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Transcendence of Digital Expansions Generated by a Generalized Thue-Morse Sequence

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Abstract

In this article, first we generalize the Thue-Morse sequence by means of a cyclic permutation and the k-adic expansion of non-negative integers, giving a sequence $(a(n))_{n=0}^{\infty}$, and consider the condition that $(a(n))_{n=0}^{\infty}$ is non-periodic. Next, we show that, if a generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$ is not periodic, then no subsequence of the form $(a(N+nl))_{n=0}^{\infty}$ (where $N \geq 0$ and $l > 0$) is periodic. We apply the combinatorial transcendence criterion established by Adamczewski, Bugeaud, Luca, and Bugeaud to find that, for a non-periodic generalized Thue-Morse sequence taking its values in $\{0, 1, \ldots, \beta - 1\}$ (where β is an integer greater than 1), the series $\sum_{n=0}^{\infty} a(N+nl)\beta^{-n-1}$ gives a transcendental number. Furthermore, for non-periodic generalized Thue-Morse sequences taking positive integer values, the continued fraction $[0, a(N), a(N+l), \ldots, a(N+nl), ...]$ gives a transcendental number.

1 Introduction

First we introduce the Thue-Morse sequence, defined by digit counting. Let k be an integer greater than 1. We define the k-adic expansion of a non-negative integer n as follows:

$$n = \sum_{q=1}^{\text{finite}} s_{n,q} k^{w_n(q)}, \tag{1}$$

where $1 \leq s_{n,q} \leq k-1, 0 \leq w_n(q) < w_n(q+1)$. For any integer s in $\{1, \ldots, k-1\}$, let $e_s(n)$ denote the number of occurrences of s in the base k representation of n. For an integer L greater than 1, we define a sequence $(e_s^L(n))_{n=0}^{\infty}$ by

$$e_s^L(n) \equiv e_s(n) \pmod{L}, \tag{2}$$

where $0 \leq e_s^L(n) \leq L-1$, $e_s(0) = 0$. Then $(e_1^2(n))_{n=0}^{\infty}$, where $k = 2$, is known as the Thue-Morse sequence. The Thue-Morse sequence has several definitions. See [11, 8, 12].

Now we introduce a new sequence. Let K be a map,

$$K : \{1, \ldots, k-1\} \longrightarrow \{0, 1, \ldots, L-1\}.$$

We define $(a(n))_{n=0}^{\infty}$ as

$$a(n) \equiv \sum_{s=1}^{k-1} K(s) e_s^L(n) \pmod{L}, \tag{3}$$

where $0 \leq a(n) \leq L-1$. Morton and Mourant [14] and Adamczewski and Bugeaud [1] proved the following result.

Theorem 1 ([14, 1])**.** *Let $\beta \geq L$ be an integer. Then $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number unless*

$$sK(1) \equiv K(s) \pmod{L} \text{ for all } 1 \leq s \leq k-1 \text{ and } K(k-1) \equiv 0 \pmod{L}. \tag{4}$$

The proof of Theorem 1 relies on the periodicity of $(a(n))_{n=0}^{\infty}$ [14] and the Cobham conjecture that was settled by Adamczewski and Bugeaud [1]. More precisely, Morton and Mourant [14] proved that $(a(n))_{n=0}^{\infty}$ is a k-automatic sequence for any map K (see Definition 23 in Section 5 for the full definition). Furthermore, they proved that $(a(n))_{n=0}^{\infty}$ is periodic if and only if $(a(n))_{n=0}^{\infty}$ is purely periodic, which enabled them to prove that $(a(n))_{n=0}^{\infty}$ is periodic if and only if the map K satisfies (4). Later, Adamczewski and Bugeaud [1] proved the Cobham conjecture by using the Schmidt subspace theorem. Thus they deduced Theorem 1 by combining the results of Morton and Mourant with the Cobham conjecture.

