table 1. All of them appear in the OEIS [12]. More information about these sequences, including their combinatorial meanings, can be found in OEIS.

n	1	2	3	4	5	6	7	8	OEIS #
$\lvert\widehat{f}_1(n)\rvert$	0	2	20	210	2520	34650	540540	9459450	A000906
$\lvert\widehat{f}_2(n)\rvert$	0	0	6	130	2380	44100	866250	18288270	A000907
$\lvert\widehat{f}_3(n)\rvert$	0	0	0	24	924	26432	705320	18858840	A001784
$\lvert\widehat{f}_4(n)\rvert$	0	0	0	0	120	7308	303660	11098780	A001785

Table 1: The first eight values of $\widehat{f}_m(n)$ for $m = 1, 2, 3, 4$.

Likewise, Theorem 9 yields

$$\begin{aligned}
\widehat{F}_1(n) &= \frac{(-1)^{n+1}(2n-1)!}{6(n-2)!\,2^{n-2}},\\
\widehat{F}_2(n) &= \frac{(-1)^{n}(2n-2)!}{24(n-3)!\,2^{n-3}} + \frac{(-1)^{n}(2n-2)!}{72(n-4)!\,2^{n-4}},\\
\widehat{F}_3(n) &= \frac{(-1)^{n+1}(2n-3)!}{120(n-4)!\,2^{n-4}} + \frac{(-1)^{n+1}(2n-3)!}{144(n-5)!\,2^{n-5}} + \frac{(-1)^{n+1}(2n-3)!}{1296(n-6)!\,2^{n-6}},\\
\widehat{F}_4(n) &= \frac{(-1)^{n}(2n-4)!}{720(n-5)!\,2^{n-5}} + \frac{(-1)^{n}13(2n-4)!}{5760(n-6)!\,2^{n-6}} + \frac{(-1)^{n}(2n-4)!}{1728(n-7)!\,2^{n-7}} + \frac{(-1)^{n}(2n-4)!}{31104(n-8)!\,2^{n-8}}.
\end{aligned}$$

See Table 2. The last sequence does not appear in the OEIS. Note that $|\widehat{f}_1(n)| = 2|\widehat{F}_1(n)|$. We leave it as an exercise to the readers to find a combinatorial explanation.

n	1	2	3	4	5	6	7	8	OEIS #
$\lvert\widehat{F}_1(n)\rvert$	0	1	10	105	1260	17325	270270	4729725	A000457
$\lvert\widehat{F}_2(n)\rvert$	0	0	1	25	490	9450	190575	4099095	A000497
$\lvert\widehat{F}_3(n)\rvert$	0	0	0	1	56	1918	56980	1636635	A000504
$\lvert\widehat{F}_4(n)\rvert$	0	0	0	0	1	119	6825	302995	—

Table 2: The first eight values of $\widehat{F}_m(n)$ for $m = 1, 2, 3, 4$.

4 Formulas for $\widehat{g}_m(n)$ and $\widehat{G}_m(n)$

Next, we study the combinatorial significance of the two sums

$$\begin{aligned}\widehat{g}_m(n) &= \sum_{k=0}^{n}(-1)^k\binom{2n-m}{n-k}s(n+k,k),\\ \widehat{G}_m(n) &= \sum_{k=0}^{n}(-1)^k\binom{2n-m}{n-k}S(n+k,k),\end{aligned}$$

where m is a nonnegative integer, an n a positive integer such that $2n \geq m$.

Recall that the unsigned Stirling number of the first kind $c(n,k) = (-1)^{n-k}s(n,k)$ counts the number of n-permutations with k disjoint cycles. We find

$$\begin{aligned}\widehat{g}_m(n) &= \sum_{k=0}^{n}(-1)^k\binom{2n-m}{n-k}s(n+k,k)\\ &= \sum_{k=0}^{n}(-1)^k\binom{2n-m}{n+k-m}s(n+k,k)\\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{2n-m}{2n-k-m}s(2n-k,n-k)\\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{2n-m}{k}s(2n-k,n-k)\\ &= \sum_{k=0}^{n}(-1)^k\binom{2n-m}{k}c(2n-k,n-k).\end{aligned}$$

Similarly, we have

$$\widehat{G}_m(n) = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{2n-m}{k}S(2n-k,n-k),$$

where the Stirling number of the second kind $S(n,k)$ counts the number of ways to partition an n-set into k nonempty subsets.

