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Ménage Numbers and Ménage Permutations

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Abstract

In this paper, we study the combinatorial structures of straight and ordinary ménage permutations. Based on these structures, we prove four formulas. The first two formulas define a relationship between the ménage numbers and the Catalan numbers. The other two formulas count the ménage permutations by number of cycles.

1 Introduction

1.1 Straight ménage permutations and straight ménage numbers

The straight ménage problem asks for the number of ways one can arrange n male-female pairs along a linearly arranged table in such a way that men and women alternate but no woman sits next to her partner.

We call a permutation $\pi \in S_n$ a *straight ménage permutation* if $\pi(i) \neq i$ and $\pi(i) \neq i+1$ for $1 \leq i \leq n$. Use V_n to denote the number of straight ménage permutations in S_n. We call V_n the *nth straight menage number*.

The straight ménage problem is equivalent to finding V_n. Label the seats along the table as $1, 2, \ldots, 2n$. Sit the men at positions with even numbers and women at positions with odd numbers. Let π be the permutation such that the man at position $2i$ is the partner of the woman at position $2\pi(i)-1$ for $1 \leq i \leq n$. Then, the requirement of the straight ménage problem is equivalent to the condition that $\pi(i)$ is neither i nor $i+1$ for $1 \leq i \leq n$.

1.2 Ordinary ménage permutations and ordinary ménage numbers

The ordinary ménage problem asks for the number of ways one can arrange n male-female pairs around a circular table in such a way that men and women alternate, but no woman sits next to her partner.

We call a permutation $\pi \in S_n$ an *ordinary ménage permutation* if $\pi(i) \neq i$ and $\pi(i) \not\equiv i+1 \pmod{n}$ for $1 \leq i \leq n$. Use U_n to denote the number of ordinary ménage permutations in S_n. We call U_n the *nth ordinary ménage number*.

The ordinary ménage problem is equivalent to finding U_n. Label the seats around the table as $1, 2, \ldots, 2n$. Sit the men at positions with even numbers and women at positions with odd numbers. Let π be the permutation such that the man at position $2i$ is the partner of the woman at position $2\pi(i) - 1$ for all $1 \leq i \leq n$. Then, the requirement of the ordinary ménage problem is equivalent to the condition that $\pi(i)$ is neither i nor $i + 1 \pmod{n}$ for $1 \leq i \leq n$.

We hold the convention that the empty permutation $\pi_\emptyset \in S_0$ is both a straight ménage permutation and an ordinary ménage permutation, so $U_0 = V_0 = 1$.

1.3 Background

Lucas [7, pp. 491–495] first posed the problem of finding ordinary ménage numbers. Touchard [11] first found the following explicit formula (1). Kaplansky and Riordan [6] also proved an explicit formula for ordinary ménage numbers. For other early work in ménage numbers, we refer interested readers to the work of Kaplansky [5] and the paper of Moser and Wyman [8] and references therein. Among more recent papers, there are some using bijective methods to study ménage numbers. For example, Canfield and Wormald [2] used graphs to address the question. The following formulas of ménage numbers are well known [1, 9, 11]:

$$U_m = \sum_{k=0}^{m} (-1)^k \frac{2m}{2m-k} \binom{2m-k}{k} (m-k)! \qquad (m \geq 2); \tag{1}$$

$$V_n = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} (n-k)! \qquad (n \geq 0). \tag{2}$$

In particular, the interesting method of Bogart and Doyle [1] can also induce (4) and (5) of Theorem 1 in a different way as this paper.

We remark that the generating function [4, p. 372]

$$\sum_{n=0}^{\infty} U_n x^n = x + \frac{1-x}{1+x} \sum_{n=0}^{\infty} n! \Big(\frac{x}{(1+x)^2} \Big)^n \tag{3}$$

is equivalent to (5) of Theorem 1.

The purpose of the current paper is to study the combinatorial structures of straight and ordinary ménage permutations and to use these structures to prove some formulas of straight and ordinary ménage numbers. We also give an analytical proof of Theorem 1 in Section 7.

1.4 Main results

Let C_k be the kth *Catalan number*:

$$C_k = \frac{(2k)!}{k!\,(k+1)!}$$

and $c(x) = \sum_{k=0}^{\infty} c_k x^k$. Our first main result is the following theorem.

Theorem 1.

$$\sum_{n=0}^{\infty} n!\,x^n = \sum_{n=0}^{\infty} V_n x^n c(x)^{2n+1}, \tag{4}$$

$$\sum_{n=0}^{\infty} n!\,x^n = c(x) + c'(x) \sum_{n=1}^{\infty} U_n x^n c(x)^{2n-2}. \tag{5}$$

Our second main result counts the straight and ordinary ménage permutations by the number of cycles.

For $k \in \mathbb{N}$, use $(\alpha)_k$ to denote $\alpha(\alpha+1)\cdots(\alpha+k-1)$. Define $(\alpha)_0 = 1$. For $k \le n$, use C_n^k (D_n^k) to denote the number of straight (ordinary) ménage permutations in S_n with k cycles.

Theorem 2.

$$1 + \sum_{n=1}^{\infty} \sum_{j=1}^{n} C_n^j \alpha^j x^n = \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{(1+x)^n (1+\alpha x)^{n+1}}, \tag{6}$$

$$1 + \sum_{n=1}^{\infty} \sum_{j=1}^{n} D_n^j \alpha^j x^n = \frac{x+\alpha x^2}{1+x} + (1-\alpha x^2) \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{(1+x)^{n+1}(1+\alpha x)^{n+1}}. \tag{7}$$

1.5 Outline

We give some preliminary concepts and facts in Section 2. In Section 3, we define three types of reductions and the nice bijection. Then, we study the structure of straight ménage permutations and prove (4) in Section 4. In Section 5, we study the structure of ordinary ménage permutations and prove (5). Finally, we count the straight and ordinary ménage permutations by number of cycles and prove (6) and (7) in Section 6.

2 Preliminaries

For $n \in \mathbb{N}$, we use $[n]$ to denote $\{1, \ldots, n\}$. Define $[0]$ to be $\emptyset$.

Definition 3. Suppose $n > 0$ and $\pi \in S_n$. If $\pi(i) = i+1$, then we call $\{i, i+1\}$ a *succession* of π. If $\pi(i) \equiv i+1 \pmod{n}$, then we call $\{i, \pi(i)\}$ a *generalized succession* of π.

2.1 Partitions and Catalan numbers

Suppose $n > 0$. A *partition* ϵ of $[n]$ is a collection of disjoint subsets of $[n]$ whose union is $[n]$. We call each subset a *block* of ϵ. We also describe a partition as an equivalence relation: $p \sim_\epsilon q$ if and only if p and q belong to a same block of ϵ.

If a partition ϵ satisfies that for any $p \sim_\epsilon p'$ and $q \sim_\epsilon q'$, $p < q < p' < q'$ implies $p \sim_\epsilon q$; then, we call ϵ a *noncrossing partition*.

For $n \in \mathbb{N}$, suppose $\epsilon = \{V_1, \ldots, V_k\}$ is a noncrossing partition of $[n]$ and $V_i = \{a^i_1, \ldots, a^i_{j_i}\}$, where $a^i_1 < \cdots < a^i_{j_i}$. Then, ϵ induces a permutation $\pi \in S_n$: $\pi(a^i_{r(i)}) = a^i_{r(i)+1}$ for $1 \leq r(i) \leq j_i - 1$ and $\pi(a^i_{j_i}) = a^i_1$. It is not difficult to see that different noncrossing partitions induce different permutations.

The following lemma is well known [10, pp. 176–178].