Let us define a generalized Thue-Morse sequence as follows: For any integer s in $\{1, \ldots, k-1\}$ and any non-negative integer y, letting $d(n; sk^y)$ be 1 or 0, and $d(n; sk^y)$ satisfies that

$d(n; sk^y) = 1$ if and only if there exists an integer q such that $s_{n,q}k^{w_n(q)} = sk^y$. Let κ be a map,

$$\kappa : \{1, \ldots, k-1\} \times \mathbb{N} \longrightarrow \{0, 1, \ldots, L-1\},$$

where $\mathbb{N}$ denotes the set of non-negative integers. We define $(a(n))_{n=0}^{\infty}$ as

$$a(n) \equiv \sum_{y=0}^{\infty} \sum_{s=1}^{k-1} \kappa(s, y) d(n; sk^y) \pmod{L}, \tag{5}$$

where $0 \leq a(n) \leq L-1$ and $a(0) = 0$. We call $(a(n))_{n=0}^{\infty}$ a generalized Thue-Morse sequence of type (L, k, κ). Thus the Thue-Morse sequence is the generalized Thue-Morse sequence of type $(2, 2, \kappa)$ with $\kappa(1, y) = 1$ for all $y \in \mathbb{N}$. Moreover, if a generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$ is of type (L, k, κ) with

$$\kappa(s, y) = \kappa(s, y+1) \tag{6}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $(a(n))_{n=0}^{\infty}$ coincides with the sequence defined by (3), which satisfies the conditions $K(s) = \kappa(s, y)$ for all s with $1 \leq s \leq k-1$. In this article, we generalize Theorem 1, as follows.

Theorem 2. *Let $(a(n))_{n=0}^{\infty}$ be a generalized Thue-Morse sequence of type (L, k, κ). Let $\beta \geq L$ be an integer. If there is not an integer A such that*

$$\kappa(s, A+y) \equiv \kappa(1, A) sk^y \pmod{L} \tag{7}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is a transcendental number.

By Theorem 2, one can find an uncountable quantity of transcendental numbers. Moreover, if $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number, then $\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is also a transcendental number. The proof of Theorem 2 does not rely on the pure periodicity of the periodic generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$ and the Cobham conjecture. Here we study non-periodicity of the subsequence $(a(N+nl))_{n=0}^{\infty}$ (for all $N \geq 0$ and for all $l > 0$) of a generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$. See also Morgenbesser, Shallit, and Stoll [15]. Almost no generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$ is k-automatic (see Proposition 28 in Section 5). Therefore, the proof of Theorem 2 is different from the proof of Theorem 1. We prove Theorem 2 by combining Theorem 15 in Section 3 with the combinatorial transcendence criterion established by Adamczewski, Bugeaud, and Luca [2].

This paper is organized as follows. In Section 2, we review the basic concepts of the periodicity of sequences, and give the formal definition of the generalized Thue-Morse sequences. For a sequence $(a(n))_{n=0}^{\infty}$, we set its generating function $g(z) \in \mathbb{C}[[z]]$ to be

$$g(z) := \sum_{n=0}^{\infty} a(n) z^n.$$

For a generalized Thue-Morse sequence, one can prove that the generating function is convergent on the open unit disk and that it has an infinite product expansion. In Section 3, first we prove the key lemma on the k-adic expansion of non-negative integers. Next, we use this lemma and the infinite product expansion of the generating function of a generalized Thue-Morse sequence to prove a necessary-sufficient condition for the non-periodicity of the generalized Thue-Morse sequence. Furthermore, we prove that if the generalized Thue-Morse sequence is not periodic, then no subsequence $(a(N+nl))_{n=0}^{\infty}$ (for all $N \geq 0$ and for all $l > 0$) of the generalized Thue-Morse sequences is periodic. In Section 4, we introduce the concept of the stammering sequence, introduced by Adamczewski, Bugeaud, and Luca [2], and the combinatorial transcendence criterion, established by Adamczewski, Bugeaud, Luca [2] and Bugeaud [9]. By applying this combinatorial transcendence criterion to the generalized non-periodic Thue-Morse sequence $(a(n))_{n=0}^{\infty}$, which takes its values from $\{0, 1, \ldots, \beta - 1\}$, we show that $\sum_{n=0}^{\infty} a(N + nl)\beta^{-n-1}$ is a transcendental number. Furthermore by applying this combinatorial transcendence criterion to the generalized non-periodic Thue-Morse sequence $(a(n))_{n=0}^{\infty}$, which takes its values in bounded positive integers, we show that the continued fraction $[0, a(N), a(N + l), \ldots, a(N + nl), \ldots]$ is also transcendental number. This result includes Theorem 2. In Section 5, we consider the necessary-sufficient condition that a generalized Thue-Morse sequence is a k-automatic sequence. Then we find many transcendental numbers whose irrationality exponent is finite in all arithmetical subsequences of the corresponding generalized Thue-Morse sequence by applying the Adamczewski and Cassaigne result on k-automatic irrational numbers [3]. Furthermore, we consider the transcendence of the value at the algebraic point of the generating function $\sum_{n=0}^{\infty} a(N+nl)z^{-n-1}$ by applying Becker's result on k-automatic power series.