For any positive integer m, define $[m]=\{1,2,\ldots,m\}$. Let n be a fixed positive integer. For any nonnegative integer m, define $\mathcal{S}_m$ as the set of permutations of $[2n]$ with n cycles and no fixed points among $[2n-m]$. Recall that if a permutation in $\mathcal{S}_m$ has no fixed point, it must be a permutation with exactly n transpositions (that is, 2-cycles). In addition, the fixed points in any permutation from $\mathcal{S}_m$ must belong to $[2n]\setminus[2n-m]$; this implies that the permutations in $\mathcal{S}_m$ has at most m fixed points.

In an analogous manner, define $\widetilde{\mathcal{S}}_m$ as the set of partitions of $[2n]$ into n nonempty subsets none of which is a singleton subset of $[2n-m]$. If a partition in $\widetilde{\mathcal{S}}_m$ has no singleton subset, it must have n parts, each of which a 2-subset. If a partition in $\widetilde{\mathcal{S}}_m$ has a singleton subset, its element must be among $[2n]\setminus[2n-m]$, hence it has at most m singleton subsets as its parts.

We first us the same argument in Theorem 1 to derive two preliminary results about $|\mathcal{S}_m|$ and $\big|\widetilde{\mathcal{S}}_m\big|$.

Lemma 10. *For positive integers m and n that satisfy $2n\geq m$,*

$$\widehat{g}_m(n)=\sum_{k=0}^{n}(-1)^k\binom{2n-m}{n-k}s(n+k,k)=|\mathcal{S}_m|.$$

Proof. In light of our earlier discussion, it suffices to show that

$$|\mathcal{S}_m|=\sum_{k=0}^{n}(-1)^k\binom{2n-m}{k}c(2n-k,n-k).$$

Let S be the set of all permutations of $[2n]$ with n cycles, without any restriction. For each $j\in[2n-m]$, define A_j to be the set of permutations of $[2n]$ with j as a fixed point. If a permutation belongs to $A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}$, it still has $n-k$ cycles whose elements come from $[2n]\setminus\{i_1,i_2,\ldots,i_k\}$. Thus,

$$\big|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}\big|=c(2n-k,n-k),$$

and there are $\binom{2n-m}{k}$ choices for $\{i_1,i_2,\ldots,i_k\}$. Since the permutations in S comprise of n cycles, we obviously need $0\leq k\leq n$. The result now follows from the principle of inclusion-exclusion. $\square$

It is clear that a similar result for $\big|\widetilde{\mathcal{S}}_m\big|$ also holds.

Lemma 11. *For positive integers m and n that satisfy $2n\geq m$,*

$$\widehat{G}_m(n)=\sum_{k=0}^{n}(-1)^k\binom{2n-m}{n-k}S(n+k,k)=(-1)^n\big|\widetilde{\mathcal{S}}_m\big|.$$

Theorem 12. *For positive integers m and n that satisfy $2n \geq m$,*

$$\widehat{g}_m(n) = \sum_{j=0}^{m} \binom{m}{j} c^*(2n-j, n-j) = \sum_{j=0}^{m} \binom{m}{j} \sum \frac{(2n-j)!}{\prod_{i\geq 2} n_i!\, i^{n_i}},$$

where the inner sum $\sum$ is taken over all integers $n_2, n_3, n_4, \ldots \geq 0$ such that $\sum_{i\geq 2} n_i = n-j$, and $\sum_{i\geq 2} i n_i = 2n-j$.

Proof. We need to determine $|\mathcal{S}_m|$. Let j be the number of fixed points in a permutation from $\mathcal{S}_m$. Since the fixed points come from $[2n] \setminus [2n-m]$, there are $\binom{m}{j}$ ways to choose these fixed points. The other $2n-j$ elements form $n-j$ cycles, all with length at least 2. Assume there are n_i cycles of length i, then $n_2, n_3, \ldots \geq 0$, and

$$\begin{aligned} n_2 + n_3 + n_4 + \cdots &= n - j, \\ 2n_2 + 3n_3 + 4n_4 + \cdots &= 2n - j, \end{aligned}$$

and there are

$$\frac{(2n-j)!}{\prod_{i\geq 2} n_i!\, i^{n_i}}$$

such permutations. The proof is completed by applying the addition principle, and recalling that $0 \leq j \leq m$. ◻