Lemma 4. *For $n \in \mathbb{N}$, there are C_n noncrossing partitions of $[n]$.*

It is well known that the generating function of the Catalan numbers is

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

It is also well known that one can define the Catalan numbers by recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \tag{8}$$

with initial condition $C_0 = 1$.

Lemma 5. *The generating function of the Catalan numbers $c(x)$ satisfies*

$$c(x) = \frac{1}{1 - xc(x)} = 1 + xc^2(x) \quad \textit{and} \quad \frac{c^3(x)}{1 - xc^2(x)} = c'(x).$$

Proof of Lemma 5. The first formula is well known. By the first formula,

$$c'(x) = c^2(x) + 2xc(x)c'(x) = \frac{c^2(x)}{1 - 2xc(x)}. \tag{9}$$

Thus, to prove the second formula, we only have to show that $\dfrac{c(x)}{1 - xc^2(x)} = \dfrac{1}{1 - 2xc(x)}$ which is equivalent to $c(x) - 2xc^2(x) = 1 - xc^2(x)$. This follows from the first formula. $\square$

2.2 Diagram representation of permutations

2.2.1 Diagram of horizontal type

For $n > 0$ and $\pi \in S_n$, we use a diagram of horizontal type to represent π. To do this, draw n points on a horizontal line. The points represent the numbers $1, \ldots, n$ from left to right. For each $i \in [n]$, we draw a directed arc from i to $\pi(i)$. The permutation uniquely determines the diagram. For example, if $\pi = (1, 5, 4)(2)(3)(6)$, then its diagram is

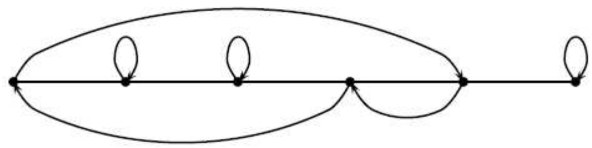

Figure 1: Circular diagram of $\pi = (1, 5, 4)(2)(3)(6)$.

2.2.2 Diagram of circular type

For $n > 0$ and $\pi \in S_n$, we also use a diagram of circular type to represent π. To do this, draw n points uniformly distributed on a circle. Specify a point that represents the number 1. The other points represent $2, \ldots, n$ in counter-clockwise order. For each i, draw a directed arc from i to $\pi(i)$. The permutation uniquely determines the diagram (up to rotation). For example, if $\pi = (1, 5, 4)(2)(3)(6)$, then its diagram is

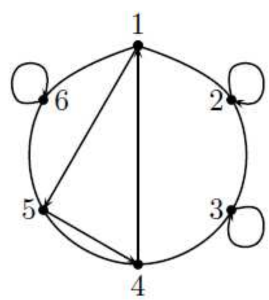

Figure 2: Horizontal diagram of $\pi = (1, 5, 4)(2)(3)(6)$.

2.3 The empty permutation

The empty permutation $\pi_\emptyset \in S_0$ is a permutation with no fixed points, no (generalized) successions and no cycles.

3 Reductions and nice bijections

In this section, we introduce reductions and nice bijections which serve as our main tools to study ménage permutations.

3.1 Reduction of type 1

Intuitively speaking, to perform a reduction of type 1 is to remove a fixed point from a permutation. Suppose $n \geq 1$, $\pi \in S_n$ and $\pi(i) = i$. Define $\pi' \in S_{n-1}$ such that:

$$\pi'(j) = \begin{cases} \pi(j), & \text{if} \quad j < i \quad \text{and} \quad \pi(j) < i; \\ \pi(j) - 1, & \text{if} \quad j < i \quad \text{and} \quad \pi(j) > i; \\ \pi(j+1), & \text{if} \quad j \geq i \quad \text{and} \quad \pi(j+1) < i; \\ \pi(j+1) - 1, & \text{if} \quad j \geq i \quad \text{and} \quad \pi(j+1) > i \end{cases} \tag{10}$$

when $n > 1$. When $n = 1$, define π' to be $\pi_\emptyset$. If we represent π by a diagram (of either type), erase the point corresponding to i and the arc connected to the point (and number other points appropriately for the circular case); then we obtain the diagram of π'. We call this procedure of obtaining a new permutation by removing a fixed point a *reduction of type* 1. For example, if

$$\pi = (1, 5, 6, 4)(2)(3)(7) \in S_7,$$

then by removing the fixed point 3 we obtain $\pi' = (1, 4, 5, 3)(2)(6) \in S_6$.

3.2 Reduction of type 2

Intuitively speaking, to do a reduction of type 2 is to glue a succession $\{k, k + 1\}$ together. Suppose $n \geq 2$, $\pi \in S_n$ and $\pi(i) = i + 1$. Define $\pi' \in S_{n-1}$ such that:

$$\pi'(j) = \begin{cases} \pi(j), & \text{if} \quad j < i \quad \text{and} \quad \pi(j) \leq i; \\ \pi(j) - 1, & \text{if} \quad j < i \quad \text{and} \quad \pi(j) > i + 1; \\ \pi(j+1), & \text{if} \quad j \geq i \quad \text{and} \quad \pi(j+1) \leq i; \\ \pi(j+1) - 1, & \text{if} \quad j \geq i \quad \text{and} \quad \pi(j+1) > i + 1. \end{cases} \tag{11}$$

If we represent π by the diagram of the *horizontal* type, erase the arc from i to $i + 1$, and glue the points corresponding to i and $i + 1$ together; then, we obtain the diagram of π'. We call this procedure of obtaining a new permutation by gluing a succession together a *reduction of type* 2. For example, if

$$\pi = (1, 5, 6, 4)(2)(3)(7) \in S_7,$$

then by gluing 5 and 6 together, we obtain $\pi' = (1, 5, 4)(2)(3)(6) \in S_6$.

3.3 Reduction of type 3

Intuitively speaking, to perform a reduction of type 3 is to glue a generalized succession $\{k, k+1 \pmod{n}\}$ together. Suppose $n \geq 1$, $\pi \in S_n$ and $\pi(i) \equiv i+1 \pmod{n}$. Define π' to be the same as in (11) when $i \neq n$. When $i = n > 1$, define π' to be

$$\pi'(j) = \begin{cases} \pi(j), & \text{if } j \neq \pi^{-1}(n); \\ 1, & \text{if } j = \pi^{-1}(n). \end{cases}$$

When $i = n = 1$, define π' to be $\pi_\emptyset$. If we represent π by a diagram of the *circular* type, erase the arc from i to $i+1$ (mod n), glue the points corresponding to i and $i+1$ (mod n) together and number the points appropriately; then, we obtain the diagram of π'. We call this procedure of obtaining a new permutation by gluing a generalized succession together a *reduction of type* 3. For example, if

$$\pi = (1, 5, 6, 7)(2)(3)(4) \in S_7,$$

then by gluing 1 and 7 together, we obtain $\pi' = (1, 5, 6)(2)(3)(4) \in S_6$.

3.4 Nice bijections

Suppose $n \geq 1$ and f is a bijection from $[n]$ to $\{2, \ldots, n+1\}$. We can also represent f by a diagram of horizontal type as for permutations. The bijection uniquely determines the diagram. If f has a fixed point or there exists i such that $f(i) = i+1$, then we can also perform reductions of type 1 or type 2 on f as above. In the latter case, we also call $\{i, i+1\}$ a *succession* of f. We can reduce f to a bijection with no fixed points and no successions by a series of reductions. It is easy to see that the resulting bijection does not depend on the order of the reductions.

The following diagram shows an example of reduction of type 2 on the bijection by gluing the succession 2 and 3 together.