2 Generalized Thue-Morse sequences and their generating functions

Let $(a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$. The sequence $(a(n))_{n=0}^{\infty}$ is called *ultimately periodic* if there exist non-negative integers N and $l > 0$ such that

$$a(n) = a(n + l) \qquad (\forall n \geq N). \tag{8}$$

An *arithmetical subsequence* of $(a(n))_{n=0}^{\infty}$ is defined to be a subsequence such as $(a(N + tl))_{t=0}^{\infty}$, where $N \geq 0$ and $l > 0$.

Definition 3. Let $(a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$. The sequence $(a(n))_{n=0}^{\infty}$ is called *everywhere non-periodic* if no arithmetical subsequence of $(a(n))_{n=0}^{\infty}$ takes on only one value.

Now we present some lemmas about the everywhere non-periodic sequences.

Lemma 4. *If $(a(n))_{n=0}^{\infty}$ is everywhere non-periodic, then $(a(n))_{n=0}^{\infty}$ is not ultimately periodic.*

Proof. We prove the contrapositive. Assume that $(a(n))_{n=0}^{\infty}$ is ultimately periodic. From the definition of everywhere non-periodic, there exist non-negative integers N and $l > 0$ such that

$$a(n) = a(n+l) \qquad (\forall n \geq N). \tag{9}$$

It follows from (9) that the arithmetical subsequence $(a(N+tl))_{t=0}^{\infty}$ takes on only one value. □

Lemma 5. *If $(a(n))_{n=0}^{\infty}$ is everywhere non-periodic, then all arithmetical subsequences of $(a(n))_{n=0}^{\infty}$ are everywhere non-periodic.*

Proof. We prove contraposition. If $(a(N+tl))_{t=0}^{\infty}$ is not everywhere non-periodic, then there exist non-negative integers k and $J > 0$ such that $(a(N+kl+mJl))_{m=0}^{\infty}$ takes on only one value. The subsequence $(a(N+kl+mJl))_{m=0}^{\infty}$ is also an arithmetical subsequence of $(a(n))_{n=0}^{\infty}$. Therefore, $(a(n))_{n=0}^{\infty}$ is not everywhere non-periodic. □

Corollary 6. *$(a(n))_{n=0}^{\infty}$ is everywhere non-periodic if and only if no arithmetical subsequence of $(a(n))_{n=0}^{\infty}$ is ultimately periodic.*

Proof. Assume $(a(n))_{n=0}^{\infty}$ is everywhere non-periodic. By Lemma 4 and Lemma 5, no arithmetical subsequence of the sequence $(a(n))_{n=0}^{\infty}$ is periodic.

We show the sufficient condition. Assume $(a(n))_{n=0}^{\infty}$ is not everywhere non-periodic. Then there exist non-negative integers N and $l > 0$ such that $(a(N+tl))_{t=0}^{\infty}$ takes on only one value. This sequence is ultimately periodic. □

Next, we generalize the Thue-Morse sequence of Emmanuel [11].

Definition 7. Let L be an integer greater than 1, and let $a_0, a_1, \ldots, a_{L-1}$ be L distinct complex numbers. We let $\{a_0, a_1 \ldots a_{L-1}\}^*$ denote the free monoid generated by $\{a_0, a_1 \ldots a_{L-1}\}$. We define a morphism f from $\{a_0, a_1 \ldots a_{L-1}\}^*$ to $\{a_0, a_1 \ldots a_{L-1}\}^*$ as follows:

$$f(a_i) = a_{i+1}, \tag{10}$$

where the index i is computed modulo L. Let f^j be the j times composed mapping of f, and let f^0 be the identity mapping. Let A and B be two finite words on $\{a_0, a_1, \ldots, a_{L-1}\}$, and let AB denote the concatenation of A and B.