Notice that the sum $\sum (2n-j)!/\prod_{i\geq 2} n_i!\, i^{n_i}$ also appeared in the last section. It is equal to $c^*(2n-j, n-j)$. Then, according to Theorem 12,

$$\begin{aligned}
\widehat{g}_1(n) &= \frac{(2n)!}{n!\,2^n} + \frac{(2n-1)!}{3(n-2)!\,2^{n-2}}, \\
\widehat{g}_2(n) &= \frac{(2n)!}{n!\,2^n} + \frac{2(2n-1)!}{3(n-2)!\,2^{n-2}} + \frac{(2n-2)!}{4(n-3)!\,2^{n-3}} + \frac{(2n-2)!}{18(n-4)!\,2^{n-4}}, \\
\widehat{g}_3(n) &= \frac{(2n)!}{n!\,2^n} + \frac{(2n-1)!}{(n-2)!\,2^{n-2}} + \frac{3(2n-2)!}{4(n-3)!\,2^{n-3}} + \frac{(2n-2)!}{6(n-4)!\,2^{n-4}} \\
&\quad + \frac{(2n-3)!}{5(n-4)!\,2^{n-4}} + \frac{(2n-3)!}{12(n-5)!\,2^{n-5}} + \frac{(2n-3)!}{162(n-6)!\,2^{n-6}}, \\
\widehat{g}_4(n) &= \frac{(2n)!}{n!\,2^n} + \frac{4(2n-1)!}{3(n-2)!\,2^{n-2}} + \frac{3(2n-2)!}{2(n-3)!\,2^{n-3}} + \frac{(2n-2)!}{3(n-4)!\,2^{n-4}} \\
&\quad + \frac{4(2n-3)!}{5(n-4)!\,2^{n-4}} + \frac{(2n-3)!}{3(n-5)!\,2^{n-5}} + \frac{2(2n-3)!}{81(n-6)!\,2^{n-6}} \\
&\quad + \frac{(2n-4)!}{6(n-5)!\,2^{n-5}} + \frac{47(2n-4)!}{480(n-6)!\,2^{n-6}} + \frac{(2n-4)!}{72(n-7)!\,2^{n-7}} \\
&\quad + \frac{(2n-4)!}{1944(n-8)!\,2^{n-8}}.
\end{aligned}$$

It is easy to verify that

$$\widehat{g}_1(n) = \frac{(2n+1)!!}{3}.$$

Since $\widehat{g}_0(n) = (2n-1)!!$, we find

$$3\widehat{g}_1(n) = \widehat{g}_0(n+1).$$

We invite the readers to find a combinatorial proof of this simple identity. We also find

$$\begin{aligned}\widehat{g}_2(n) &= \frac{1}{9}(4n^3+9n^2-n-3)(2n-3)!!\\ &= \frac{1}{18}(2n+3)!! + \frac{1}{12}(2n+1)!! - \frac{1}{12}(2n-3)!!.\end{aligned}$$

It becomes clear that $\widehat{g}_m(n)$ can be written as a linear combination of the double falling factorials of the form $m!!$ for some odd integers m. We invite the readers to devise a combinatorial argument to find its coefficients.

The sequence $\{\widehat{g}_1(n)\}_{n\geq 1}$ appears in the OEIS [12] as Sequence A051577, but the other sequences $\{\widehat{g}_2(n)\}_{n\geq 1}$, $\{\widehat{g}_3(n)\}_{n\geq 1}$, and $\{\widehat{g}_4(n)\}_{n\geq 1}$ do not appear in the OEIS. Their numeric values for $n \leq 8$ are listed in Table 3.

n	1	2	3	4	5	6	7	8	OEIS #
$\widehat{g}_1(n)$	1	5	35	315	3465	45045	675675	11486475	A051577
$\widehat{g}_2(n)$	1	7	61	655	8365	123795	2082465	39234195	—
$\widehat{g}_3(n)$	1	9	93	1149	16569	273077	5060825	104129025	—
$\widehat{g}_4(n)$	1	11	131	1821	29121	526631	10619735	236128585	—

Table 3: The first eight values of $\widehat{g}_m(n)$ for $m = 1, 2, 3, 4$.

A similar argument yields the next result.