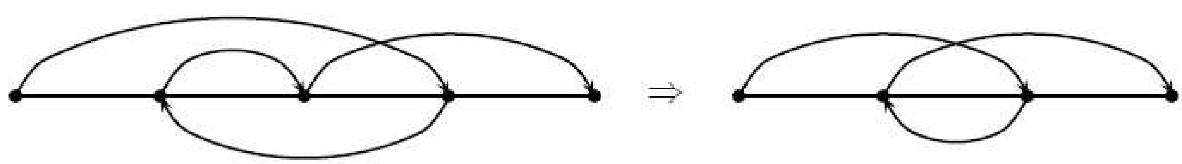

Figure 3: Reduction of type 2 by gluing the succession 2 and 3 together.

Definition 6. Suppose f is a bijection from $[n]$ to $\{2, \ldots, n+1\}$. If there exist a series of reductions of type 1 or type 2 by which one can reduce f to the simplest bijection $1 \mapsto 2$, then we call f a *nice bijection*.

Suppose $\pi \in S_n$ and p is a point of π. We can replace p by a bijection f from $[k]$ to $\{2, \ldots, k+1\}$ and obtain a new permutation $\pi' \in S_{n+k}$ by the following steps:

(1) represent π by the horizontal diagram;

(2) add a point q right before p and add an arc from q to p ;

(3) replace the arc from $\pi^{-1}(p)$ to p by an arc from $\pi^{-1}(p)$ to q ;

(4) replace the arc from q to p by the diagram of f.

For example, if π, p and f are as below,

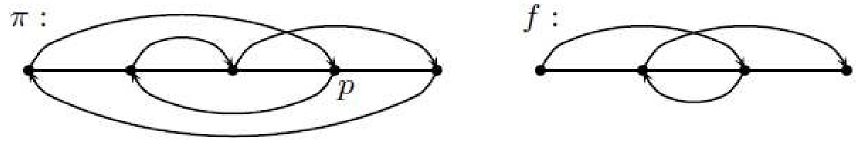

Figure 4: Example: π, p and f.

then, we can replace p by f and obtain the following permutation:

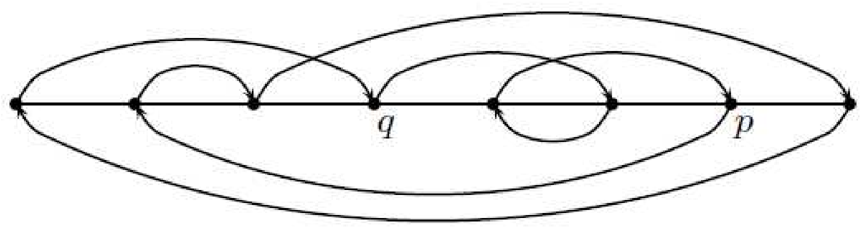

Figure 5: Replace p by f.

It is easy to see, by the definition of the nice bijection, that if f is a nice bijection, then we can reduce π' back to π: first, we can reduce π' to the permutation obtained in Step (3) because f is nice; then, by gluing the succession p and q together, we obtain π. Notice that the bijection f in the above example is not nice.

For the circular diagram of a permutation π, we can also use Steps (2)–(4) shown above to obtain a new diagram. However, to obtain a permutation π', we need to specify the point representing number 1 in the new diagram. See the example in the proof of Theorem 11.

Lemma 7. *For $n \geq 2$, let a_n be the number of nice bijections from $[n-1]$ to $\{2, \ldots, n\}$. Define $a_1 = 1$. The generating function of a_n satisfies the following equation:*

$$g(x) := a_1 x + a_2 x^2 + a_3 x^3 + \cdots = xc(x)$$

where $c(x) = \sum_{n=0}^{\infty} C_k x^k$ is the generating function of the Catalan numbers.

Proof of Lemma 7. Suppose $n \geq 2$ and f is a nice bijection from $[n-1]$ to $\{2, \dots, n\}$ such that $f(1) = k$.

Suppose $p < k$. We claim that $f(p) < k$. If not, suppose $q = f(p) > k$. Then, neither $\{1, k\}$ nor $\{p, q\}$ is a succession. Consider the horizontal diagram of f. If we perform a reduction of type 1 or type 2 on f, then the arc we remove is neither the arc from 1 to k nor the arc from p to q. By induction, no matter how many reductions we perform, there always exists an arc from 1 to k' and an arc from p' to q' such that $p' < k' < q'$. In other words, we can never reduce the bijection to $1 \mapsto 2$. Thus, we proved the claim.

Thus, the image of $\{2, \dots, k-1\}$ under f must be $\{2, \dots, k-1\}$, and therefore, the image of $\{k, \dots, n-1\}$ under f must be $\{k+1, \dots, n\}$.

This implies that $f|_{\{2,\dots,k-1\}}$ has the same diagram as a permutation $\tau \in S_{k-2}$ and we can reduce τ to $\pi_\emptyset$. This also implies that $f|_{\{k,\dots,n-1\}}$ has the same diagram as a nice bijection from $[n-k]$ to $\{2, \dots, n-k+1\}$. By Lemma 8, the number of nice bijections from $[n-1]$ to $\{2, \dots, n\}$ such that $f(1) = k$ is $C_{k-2}a_{n-k+1}$. Letting k vary, we obtain $a_n = \sum_{k=2}^{n} C_{k-2}a_{n-k+1}$.

By (8), $(C_n)_{n\geq 0}$ and $(a_{n+1})_{n\geq 0}$ have the same recurrence relation and the same initial condition $C_0 = a_1 = 1$, so $C_n = a_{n+1} (n \geq 0)$. Therefore $g(x) = xc(x)$. □

4 Structure of straight ménage permutations

In Section 4, when we mention a reduction, we mean a reduction of **type 1 or type 2**.

If a permutation π is not a straight ménage permutation, then π has at least one fixed point or succession. Thus, we can apply a reduction to π. By induction, we can reduce π to a straight ménage permutation π' by a series of reductions. For example, we can reduce $\pi_1 = (1,3)(2)(4,5,6)$ to $\pi_\emptyset$: $(1,3)(2)(4,5,6) \to (1,2)(3,4,5) \to (1)(2,3,4) \to (1,2,3) \to (1,2) \to (1) \to \pi_\emptyset$. We can reduce $\pi_2 = (1,5,4)(2)(3)(6) \in S_6$ to $(1,3,2)$: $(1,5,4)(2)(3)(6) \to (1,5,4)(2)(3) \to (1,4,3)(2) \to (1,3,2)$. It is easy to see that the resulting straight ménage permutation does not depend on the order of the reductions. Recall that we defined the permutation induced from a noncrossing partition in Section 2.1.

Lemma 8. *Suppose $\pi \in S_n$. We can reduce π to $\pi_\emptyset$ by reductions of type 1 and type 2 if and only if there is a noncrossing partition inducing π. In particular, there are C_n such permutations in S_n.*

Proof. $\Rightarrow$: Suppose we can reduce $\pi \in S_n$ to $\pi_\emptyset$. Then, π has at least one fixed point or succession. We use induction on n. If $n = 1$, the conclusion is trivial. Suppose $n > 1$.

If π has a fixed point i, then by reduction of type 1 on i we obtain π' satisfying (10). By induction assumption, there is a noncrossing partition $\Phi = \{V_1, \dots, V_k\}$ inducing π'. Now, we define a new noncrossing partition $\Pi_1(\Phi, i)$ as follows. Set

$$\tilde{V}_r = \{x+1 | x \in V_r \text{ and } x \geq i\} \cup \{x | x \in V_r \text{ and } x < i\}$$

for $1 \le r \le k$ and $\tilde{V}_{k+1} = \{i\}$. Define $\Pi_1(\Phi, i) = \{\tilde{V}_1, \dots, \tilde{V}_{k+1}\}$. It is not difficult to check that $\Pi_1(\Phi, i)$ is a noncrossing partition inducing π.