Let $A_0 = a_0$, k be an integer greater than 1, and let κ be a map $\kappa{:}\{1, \ldots, k-1\} \times \mathbb{N} \to \{0, \ldots, L-1\}$. For a non-negative integer m, we define a space of words W_m by

$$W_m := \{a_{i_1} a_{i_2} \ldots a_{i_m} \,| a_{i_1}, a_{i_2}, \cdots, a_{i_m} \in \{a_0, a_1, \ldots, a_{L-1}\}\}. \tag{11}$$

We define $A_{n+1} \in W_{k^{n+1}}$ recursively as

$$A_{n+1} := A_n f^{\kappa(1,n)}(A_n) \cdot \cdots \cdot f^{\kappa(k-1,n)}(A_n), \tag{12}$$

and we let

$$A_\infty := \lim_{n\to\infty} A_n \tag{13}$$

denote the limit of A_n. The sequence (or infinite word) A_∞ is called the *generalized Thue-Morse sequence of type* (L, k, κ), abbreviated as the (L, k, κ)-TM sequence.

Example 8 ([11]). Let $L = 2$, $a_0 = 0$, $a_1 = 1$ and $\kappa(1, y) = 1$ for all $y \in \mathbb{N}$. The $(2, 2, 1)$-TM sequence is as follows:

$$\begin{aligned} A_0 = 0,\ A_1 = 01,\ A_2 = 0110,\ A_3 = 01101001,\\ A_\infty = 0110100110010110100101100110100110010110011010010110100110\cdots. \end{aligned}$$

This example is the Thue-Morse sequence of Emmanuel [11].

Example 9. Let $L = 2$, $a_0 = 0$, $a_1 = 1$ and

$$\kappa(1, y) = \begin{cases} 1, & y \text{ is a prime number;} \\ 0, & \text{otherwise.} \end{cases}$$

The $(2, 2, \kappa)$-TM sequence is

$$\begin{aligned} &A_0 = 0,\ A_1 = 00,\ A_2 = 0000,\ A_3 = 00001111,\\ &A_\infty = 00001111000011111111000011110000\cdots. \end{aligned}$$

Example 10. Let $L = 2$, $a_0 = 0$, $a_1 = 1$ and

$$\kappa(1, y) = \begin{cases} 1, & y \text{ is a square number and } s = 2; \\ 0, & \text{otherwise} \end{cases}.$$

The $(2, 3, \kappa)$-TM sequence is

$$\begin{aligned} A_0 = 0,\ A_1 = 001,\ A_2 = 001001001,\\ A_\infty = 001001001001001001001001001001001001001001001001001001110110\cdots. \end{aligned}$$

Let $(a(n))_{n=0}^\infty$ be a sequence with values in $\mathbb{C}$. The *generating function* of $(a(n))_{n=0}^\infty$ is the formal power series $g(z) \in \mathbb{C}[[z]]$, defined as

$$g(z) := \sum_{n=0}^{\infty} a(n) z^n.$$

The following lemma clarifies the meaning of an (L, k, κ)-TM sequence.

Lemma 11. *Let $A_\infty = (b(n))_{n=0}^{\infty}$ be a (L,k,κ)-TM sequence with $a_j = \exp\frac{2\pi\sqrt{-1}j}{L}$ (for all j with $0 \leq\ j \leq L-1$). Let $G_{A_\infty}(z)$ be the generating function of $(b(n))_{n=0}^{\infty}$,*

$$G_{A_\infty}(z) := \sum_{n=0}^{\infty} b(n) z^n.$$

The generating function $G_{A_\infty}(z)$ will have the infinite product on $|z|<1$,

$$G_{A_\infty}(z) = \prod_{y=0}^{\infty}(1+\sum_{s=1}^{k-1}\exp\frac{2\pi\sqrt{-1}\kappa(s,y)}{L} z^{sk^y}). \tag{14}$$

Proof. From the assumption $a_j = \exp\frac{2\pi\sqrt{-1}j}{L}$ for all j with $0 \leq\ j \leq L-1$, we have

$$f(a_j) = \exp\frac{2\pi\sqrt{-1}}{L} a_j \tag{15}$$

for all j with $0 \leq\ j \leq L-1$. The (L,k,κ)-TM sequence takes on only finite values, and by the Cauchy-Hadamard theorem, $G_{A_\infty}(z)$ and $\prod_{y=0}^{\infty}(1+\sum_{s=1}^{k-1}\exp\frac{2\pi\sqrt{-1}\kappa(s,y)}{L} z^{sk^y})$ converge absolutely on the unit disk. Let $G_{A_n}(z)$ be the generating function of A_n; We identify the infinite word $A_n 0\cdots 0\cdots =: A_n 0^\infty$ with A_n.