Theorem 13. *For positive integers m and n that satisfy $2n \geq m$,*

$$\widehat{G}_m(n) = \sum_{j=0}^{m}(-1)^n\binom{m}{j}S^*(2n-j,n-j) = \sum_{j=0}^{m}\binom{m}{j}\sum\frac{(-1)^n(2n-j)!}{\prod_{i\geq 2} n_i!\,(i!)^{n_i}},$$

where the inner sum $\sum$ is taken over all integers $n_2, n_3, n_4, \ldots \geq 0$ such that $\sum_{i\geq 2} n_i = n-j$, and $\sum_{i\geq 2} in_i = 2n-j$.

Theorem 13 yields the following:

$$\begin{aligned}
\widehat{G}_1(n) &= \frac{(-1)^n(2n)!}{n!\,2^n} + \frac{(-1)^n(2n-1)!}{6(n-2)!\,2^{n-2}},\\
\widehat{G}_2(n) &= \frac{(-1)^n(2n)!}{n!\,2^n} + \frac{(-1)^n(2n-1)!}{3(n-2)!\,2^{n-2}} + \frac{(-1)^n(2n-2)!}{24(n-3)!\,2^{n-3}} + \frac{(-1)^n(2n-2)!}{72(n-4)!\,2^{n-4}},\\
\widehat{G}_3(n) &= \frac{(-1)^n(2n)!}{n!\,2^n} + \frac{(-1)^n(2n-1)!}{2(n-2)!\,2^{n-2}} + \frac{(-1)^n(2n-2)!}{8(n-3)!\,2^{n-3}} + \frac{(-1)^n(2n-2)!}{24(n-4)!\,2^{n-4}}\\
&\quad + \frac{(-1)^n(2n-3)!}{120(n-4)!\,2^{n-4}} + \frac{(-1)^n(2n-3)!}{144(n-5)!\,2^{n-5}} + \frac{(-1)^n(2n-3)!}{1296(n-6)!\,2^{n-6}},
\end{aligned}$$

$$\begin{aligned}
\widehat{G}_4(n) &= \frac{(-1)^n(2n)!}{n!\,2^n} + \frac{(-1)^n 2(2n-1)!}{3(n-2)!\,2^{n-2}} + \frac{(-1)^n(2n-2)!}{4(n-3)!\,2^{n-3}} + \frac{(-1)^n(2n-2)!}{12(n-4)!\,2^{n-4}}\\
&\quad + \frac{(-1)^n(2n-3)!}{30(n-4)!\,2^{n-4}} + \frac{(-1)^n(2n-3)!}{36(n-5)!\,2^{n-5}} + \frac{(-1)^n(2n-3)!}{324(n-6)!\,2^{n-6}}\\
&\quad + \frac{(-1)^n(2n-4)!}{720(n-5)!\,2^{n-5}} + \frac{(-1)^n 13(2n-4)!}{5760(n-6)!\,2^{n-6}} + \frac{(-1)^n(2n-4)!}{1728(n-7)!\,2^{n-7}}\\
&\quad + \frac{(-1)^n(2n-4)!}{31104(n-8)!\,2^{n-8}}.
\end{aligned}$$

The first eight absolute values of each sequence are tabulated in Table 4. None of these sequences appear in the OEIS.

n	1	2	3	4	5	6	7	8	OEIS #
$\lvert\widehat{G}_1(n)\rvert$	1	4	25	210	2205	27720	405405	6756750	—
$\lvert\widehat{G}_2(n)\rvert$	1	5	36	340	3955	54495	866250	15585570	—
$\lvert\widehat{G}_3(n)\rvert$	1	6	48	496	6251	92638	1574650	30150120	—
$\lvert\widehat{G}_4(n)\rvert$	1	7	61	679	9150	144186	2594410	52390030	—

Table 4: The first eight values of $\widehat{G}_m(n)$ for $m = 1, 2, 3, 4$.