If π has a succession $\{i, i+1\}$, then by reduction of type 2 on $\{i, i+1\}$, we obtain π'' satisfying (11). By induction assumption, there is a noncrossing partition $\Phi = \{U_1, \dots, U_s\}$ inducing π''. Now, we define a new noncrossing partition $\Pi_2(\Phi, i)$ as follows. Set

$$\tilde{U}_t = \begin{cases} \{x+1 | x \in U_t \text{ and } x > i\} \cup \{x | x \in U_t \text{ and } x < i\}, & \text{if } t \ne t_0; \\ \{x+1 | x \in U_t \text{ and } x > i\} \cup \{x | x \in U_t \text{ and } x < i\} \cup \{i, i+1\}, & \text{if } t = t_0. \end{cases}$$

Define $\Pi_2(\Phi, i) = \{\tilde{U}_1, \dots, \tilde{U}_s\}$. It is not difficult to check that $\Pi_2(\Phi, i)$ is a noncrossing partition inducing π.

$\Leftarrow$: Suppose there is a noncrossing partition $\{V_1, \dots, V_k\}$ inducing π where $V_r = \{a_1^r, \dots, a_{j_r}^r\}$ and $a_1^r < \cdots < a_{j_r}^r$. We prove by using induction on n. The case $n = 1$ is trivial. Suppose $n > 1$. For $r_1 \ne r_2$, if $[a_1^{r_1}, a_{j_{r_1}}^{r_1}] \cap [a_1^{r_2}, a_{j_{r_2}}^{r_2}] \ne \emptyset$, then either $[a_1^{r_1}, a_{j_{r_1}}^{r_1}] \subset [a_1^{r_2}, a_{j_{r_2}}^{r_2}]$ or $[a_1^{r_2}, a_{j_{r_2}}^{r_2}] \subset [a_1^{r_1}, a_{j_{r_1}}^{r_1}]$; otherwise, the partition cannot be noncrossing. Thus, there exists p such that $[a_1^p, a_{j_p}^p] \cap [a_1^q, a_{j_q}^q] = \emptyset$ for all $q \ne p$. If $j_p = 1$, then a_1^p is a fixed point of π. If $j_p > 1$, then $\{a_1^p, a_2^p\}$ is a succession of π.

For the case $j_p = 1$, perform a reduction of type 1 on a_1^p and obtain $\pi' \in S_{n-1}$. Then, $\{\tilde{V}_r | r \ne p\}$ is a noncrossing partition inducing π', where

$$\tilde{V}_r = \{x - 1 | x \in V_r \text{ and } x > a_1^p\} \cup \{x | x \in V_r \text{ and } x < a_1^p\}. \tag{12}$$

By induction assumption, we can reduce π' to $\pi_\emptyset$; then, we can also reduce π to $\pi_\emptyset$.

For the case $j_p > 1$, perform a reduction of type 2 on $\{a_1^p, a_2^p\}$ and get $\pi' \in S_{n-1}$. Then, $\{\tilde{V}_r | 1 \le r \le k\}$ is a noncrossing partition inducing π', where $\tilde{V}_r$ is the same as in (12). By induction assumption, we can reduce π' to $\pi_\emptyset$; then, we can also reduce π to $\pi_\emptyset$. $\square$

Conversely, for a given straight ménage permutation $\pi \in S_m$, what is the cardinality of the set

$$\{\tau \in S_{m+n} \big| \text{ we can reduce } \tau \text{ to } \pi \text{ by reductions of type 1 and type 2}\}? \tag{13}$$

Interestingly the answer only depends on m and n; it does not depend on the choice of π. In fact, we have

Theorem 9. *Suppose $m \ge 0$ and $\pi \in S_m$ is a straight ménage permutation. Suppose $n \ge 0$ and w_m^n is the cardinality of the set in (13). Set $W_m(x) = w_m^0 + w_m^1 x + w_m^2 x^2 + w_m^3 x^3 + \cdots$; then,*

$$W_m(x) = c(x)^{2m+1}.$$

Proof of Theorem 9. If $m = 0$, then $\pi = \pi_\emptyset$, and the conclusion follows from Lemma 8. Thus, we only consider the case that $m > 0$.

Obviously, $w_m^0 = 1$. Now, suppose $n \ge 1$.

Represent π by a horizontal diagram. The diagram has $m+1$ gaps: one gap before the first point, one gap after the last point and one gap between each pair of adjacent points.

Let A be the set in (13). To obtain a permutation in A, we add points to π in the following two ways:

(a) add a permutation induced by a noncrossing partition Φ_p of $[d_p]$ into the pth gap of π, where $1 \le p \le m+1$ and $d_p \ge 0$ ($d_p = 0$ means that we add nothing into the pth gap);

(b) replace the qth point of π by a nice bijection f_q from $[r_q]$ to $\{2, \ldots, r_q + 1\}$, where $1 \le q \le m$ and $r_q \ge 0$ ($r_q = 0$ means that we do not change the qth point).

For example, if π and f are as below,

Figure 6: Example: π and f.

then we can add the permutation $(1,2)$ between p and q, add the permutation $(1)(2,3)$ after the last point and replace p by f. Then, we obtain a permutation in A that is:

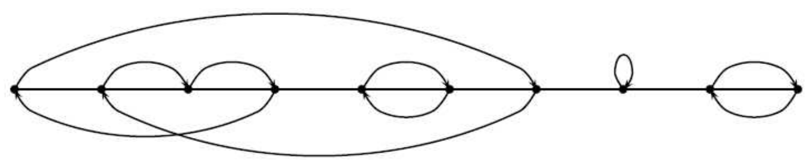

Figure 7: Add the permutation $(1,2)$ between p and q, add the permutation $(1)(2,3)$ after the last point and replace p by f.

Statement: The set of permutations constructed from (a) and (b) equals A**.** It is easy to see that we can reduce a permutation constructed through (a) and (b) to π. Conversely, suppose we can reduce $\pi' \in S_{m+n}$ to π. Now, we show that one can construct π' through (a) and (b). Use induction on n. The case that $n = 0$ is trivial. Suppose $n > 0$. Then, π' has at least one fixed point or succession. For a nice bijection f from $[s]$ to $\{2, \ldots, s+1\}$ and $1 < w_1 < s+1$, $1 \le w_2 \le s+1$, define nice bijections $B_1(f, w_1)$ and

$B_2(f, w_2)$ as

$$B_1(f, w_1) = \begin{cases} f(x), & \text{if } x < w_1 \text{ and } f(x) < w_1; \\ f(x)+1, & \text{if } x < w_1 \text{ and } f(x) \geq w_1; \\ f(x-1), & \text{if } x > w_1 \text{ and } f(x) < w_1; \\ f(x-1)+1, & \text{if } x > w_1 \text{ and } f(x) \geq w_1; \\ w_1, & \text{if } x = w_1; \end{cases} \tag{14}$$

$$B_2(f, w_2) = \begin{cases} f(x), & \text{if } x < w_2 \text{ and } f(x) \leq w_2; \\ f(x)+1, & \text{if } x < w_2 \text{ and } f(x) > w_2; \\ f(x-1), & \text{if } x > w_2 \text{ and } f(x) \leq w_2; \\ f(x-1)+1, & \text{if } x > w_2 \text{ and } f(x) > w_2; \\ w_2+1, & \text{if } x = w_2. \end{cases} \tag{15}$$

We can reduce $B_1(f, w_1)$ to f by a reduction of type 1 on the fixed point w_1. We can reduce $B_2(f, w_2)$ to f by a reduction of type 2 on the succession $\{w_2, w_2+1\}$.