We will show by induction that the following equality holds for n,

$$G_{A_n}(z) = \prod_{y=0}^{n-1}(1+\sum_{s=1}^{k-1}\exp\frac{2\pi\sqrt{-1}\kappa(s,y)}{L} z^{sk^y}). \tag{16}$$

First, we check the case $n=1$. From the definition of A_1, we have

$$G_{A_1}(z) = 1+\sum_{s=1}^{k-1}\exp\frac{2\pi\sqrt{-1}\kappa(s,0)}{L} z^{s}. \tag{17}$$

Thus, the $n=1$ case is true. By the induction hypothesis we may assume that

$$G_{A_j}(z) = \prod_{y=0}^{j-1}(1+\sum_{s=1}^{k-1}\exp\frac{2\pi\sqrt{-1}\kappa(s,y)}{L} z^{sk^y}). \tag{18}$$

Therefore, we have

$$G_{A_{j+1}}(z) = G_{A_j}(z) + \sum_{s=1}^{k-1} G_{f^{\kappa(s,j)}(A_j)}(z) z^{sk^j}. \tag{19}$$

Alternatively,

$$G_{f^{\kappa(s,j)}(A_j)}(z) = \exp\frac{2\pi\sqrt{-1}\kappa(s,j)}{L} G_{A_j}(z). \tag{20}$$

From (18)-(20), we get

$$
\begin{aligned}
G_{A_{j+1}}(z) &= G_{A_j}(z)(1+\sum_{s=1}^{k-1}\exp\frac{2\pi\sqrt{-1}\kappa(s,y)}{L}z^{sk^j}) \\
&= \prod_{y=0}^{j}(1+\sum_{s=1}^{k-1}\exp\frac{2\pi\sqrt{-1}\kappa(s,y)}{L}z^{sk^y}).
\end{aligned} \tag{21}
$$

Therefore (14) is true. Finally, we will compare the coefficients of z^j on both sides of (14). On the right-hand side of (14), the coefficient of z^j are determined by $G_{A_N}(z)$ for sufficiently large N. From the definition of A_∞, the prefix word, p^N, of A_∞ is A_N. From the above argument and (16), the coefficients of z^j on both sides of (14) must coincide. ☐

Proposition 12. *Let $A_\infty = (b(n))_{n=0}^{\infty}$ be a (L,k,κ)-TM sequence with $a_j = \exp\frac{2\pi\sqrt{-1}j}{L}$ (for all j with $0 \leq\ j \leq L-1$). Let $(a(n))_{n=0}^{\infty}$ be a sequence defined by (5). Then*

$$
\frac{L}{2\pi\sqrt{-1}}\log b(n) \equiv a(n) \pmod{L}. \tag{22}
$$

Proof. Let the k-adic expansion of n be as follows:

$$
n = \sum_{q=1}^{n(k)} s_{n,q}k^{w_n(q)}, \tag{23}
$$

where $1 \leq s_{n,q} \leq k-1$, $0 \leq w_n(q) < w_n(q+1)$. By uniqueness of the k-adic expansion and Lemma 11, we have

$$
\begin{aligned}
b(n) &= \textstyle\prod_{q=1}^{n(k)}\exp\frac{2\pi\sqrt{-1}\kappa(s_{n,q},w_n(q))}{L} \\
&= \exp\frac{2\pi\sqrt{-1}(\sum_{q=1}^{n(k)}\kappa(s_{n,q},w_n(q)))}{L} \\
&= \exp\frac{2\pi\sqrt{-1}(\sum_{q=1}^{n(k)}\kappa(s_{n,q},w_n(q)) \pmod{L})}{L}.
\end{aligned} \tag{24}
$$

By (23), (24) and the definition of $a(n)$, the equality (22) is obtained. ☐

Now we give other representations of Example 9 and Example 10 by using Proposition 12.