5 Connections between the sums

Theorem 12 states that $\widehat{g}_m(n) = \sum_{j=0}^{m} \binom{m}{j} c^*(2n-j, n-j)$. Together with Lemma 2 which implies $c^*(2n-j, n-j) = (-1)^j \widehat{f}_j(n)$, we immediately obtain the identity

$$\widehat{g}_m(n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \widehat{f}_j(n).$$

Likewise, we also have

$$\widehat{G}_m(n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \widehat{F}_j(n).$$

Using the binomial inversion formula (see, for example, [1]), we also obtain

$$\begin{aligned} \widehat{f}_m(n) &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \widehat{g}_j(n), \\ \widehat{F}_m(n) &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \widehat{G}_j(n). \end{aligned}$$

Symbolically, we can apply the idea from umbral calculus [9] to abbreviate these results as

$$\begin{aligned} \widehat{g}_m(n) &= L\Big(\big(1 - \widehat{f}(n)\big)^m\Big), \\ \widehat{G}_m(n) &= L\Big(\big(1 - \widehat{F}(n)\big)^m\Big), \\ \widehat{f}_m(n) &= L\Big(\big(1 - \widehat{g}(n)\big)^m\Big), \\ \widehat{F}_m(n) &= L\Big(\big(1 - \widehat{G}(n)\big)^m\Big). \end{aligned}$$

where L is the linear operator that maps $t(n)^j$ to $t_j(n)$.

6 Formulas for $g_m(n)$ and $G_m(n)$

Let $S_1(n,k)$ denote the sum of the $\binom{n}{k}$ products composed of k distinct factors from $[n]$, and $S_2(n,k)$ the sum of the $\binom{n-k+1}{k}$ possible products (repetition allowed) of k factors from $[n]$. It is obvious that

$$\sum_{k=0}^{\infty} S_1(n,k)\, x^k = \prod_{j=1}^{n} (1 + jx), \qquad \text{and} \qquad \sum_{k=0}^{\infty} S_2(n,k)\, x^k = \prod_{j=1}^{n} \frac{1}{1 - jx}.$$

Comparing them to the well-known identities

$$\sum_{k=0}^{\infty} s(n,k)\, x^k = \prod_{j=0}^{n-1}(x-j), \qquad \text{and} \qquad \sum_{n=k}^{\infty} S(n,k)\, x^n = \prod_{j=1}^{k} \frac{x}{1-jx},$$

it is not difficult to see that

$$S_1(n,k) = (-1)^k s(n+1, n+1-k), \tag{3}$$

and

$$S_2(n,k) = S(n+k, n). \tag{4}$$

Their equivalent forms also appear on [4, pages 71 and 72].

Gould obtained [3, Equation 1.9]

$$\sum_{k=0}^{n} \binom{n-\ell}{n+k}\binom{n+\ell}{n-k} S_1(n+k-1, n) = S_2(\ell-n, n),$$

and proved the following identity [3, Equation 1.4] from [10]:

$$\sum_{k=0}^{n} \binom{n-\ell}{n+k}\binom{n+\ell}{n-k} S_2(k, n) = S_1(\ell-1, n).$$

Applying (3) and (4) to them yields the identities

$$\sum_{k=0}^{n} \binom{n-\ell}{n+k}\binom{n+\ell}{n-k} s(n+k, k) = (-1)^n S(\ell, \ell-n),$$

and

$$\sum_{k=0}^{n} \binom{n-\ell}{n+k}\binom{n+\ell}{n-k} S(n+k, k) = (-1)^n s(\ell, \ell-n).$$

These are the two identities mentioned in the Introduction. Sun recently derived similar results [14, Theorem 2.3] that relate the Stirling numbers of the same kind. We note that his results are implied by those found in [3].

Setting $\ell = n+m$ leads to the next key result.

Lemma 14. *The following identities*

$$\sum_{k=0}^{n} \binom{-m}{n+k}\binom{2n+m}{n-k} s(n+k, k) = (-1)^n S(n+m, m), \tag{5}$$

$$\sum_{k=0}^{n} \binom{-m}{n+k}\binom{2n+m}{n-k} S(n+k, k) = (-1)^n s(n+m, m), \tag{6}$$

hold for all positive integers m.

Since $\binom{-1}{n+k} = (-1)^{n+k}$, $S(n+1,1) = 1$, and $s(n+1,1) = (-1)^n n!$, Lemma 14 immediately yields the formulas for $g_1(n)$ and $G_1(n)$.

Theorem 15. *For all positive integers n,*

$$g_1(n) = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} s(n+k,k) = 1,$$

and

$$G_1(n) = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} S(n+k,k) = (-1)^n n!.$$

For $m > 1$, the simplification becomes more complicated.