Case 1: π' has a fixed point i. Using a reduction of type 1 on i, we obtain $\pi'' \in S_{m+n-1}$. By induction assumption, we can construct π'' from π by (a) and (b). According to the value of i, there is either a k such that

$$1 \leq i - \sum_{j<k}(d_j + r_j + 1) \leq d_k + 1 \tag{16}$$

or a k' such that

$$1 < i - (\sum_{j<k'}(d_j + r_j + 1) + d_{k'}) \leq r_{k'} + 1. \tag{17}$$

If there is a k such that (16) holds, then we can construct π' from π by (a) and (b), except that we add the permutation induced by $\Pi_1(\Phi_k, i - \sum_{j<k}(d_j + r_j + 1))$ instead of Φ_k into the kth gap, where Π_1 is the same as in the proof of Lemma 8. Conversely, if there exists a k' such that (17) holds, then we can construct π' from π by (a) and (b), except that we replace the k'th point of π by $B_1(f_{k'}, i - \sum_{j<k'}(d_j + r_j + 1))$ instead of $f_{k'}$, where B_1 is the same as in (14).

Case 2: π' has a succession $\{i, i+1\}$. By reduction of type 2 on $\{i, i+1\}$ we obtain $\pi'' \in S_{m+n-1}$. By induction assumption, we can construct π'' from π by (a) and (b). According to the value of i, there is either a k such that

$$0 < i - \sum_{j<k}(d_j + r_j + 1) \leq d_k \tag{18}$$

or a k' such that

$$0 < i - (\sum_{j<k'}(d_j + r_j + 1) + d_{k'}) \leq r_{k'} + 1. \tag{19}$$

If there is a k such that (18) holds, then we can construct π' from π by (a) and (b), except that we add the permutation induced by $\Pi_2(\Phi_k, i - \sum_{j<k}(d_j + r_j + 1))$ instead of Φ_k into the kth gap, where Π_2 is the same as in the proof of Lemma 8. Conversely, if there exists a k' such that (19) holds, then we can construct π' from π by (a) and (b), except that we replace the k'th point of π by $B_2(f_{k'}, i - \sum_{j<k'}(d_j + r_j + 1))$ instead of $f_{k'}$, where B_2 is the same as in (15).

Thus, we have proved the statement.

Now, add points to π by (a) and (b). The total number of points added to π is $d_1 + \cdots + d_{m+1} + r_1 + \cdots + r_m$. To obtain a permutation in $S_{m+n} \cap A$, $d_1 + \cdots + d_{m+1} + r_1 + \cdots + r_m$ should be n. Therefore, the number of permutations in $S_{m+n} \cap A$ is

$$w_m^n = \sum_{\substack{d_1,\ldots,d_{m+1} \\ r_1,\ldots,r_m}} C_{d_1} \cdots C_{d_{m+1}} a_{1+r_1} \cdots a_{1+r_m}$$

where C_k is the kth Catalan number and a_k is the same as in Lemma 7 and the sum runs over all $(2m+1)$-triples $(d_1, \ldots, d_{m+1}, r_1, \ldots, r_m)$ of nonnegative numbers with sum n.

By Lemma 7, the generating function of w_m^n is

$$c(x)^{m+1}\big(\frac{g(x)}{x}\big)^m = c(x)^{2m+1}.$$

□

Proof of (4). We can reduce each permutation in S_n to a straight ménage permutation in S_i $(0 \leq i \leq n)$. Thus, we have

$$n! = \sum_{i=0}^{n} w_i^{n-i} V_i$$

where V_i is the ith straight ménage number and w_i^{n-i} is the same as in Theorem 9. Thus,

$$\sum_{n=0}^{\infty} n!\, x^n = \sum_{n=0}^{\infty}\sum_{i=0}^{n} w_i^{n-i} V_i x^n = \sum_{i=0}^{\infty}\sum_{n=i}^{\infty} w_i^{n-i} V_i x^n = \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} w_i^{n} V_i x^n x^i = \sum_{i=0}^{\infty} c(x)^{2i+1} V_i x^i$$

where the last equality is from Theorem 9. □

5 Structure of ordinary ménage permutations

By definition, a permutation τ is an ordinary ménage permutation if and only if we cannot apply reductions of either type 1 or type 3 on τ. Similarly as in Section 4, we can reduce each permutation π to an ordinary ménage permutation by reductions of type 1 and type 3. The resulting permutation does not depend on the order of the reductions.

By the circular diagram representation of permutations, it is not difficult to see that we can reduce a permutation π to $\pi_\emptyset$ by reductions of type 1 and type 2 if and only if we can reduce π to $\pi_\emptyset$ by reductions of type 1 and type 3. Thus, we have the following lemma:

Lemma 10. *Suppose $\pi \in S_n$. We can reduce π to $\pi_\emptyset$ by reductions of type 1 and type 3 if and only if there is a noncrossing partition inducing π. In particular, there are C_n permutations of this type in S_n.*

In the following parts of Section 5, when we mention reductions, we mean reductions of **type 1 or type 3** unless otherwise specified.

Theorem 11. *Suppose $m \geq 0$ and $\pi \in S_m$ is an ordinary ménage permutation. Let r_m^n denote the cardinality of the set*

$$\{\tau \in S_{m+n} | \textit{we can reduce } \tau \textit{ to } \pi \textit{ by reductions of type 1 and type 3}\}. \tag{20}$$

Then, the generating function of r_m^n satisfies

$$R_m(x) := r_m^0 + r_m^1 x + r_m^2 x^2 + r_m^3 x^3 + \cdots = \begin{cases} c'(x)c(x)^{2m-2}, & \textit{if} \quad m > 0; \\ c(x), & \textit{if} \quad m = 0. \end{cases}$$

Proof. When $m = 0$, $r_m^n = C_n$ by Lemma 10. So $R_0(x) = c(x)$. Now, suppose $m > 0$.

Obviously $r_m^0 = 1$. Suppose $n > 0$.

Represent π by a circular diagram. The diagram has m gaps: one gap between each pair of adjacent points. Call the point corresponding to number i *point* i.

Let A denote the set in (20). To obtain a permutation in A we can add points into π by the following steps:

(a) Add a permutation induced by a noncrossing partition Φ_i of $[d_i]$ into the gap between point i and point $i+1$ (mod m), where $1 \leq i \leq m$ and $d_p \geq 0$ ($d_p = 0$ means we add nothing into the gap). Use Q_i to denote the set of points added into the gap between point i and point $i+1$ (mod m).

(b) Replace point i by a nice bijection f_i from $[t_i]$ to $\{2, \ldots, t_i + 1\}$, where $1 \leq i \leq m$ and $t_i \geq 0$ ($t_i = 0$ means that we do not change the ith point). Use P_i to denote the set of points obtained from this replacement. Thus, P_i contains $t_i + 1$ points.

(c) Specify a point in $P_1 \bigcup Q_m$ to correspond to the number 1 of the new permutation π'.

Steps (a) and (b) are the same as in the proof of Theorem 9, but Step (c) needs some explanation.

In the proof of Theorem 9, after adding points into the permutation by (a) and (b), we defined π' from the resulting *horizontal* diagram in a natural way, that is, the left most point corresponds to 1, and the following points correspond to 2, 3, 4, ... respectively. However, now there is no natural way to define π' from the resulting *circular* diagram because we have more than one choice of the point corresponding to number 1 of π'.