We begin with Example 9. Let the 2-adic expansion of non-negative integer n be

$$
n = \sum_{q=1}^{\text{finite}} 2^{w_n(q)}, \tag{25}
$$

where $0 \leq w_n(q) < w_n(q+1)$. We define the number $A(n)$ to be

$$
A(n) = \#\{\ w_n(q) \mid w_n(q) \text{ is a prime number }\},
$$

and we define $(a(n))_{n=0}^{\infty}$ as

$$a(n) = \begin{cases} 1, & A(n) \equiv 1 \pmod 2; \\ 0, & A(n) \equiv 0 \pmod 2, \end{cases} \tag{26}$$

e.g., $a(44) = a(2^2 + 2^3 + 2^5) = 1$, $a(12) = a(2^2 + 2^3) = 0$). The sequence $(a(n))_{n=0}^{\infty}$ is the generalized Thue-Morse sequence of type $(2, 2, \kappa)$ with

$$\kappa(1, y) = \begin{cases} 1, & y \text{ is a prime number;} \\ 0, & \text{otherwise.} \end{cases}$$

Next, we give another representation of Example 10. Let the 3-adic expansion of non-negative integer n be

$$n = \sum_{q=1}^{\text{finite}} s_{n,q} 3^{w_n(q)}, \tag{27}$$

where $1 \leq s_{n,q} \leq 2, 0 \leq w_n(q) < w_n(q+1)$. We define the number $B(n)$ as

$$B(n) = \#\{\ w_n(q) \mid w_n(q) \text{ is a square number and } s_{n,q} = 2\ \},$$

and we define $(a(n))_{n=0}^{\infty}$ as

$$a(n) = \begin{cases} 1, & B(n) \equiv 1 \pmod 2; \\ 0, & B(n) \equiv 0 \pmod 2, \end{cases} \tag{28}$$

e.g., $a(169) = a(1 + 2 \times 3 + 2 \times 3^4) = 0$, $a(7) = a(1 + 2 \times 3) = 1$). The sequence $(a(n))_{n=0}^{\infty}$ is the generalized Thue-Morse sequence of type $(2, 3, \kappa)$ with

$$\kappa(s, y) = \begin{cases} 1 & y \text{ is a square number and } s = 2; \\ 0 & \text{otherwise.} \end{cases}$$

3 Necessary-sufficient condition for the non-periodicity of a generalized Thue-Morse sequence

We begin by presenting the following key lemma about the k-adic expansion of non-negative integers.

Lemma 13. *If $k > 1$ and $l > 0$ be integers and t be a non-negative integer, then there exists an integer x such that*

$$xl = \sum_{q=1}^{\text{finite}} s_{xl,q} k^{w_{xl}(q)}, \tag{29}$$

where $s_{xl,1} = 1, w_{xl}(2) - w_{xl}(1) > t, w_{xl}(q+1) > w_{xl}(q) \geq 0.$

Furthermore, if t' *be other non-negative integer, then there exists an integer* X *such that*

$$Xl = \sum_{q=1}^{\text{finite}} s_{Xl,q} k^{w_X l(q)}, \tag{30}$$

where $s_{Xl,1} = 1, w_{Xl}(2) - w_{Xl}(1) > t', w_{Xl}(q+1) > w_{Xl}(q) \geq 0, w_{xl}(1) = w_{Xl}(1).$

Proof. Let us assume the factorization of k into prime factors is

$$k = \prod_{t=1}^{N} p_t{}^{y_t}, \tag{31}$$

where $p_1, p_2, \cdots p_N$ are N distinct prime numbers and y_t for p_t $(1 \leq t \leq N)$ are N positive integers. Let l be represented as

$$l = G \prod_{u=1}^{n} p_{t_u}{}^{x_u}, \tag{32}$$

where G and k are coprime, $p_{t_u} \in \{p_t | 1 \leq t \leq N\}$ and x_u are n positive integers. As G and k are coprime, there exist integers D and E such that

$$DG = 1 - k^{t+1}E. \tag{33}$$

We set

$$F = \max\{A | x_u = y_{t_u} A + H,\ 0 \leq H < y_{t_u},\ 1 \leq u \leq n\}.$$

From the definition of F, $k^{F+1} \prod_{u=1}^{n} p_{t_u}{}^{-x_u}$ is a non-negative integer. Thus we have

$$lD^2Gk^{F+1} \prod_{u=1}^{n} p_{t_u}{}^{-x_u} = k^{F+1}D^2G^2. \tag{34}$$

On the other hand, by (33) we have

$$D^2G^2 = 1 + k^{t+1}E(k^{t+1}E - 2). \tag{35}$$

Thus $E(k^{t+1}E-2)$ is a non-negative integer. If $E(k^{t+1}E-2) > 0$, it follows from the k-adic expansion of $E(k^{t+1}E-2)$ that $k^{F+1}D^2G^2$ satisfies the Lemma. If $E(k^{t+1}E-2) = 0$, then $G = 1$. The integer $k^{F+1}(1+k^{t+1})$ also satisfies the Lemma. As $F+1$ is independent of t, the second claim is trivial. □

Now we will show the everywhere non-periodic result by the previous lemma.