Theorem 16. *For all positive integers n,*

$$g_2(n) = \sum_{k=0}^{n} (-1)^k \binom{2n+2}{n-k} s(n+k,k) = 2n+3-2^{n+1}.$$

Proof. Since $\binom{-2}{n+k} = (-1)^{n+k}(n+k+1)$, and $S(n+2,2) = 2^{n+1}-1$, we deduce from (5) that

$$\sum_{k=0}^{n} (-1)^k (n+k+1) \binom{2n+2}{n-k} s(n+k,k) = 2^{k+1} - 1.$$

From $(n+k+2)\binom{2n+2}{n-k} = (2n+2)\binom{2n+1}{n-k}$, we obtain

$$(n+k+1)\binom{2n+2}{n-k} = (2n+2)\binom{2n+1}{n-k} - \binom{2n+2}{n-k}.$$

Thus, we can further reduce the previous identity to

$$(2n+2)g_1(n) - g_2(n) = 2^{n+1} - 1,$$

which completes the proof because $g_1(n) = 1$. □

Encouraged by what we found, we used a computer algebra system to compute the value of $g_m(n)$ for $m = 1, 2, 3, 4, 5$. This led us to the following conclusion:

Theorem 17. *For any positive integer m,*

$$g_m(n) = \sum_{j=1}^{m} (-1)^{j-1} \binom{2n+m}{m-j} S(n+j,j).$$

Proof. It follows from (5) that, for $j \geq 1$,

$$\begin{aligned} S(n+j,j) &= (-1)^n \sum_{k=0}^{n} \binom{-j}{n+k} \binom{2n+j}{n-k} s(n+k,k) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n+k+j-1}{n+k} \binom{2n+j}{n-k} s(n+k,k). \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{j=1}^{m} (-1)^{j-1} \binom{2n+m}{m-j} S(n+j,j) \\ &= \sum_{j=1}^{m} (-1)^{j-1} \binom{2n+m}{m-j} \sum_{k=0}^{n} (-1)^k \binom{n+k+j-1}{n+k} \binom{2n+j}{n-k} s(n+k,k) \\ &= \sum_{k=0}^{n} (-1)^k \binom{2n+m}{n-k} s(n+k,k) \sum_{j=1}^{m} (-1)^{j-1} \binom{n+m+k}{m-j} \binom{n+k+j-1}{n+k}. \end{aligned}$$

Using Vandermonde convolution (see, for example, [5, Equation 3.1]), the inner sum simplifies to

$$\begin{aligned} \sum_{j=1}^{m} (-1)^{j-1} \binom{n+m+k}{m-j} \binom{n+k+j-1}{n+k} &= \sum_{j=1}^{m} (-1)^{j-1} \binom{n+m+k}{m-j} \binom{n+k+j-1}{j-1} \\ &= \sum_{j=1}^{m} \binom{n+m+k}{m-j} \binom{-n-k-1}{j-1} \\ &= \binom{m-1}{m-1}, \end{aligned}$$

from which the desired result follows. ∎

When $m = 1, 2$, the formulas reduce to those in Theorems 15 and 16. We also find

$$g_3(n) = \binom{2n+3}{2} S(n+1,1) - (2n+3)\, S(n+2,2) + S(n+3,3).$$

Using an analogous argument, we obtain the following result.

Theorem 18. *For any positive integer m,*

$$G_m(n) = \sum_{j=1}^{m} (-1)^{j-1} \binom{2n+m}{m-j} s(n+j,j).$$

Accordingly,

$$\begin{aligned} G_2(n) &= (2n+2)\, s(n+1,1) - s(n+2,2), \\ G_3(n) &= \binom{2n+3}{2} s(n+1,1) - (2n+3)\, s(n+2,2) + s(n+3,3). \end{aligned}$$

While the similarity between the formulas for $g_m(n)$ and $G_m(n)$ is striking, there is an important difference between them. It is well-known (see, for example, [4, Equation 77]) that

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

This suggests that it is possible to find a closed form for $g_m(n)$ without any reference to $S(n,k)$. For example, after simplification,

$$g_3(n) = 2n^2 + 7n + 4 - 2^{n+1}(2n-5) - \frac{3^n}{2}.$$

The same cannot be said of $G_m(n)$, because there does not exist a simple summation formula for $s(n,k)$.

We invite interested readers to find alternative combinatorial and/or generating function proofs which provide closed forms for $g_m(n)$ and $G_m(n)$ for $m > 0$.

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