Note that π' can become π by a series of reductions of type 1 or type 3. Then, for each $1 \leq u \leq m$, the points in P_u will become point u of π after the reductions. Thus, we can choose any point of P_1 to be the one corresponding to the number 1 of π'. Moreover, we can also choose any point of Q_m to be the one corresponding to the number 1 of π'. The reason is that if a permutation in S_k has a cycle $(1, k)$, then a reduction of type 3 will reduce $(1, k)$ to the cycle (1). Thus, we can choose any point in $P_1 \bigcup Q_m$ to be the point corresponding to number 1 of π'. To see this more clearly, let us look at an example. Suppose π and f are as below:

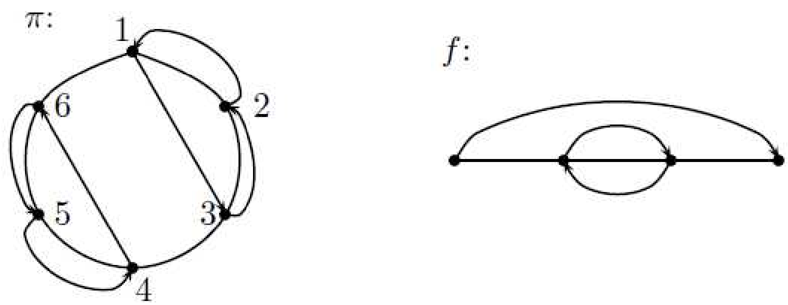

Figure 8: Example: π and f.

Then, we can add the permutation $(1, 2)$ between point 2 and point 3, add the permutation $(1)(2, 3)$ between point 6 and point 1 and replace point 2 by f. Then, we obtain a new diagram:

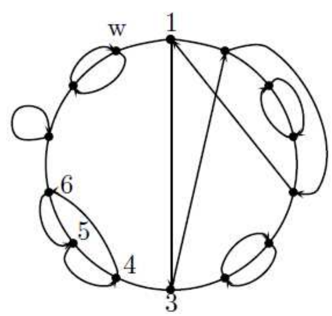

Figure 9: Add the permutation $(1, 2)$ between point 2 and point 3, add the permutation $(1)(2, 3)$ between point 6 and point 1 and replace point 2 by f.

Now, we can choose point 1 or any of the three points between point 1 and point 6 to be the point corresponding to number 1 of the new permutation π'.

For instance, if we set point 1 to be the point corresponding to number 1 of π', then

$$\pi' = (1, 8, 2, 5)(3, 4)(6, 7)(9, 11, 10)(12)(13, 14).$$

If we set point w to be the point corresponding to number 1 of π', then

$$\pi' = (1, 14)(2, 9, 3, 6)(4, 5)(7, 8)(10, 12, 11)(13).$$

Now, continue the proof. By a similar argument to the one we used to prove the statement in the proof of Theorem 9, we have that the set of permutations constructed by (a)–(c) equals A.

Now, add points to π by (a)–(c). Then, $P_1 \bigcup Q_m$ contains, in total, $d_m + (t_1 + 1)$ points. Thus, we have $d_m + (t_1 + 1)$ ways to specify the point corresponding to number 1 in π'. The total number added to π is $d_1 + \cdots + d_m + t_1 + \cdots + t_m$. Therefore, the number of permutations in $S_{m+n} \cap A$ is

$$r_m^n = \sum_{\substack{d_1,\ldots,d_m\\ t_1,\ldots,t_m}} C_{d_1} \cdots C_{d_m} a_{1+t_1} \cdots a_{1+t_m} (d_m + t_1 + 1)$$

where C_k is the kth Catalan number, a_k is the same as in Lemma 7, and the sum runs over all $2m$-triples $(d_1, \ldots, d_m, t_1, \ldots, t_m)$ of nonnegative integers such that $\sum_{u=1}^{m}(t_u + d_u) = n$.

Set $\eta_k = \sum\limits_{r=0}^{k} a_{r+1} C_{k-r}(k + 1)$; from Lemma 7, we have

$$1 + \frac{\eta_1}{2}x + \frac{\eta_2}{3}x^2 + \frac{\eta_3}{4}x^3 + \cdots = c^2(x).$$

By Lemma 5, the generating function of η_k is $1 + \eta_1 x + \eta_2 x^2 + \eta_3 x^3 + \cdots = (xc^2(x))' = c'(x)$.

Thus, the generating function of r_m^n is $c(x)^{2m-2}c'(x)$. $\square$

Proof of (5). We can reduce each permutation in S_n to an ordinary ménage permutation. Thus, we have

$$n! = \sum_{i=0}^{n} r_i^{n-i} U_i$$

where U_i is the ith ordinary ménage number and r_i^{n-i} is the same as in Theorem 11. Thus,

$$\sum_{n=0}^{\infty} n!\, x^n = \sum_{n=0}^{\infty}\sum_{i=0}^{n} r_i^{n-i} U_i x^n = \sum_{i=0}^{\infty}\sum_{n=i}^{\infty} r_i^{n-i} U_i x^n$$

$$= \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} r_i^n U_i x^n x^i = c(x) + \sum_{i=1}^{\infty} c'(x)c(x)^{2i-2} U_i x^i$$

where the last equality follows from Theorem 11. $\square$

6 Counting ménage permutations by number of cycles

We prove (6) in Section 6.2 and prove (7) in Section 6.3. Our main method is coloring. We remark that one can also prove (6) and (7) by using the inclusion-exclusion principle in a similar way as Gessel [3].

6.1 Coloring and weights

For a permutation π, we use $f(\pi)$, $g(\pi)$, $h(\pi)$ and $r(\pi)$ to denote the number of its cycles, fixed points, successions and generalized successions, respectively. The following lemma is well known.

Lemma 12. *For $n \geq 0$,*

$$\sum_{\pi \in S_n} \alpha^{f(\pi)} = (\alpha)_n.$$

For a permutation, color some of its fixed points red and color some of its *generalized* successions yellow. Then, we obtain a colored permutation. Here and in the following sections, we use the phrase *colored permutation* as follows: if two permutations are the same as maps but have different colors, then they are different colored permutations.

Define $\mathbb{S}_n$ to be the set of colored permutations on n objects. Set A_n to be the subset of $\mathbb{S}_n$ consisting of colored permutations with a colored generalized succession $\{n, 1\}$. Set $B_n = \mathbb{S}_n \backslash A_n$. So we can consider S_n as a subset of $\mathbb{S}_n$ consisting of permutations with no color. In particular, $\mathbb{S}_0 = S_0$.

Set

$$M_n^\alpha(t,u) = \sum_{\pi \in S_n} \alpha^{f(\pi)} t^{g(\pi)} u^{h(\pi)} \qquad \text{and} \qquad L_n^\alpha(t,u) = \sum_{\pi \in S_n} \alpha^{f(\pi)} t^{g(\pi)} u^{\tau(\pi)}.$$

For a colored permutation $\epsilon \in \mathbb{S}_n$, define two weights W_1 and W_2 of ϵ as:

$$W_1(\epsilon) = x^n \cdot \alpha^{f(\epsilon)} \cdot t^{\text{number of colored fixed points}} \cdot u^{\text{number of colored successions}},$$
$$W_2(\epsilon) = x^n \cdot \alpha^{f(\epsilon)} \cdot t^{\text{number of colored fixed points}} \cdot u^{\text{number of colored generalized successions}}.$$

Lemma 13. $\sum\limits_{\epsilon \in B_n} W_1(\epsilon) = \sum\limits_{\epsilon \in B_n} W_2(\epsilon) = M_n^\alpha(1+t, 1+u)x^n$, $\sum\limits_{\epsilon \in \mathbb{S}_n} W_2(\epsilon) = L_n^\alpha(1+t, 1+u)x^n$.

Proof. The lemma follows directly from the definitions of M_n^α, L_n^α, B_n, W_1 and W_2. □

6.2 Counting straight ménage permutations by number of cycles

In this subsection, we represent permutations by diagrams of **horizontal type**.

Suppose $n \geq 0$ and $\pi \in S_n$. Then, π has $n+1$ gaps: one gap before the first point, one gap after the last point and one gap between each pair of adjacent points. We can add points to π by the following steps.

(a) Add the identity permutation of $S_{(d_p)}$ into the pth gap of π, where $1 \le p \le n+1$ and $d_p \ge 0$ ($d_p = 0$ means that we add nothing into the pth gap).

(b) Replace the qth point of π by a nice bijection from $[r_q]$ to $\{2, \dots, r_q+1\}$, which sends each i to $i+1$. Here, $1 \le q \le n$ and $r_q \ge 0$ ($r_q = 0$ means that we do not change the qth point).

(c) Color the fixed points added by (a) red, and color the successions added by (b) yellow.

Then (a), (b) and (c) give a colored permutation in $\bigcup_{n=0}^{\infty} B_n$.

Lemma 14. *Suppose $\pi \in S_n$. The sum of the W_1-weights of all colored permutations constructed from π by (a)–(c) is*

$$x^n \cdot \alpha^{f(\pi)} \cdot \frac{1}{(1-\alpha t x)^{n+1}} \cdot \frac{1}{(1-ux)^n}.$$

Proof. In Step (a), we added d_p fixed points into the pth gap. They contribute $(xt\alpha)^{d_p}$ to the weight because each of them is a single cycle and a colored fixed point. Because d_p can be any nonnegative integer, the total contribution of the fixed points added into a gap is $\frac{1}{(1-\alpha tx)}$. Thus, the total contribution of the fixed points added into all the gaps is $\frac{1}{(1-\alpha tx)^{n+1}}$.

In Step (b), through the replacement on the qth point, we added r_q points and r_q successions to π (each of which received a color in Step (c)). Thus, the contribution of this replacement to the weight is $(ux)^{d_q}$. Because d_q can be any nonnegative integer, the total contribution of the nice bijections replacing the qth point is $\frac{1}{(1-ux)}$. Thus, the total contribution of the nice bijections corresponding to all points is $\frac{1}{(1-ux)^n}$.

Observing that the W_1-weight of π is $x^n \cdot \alpha^{f(\pi)}$, we complete the proof. □

Suppose ϵ is a colored permutation in $\bigcup_{n=0}^{\infty} B_n$. If we perform reductions of type 1 on the colored fixed points and perform reductions of type 2 on the colored successions, then we obtain a new permutation ϵ' with no color. This ϵ' is the only permutation in $\bigcup_{n=0}^{\infty} S_n$ from which we can obtain ϵ by Steps (a)–(c). Therefore, there is a bijection between $\bigcup_{n=0}^{\infty} B_n$ and

$$\bigcup_{n=0}^{\infty} \bigcup_{\pi \in S_n} \{\text{colored permutation constructed from } \pi \text{ through Steps (a)–(c)}\}.$$

Because of the bijection, Lemmas 13 and 14 imply that

$$\sum_{n=0}^{\infty} M_n^{\alpha}(1+t, 1+u)x^n = \sum_{n=0}^{\infty} \sum_{\pi \in S_n} \frac{x^n \alpha^{f(\pi)}}{(1-\alpha tx)^{n+1}(1-ux)^n}. \tag{21}$$

Proof of (6). By Lemma 12 and (21), the sum of the W_1-weights of colored permutations in $\bigcup\limits_{n=0}^{\infty} B_n$ is

$$\sum_{n=0}^{\infty} M_n^{\alpha}(1+t, 1+u)x^n = \sum_{n=0}^{\infty} \frac{x^n (\alpha)_n}{(1-\alpha tx)^{n+1}(1-ux)^n}. \tag{22}$$

Setting $t = u = -1$, we have

$$\sum_{n=0}^{\infty} M_n^{\alpha}(0,0)x^n = \sum_{n=0}^{\infty} \frac{x^n(\alpha)_n}{(1+\alpha x)^{n+1}(1+x)^n}.$$

Recall that straight ménage permutations are permutations with no fixed points or successions. Thus, for $n > 0$, $M_n^{\alpha}(0,0)$ is the sum of the W_1-weights of straight ménage permutations in S_n, which is $\sum_{j=1}^{n} C_n^j \alpha^j x^n$. Furthermore, $M_0^{\alpha}(0,0) = 1$. Thus, we have proved (6). $\square$

6.3 Counting ordinary ménage permutations by number of cycles

In this subsection, we represent permutations by diagrams of the **horizontal type**.

Suppose $n \geq 0$ and $\pi \in S_n$. Define $\mathbb{S}_m(\pi)$ to be the a subset of $\mathbb{S}_m$: $\tau \in \mathbb{S}_m$ is in $\mathbb{S}_m(\pi)$ if and only if when we apply reductions of type 1 on the colored fixed points of τ and apply reductions of type 3 on the colored generalized successions of τ, we obtain π. Define $A_m(\pi) = \mathbb{S}_m(\pi) \cap A_m$ and $B_m(\pi) = \mathbb{S}_m(\pi) \backslash A_m(\pi)$.

Lemma 15. *The W_2-weights of colored permutations in $\bigcup_{n=0}^{\infty} A_n$ are*

$$\sum_{n=1}^{\infty} x^n(\alpha)_n \cdot \frac{1}{(1-\alpha tx)^n} \cdot \frac{1}{(1-ux)^{n+1}} \cdot ux + \alpha utx + \frac{\alpha ux}{1-ux}.$$

Proof. We first evaluate the sum of W_2-weights of colored permutations in $\bigcup_{m=n}^{\infty} A_m(\pi)$ and then add them up with respect to $\pi \in S_n$ and $n \geq 0$.

Case 1: $n > 0$**.** In this case, for $\pi \in S_n$, we can construct a colored permutation in $\bigcup_{m=n}^{\infty} A_m(\pi)$ by the following steps.

(a') Define $\tilde{\pi}$ to be a permutation in S_{n+1} that sends $\pi^{-1}(1)$ to $n+1$, sends $n+1$ to 1 and sends all other j to $\pi(j)$. Represent $\tilde{\pi}$ by a horizontal diagram.

(b') For $1 \leq p \leq n-1$, add the identity permutation of $S_{(d_p)}$ into the gap of $\tilde{\pi}$ between number p and $p+1$, where $d_p \geq 0$ ($d_p = 0$ means we add nothing into the gap).

(c') Replace the qth point of $\tilde{\pi}$ by a nice bijection from $[r_q]$ to $\{2, \ldots, r_q+1\}$, which maps each i to $i+1$. Here, $1 \leq q \leq n$ and $r_q \geq 0$ ($r_q = 0$ means that we do not change the qth point).

(d') Color the generalized succession consisting of 1 and the largest number yellow. Color the fixed points and generalized successions added by (b')–(c') red and yellow, respectively.

Suppose $\epsilon \in \bigcup_{m=n}^{\infty} A_m(\pi)$. If we perform reductions of type 1 on its colored fixed points and perform reductions of type 3 on its colored generalized successions, we obtain π. Furthermore, π is the only permutation in $\bigcup_{k=0}^{\infty} S_k$ from which we can obtain ϵ through (a')–(d'). Therefore, there is a bijection between $\bigcup_{m=n}^{\infty} A_m(\pi)$ and

$$\{\text{colored permutations constructed from } \pi \text{ through } (a')\text{–}(d')\}.$$

Now, we can claim that the sum of W_2-weight of the colored permutations in $\bigcup_{m=n}^{\infty} A_m(\pi)$ is

$$x^n \alpha^{f(\pi)} \cdot \frac{1}{(1-\alpha t x)^n} \cdot \frac{1}{(1-ux)^{n+1}} \cdot ux. \tag{23}$$

In (23), $x^n \alpha^{f(\pi)}$ is the W_2-weight of π. The term $\frac{1}{(1-\alpha t x)^n}$ corresponds to the fixed points added to the permutation in (b'). The term $\frac{1}{(1-ux)^{n+1}}$ corresponds to the successions added to the permutation in (c'). The term ux corresponds to the generalized succession $\{n+1, 1\}$ added to the permutation in (a').

Case 2: $n = 0$ **and** $m = 0$**.** In this case, $A_0(\pi_\emptyset)$ is empty.

Case 3: $n = 0$ **and** $m \geq 2$**.** In this case, $A_m(\pi_\emptyset)$ contains one element: the cyclic permutation π_C, which maps each i to $i+1$ (mod m). Each generalized succession of π_C has a color, so the sum of the W_2-weight of the colored permutations in $A_m(\pi_\emptyset)$ is $\alpha(ux)^m$.

Case 4: $n = 0$ **and** $m = 1$**.** In this case, $A_1(\pi_\emptyset)$ contains one map: $id_1 \in S_1$. However, id_1 can have two types of color, namely, yellow and red+yellow, because id_1 has one fixed point and one generalized succession. Thus, $A_1(\pi_\emptyset)$ contains two colored permutations, and the sum of their W_2-weights is $\alpha ux + \alpha tux$.

We remark that the identity permutation id_1 actually corresponds to four colored permutations. In addition to the two in $A_1(\pi_\emptyset)$, the other two are id_1, with a red color for its fixed point, and id_1, with no color. We have considered these colored permutations in $B_1(\pi_\emptyset)$ and $B_1(id_1)$, respectively.

Because $\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} \bigcup_{\pi \in S_n} \bigcup_{m=n}^{\infty} A_m(\pi)$, the sum of the W_2-weights of the colored permutations in $\bigcup_{n=0}^{\infty} A_n$ equals the sum of the weights found in Cases 1–4. Thus, the sum is

$$\left(\sum_{n=1}^{\infty} \sum_{\pi \in S_n} x^n \alpha^{f(\pi)} \cdot \frac{1}{(1-\alpha t x)^n} \cdot \frac{1}{(1-ux)^{n+1}} \cdot ux \right) + \big(\sum_{m=2}^{\infty} \alpha (ux)^m \big) + \big(\alpha ux + \alpha tux \big)$$
$$= \sum_{n=1}^{\infty} x^n (\alpha)_n \cdot \frac{1}{(1-\alpha t x)^n} \cdot \frac{1}{(1-ux)^{n+1}} \cdot ux + \alpha utx + \frac{\alpha ux}{1-ux}$$

where we used Lemma 12. $\square$

Proof of (7). By Lemma 13, $\sum\limits_{n=0}^{\infty} L_n^{\alpha}(1+t, 1+u)x^n$ is the sum of the W_2-weights of the colored permutations in $\bigcup_{n=0}^{\infty} \mathbb{S}_n$. By Lemma 13 and (22), the sum of the W_2-weights of all colored permutations in $\bigcup_{n=0}^{\infty} B_n$ is

$$\sum_{n=0}^{\infty} \frac{x^n (\alpha)_n}{(1-\alpha t x)^{n+1}(1-ux)^n}.$$

Because $\bigcup_{n=0}^{\infty} \mathbb{S}_n = (\bigcup_{n=0}^{\infty} B_n) \bigcup (\bigcup_{n=0}^{\infty} A_n)$, Lemma 15 implies

$$\sum_{n=0}^{\infty} L_n^{\alpha}(1+t,1+u)x^n$$
$$=\left(\sum_{n=0}^{\infty} \frac{x^n(\alpha)_n}{(1-\alpha tx)^{n+1}(1-ux)^n}\right)+\left(\sum_{n=1}^{\infty} x^n(\alpha)_n\cdot\frac{1}{(1-\alpha tx)^n}\cdot\frac{1}{(1-ux)^{n+1}}\cdot ux+\alpha utx+\frac{\alpha ux}{1-ux}\right)$$
$$=\sum_{n=0}^{\infty}\left[\frac{x^n(\alpha)_n}{(1-\alpha tx)^{n+1}(1-ux)^{n+1}}(1-\alpha tux^2)\right]+\alpha utx+\frac{(\alpha-1)ux}{1-ux}. \tag{24}$$

Recall that ordinary ménage permutations are permutations with no fixed points and no generalized successions. By definition, $L_0^{\alpha}(0,0)x^0=1$. When $n\geq 1$, $L_n^{\alpha}(0,0)x^n$ is the sum of the W_2-weights of all ordinary ménage permutations in S_n, which equals $\sum_{j=1}^{n} D_n^j\alpha^j x^n$. Thus, $\sum_{n=0}^{\infty} L_n^{\alpha}(0,0)x^n$ equals the left side of (7). When we set $t=u=-1$, the left side of (24) equals the left side of (7), and the right side of (24) equals the right side of (7). We have proved (7). □

7 An analytical proof of (4) and (5)

Now, we derive (4) and (5) from (1) and (2). By (2),

$$\begin{aligned}\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(n-k)!\,x^n=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(-1)^k\binom{2n-k}{k}(n-k)!\,x^n\\ &=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^k\binom{2n+k}{k}n!\,x^{n+k}=\sum_{n=0}^{\infty}n!\,x^n\left[\sum_{k=0}^{\infty}(-1)^k\binom{2n+k}{k}x^k\right]\\ &=\sum_{n=0}^{\infty}n!\,\frac{x^n}{(1+x)^{2n+1}}.\end{aligned} \tag{25}$$

Letting $x=zc^2(z)$, from Lemma 5 we have $1+x=c(z)$, $\dfrac{x}{(1+x)^2}=z$ and

$$\sum_{n=0}^{\infty} V_n z^n c^{2n}(z)=\sum_{n=0}^{\infty} V_n x^n=\sum_{n=0}^{\infty} n!\,\frac{x^n}{(1+x)^{2n+1}}=\sum_{n=0}^{\infty} n!\,\frac{z^n}{c(z)}.$$

which implies (4). From (1),

$$U_n=\begin{cases}1, & \text{if } n=0;\\ 0, & \text{if } n=1;\\ \sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(n-k)!+\sum_{k=1}^{n}(-1)^k\binom{2n-k-1}{k-1}(n-k)!, & \text{if } n>1.\end{cases}$$

Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} U_n x^n &= x + \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(n-k)!\,x^n + \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^k\binom{2n-k-1}{k-1}(n-k)!\,x^n \\
&= x + \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(n-k)!\,x^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{k+1}\binom{2n-k}{k}(n-k)!\,x^{n+1} \\
&= x + (1-x)\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(n-k)!\,x^n.
\end{aligned}$$

Then, by (25),

$$\sum_{n=0}^{\infty} U_n x^n = x + (1-x)\sum_{n=0}^{\infty} n!\,\frac{x^n}{(1+x)^{2n+1}}.$$

Noticing $U_0 = 1$ we have

$$1 - x + \sum_{n=1}^{\infty} U_n x^n = (1-x)\sum_{n=0}^{\infty} n!\,\frac{x^n}{(1+x)^{2n+1}}$$

and

$$1 + x + \frac{1+x}{1-x}\sum_{n=1}^{\infty} U_n x^n = \sum_{n=0}^{\infty} n!\,\frac{x^n}{(1+x)^{2n}}. \tag{26}$$

Letting $x = zc^2(z)$, from Lemma 5 and (26), we have $1 + x = c(z)$, $\dfrac{x}{(1+x)^2} = z$ and

$$\sum_{n=0}^{\infty} n!\,z^n = c(z) + \frac{c(z)}{1 - zc^2(z)}\sum_{n=1}^{\infty} U_n z^n (c(z))^{2n} = c(z) + \frac{(c(z))^3}{1 - zc^2(z)}\sum_{n=1}^{\infty} U_n z^n (c(z))^{2n-2}.$$

Then, (5) follows from the above equation and Lemma 5.

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