Proposition 14. *Let $A_\infty = (a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$, and let $G_{A_\infty}(z)$ denote the generating function of $(a(n))_{n=0}^{\infty}$. Assume that $G_{A_\infty}(z)$ has the following infinite product expansion for an integer k greater than 1 and $t_{s,y} \neq 0$ for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$,*

$$G_{A_\infty}(z) = \prod_{y=0}^{\infty}(1 + \sum_{s=1}^{k-1} t_{s,y} z^{sk^y}). \tag{36}$$

If there exists a periodic arithmetical subsequence of $(a(n))_{n=0}^{\infty}$, then $G_{A_\infty}(z)$ has the following infinite product expansion

$$G_{A_\infty}(z) = (\sum_{n=0}^{k^A-1} a(n)z^n) \prod_{y=0}^{\infty}(1 + \sum_{s=1}^{k-1} h^{sk^y} z^{sk^{A+y}}), \tag{37}$$

where A is a non-negative integer and h is a complex number.

Proof. Let n and m be two non-negative integers and their respective k-adic expansions are as follows:

$$n = \sum_{q}^{\text{finite}} s_{n,q} k^{w_n(q)}, \quad m = \sum_{p}^{\text{finite}} s_{m,p} k^{w_m(p)}, \tag{38}$$

where $1 \leq s_{n,q}, s_{m,p} \leq k-1$, $0 \leq w_n(q) < w_n(q+1)$, and $0 \leq w_m(p) < w_m(p+1)$. If $w_n(q) \neq w_n(p)$ for all pairs (q,p), then

$$a(n+m) = a(n)a(m) \tag{39}$$

by the assumption of $G_{A_\infty}(z)$ and the uniqueness of the k-adic expansion of non-negative integers. If $(a(n))_{n=0}^{\infty}$ has a periodic arithmetical subsequence, then by Corollary 6 $(a(n))_{n=0}^{\infty}$ is not everywhere non-periodic. Thus there exist two non-negative integers, N and $l > 0$, such that

$$a(N) = a(N+tl) \qquad (\forall t \in \mathbb{N}). \tag{40}$$

Let the k-adic expansion of N be

$$N = \sum_{q=1}^{N(k)} s_{N,q} k^{w_N(q)} \qquad \textit{where } 1 \leq s_{N,q} \leq k-1, 0 \leq w_N(q) < w_N(q+1). \tag{41}$$

By the assumption of $G_{A_\infty}(z)$ and (39), we have

$$a(N) = a(N + k^r tl) = a(N)a(k^r tl) \qquad (\forall r > w_N(N(k))). \tag{42}$$

$$a(N) \neq 0. \tag{43}$$

From (42) and (43), we get

$$a(k^r tl) = 1 \qquad (\forall r > w_N(N(k))). \tag{44}$$

By Lemma 13, there exists an integer x greater than zero such that

$$xl = \sum_{q=1}^{xl(k)} s_{xl,q} k^{w_{xl}(q)}, \tag{45}$$

where $s_{xl,1} = 1$ and $w_{xl}(2) - w_{xl}(1) > 1$.

Moreover, there exists an integer X greater than zero such that

$$Xl = \sum_{q=1}^{Xl(k)} s_{Xl_q} k^{w_{Xl}(q)}, \tag{46}$$

where $s_{Xl,1} = 1$, $w_{Xl}(2) - w_{Xl}(1) > w_{xl}(xl(k))$ and $w_{Xl}(1) = w_{xl}(1)$.

Let $xlk^{-w_{xl}(1)}$ and $Xlk^{-w_{xl}(1)}$ be replaced by xl and Xl, respectively. Let r be an integer greater than $w(N(k)) + w_{xl}(1)$ and s be an integer in $\{1, \ldots, k-1\}$.

By the definition of Xl and (39), we have

$$a(k^r sXl) = a(sk^r)a(k^r sXl - sk^r). \tag{47}$$

From (43), we get

$$1 = a(k^r xl), \tag{48}$$
$$1 = a(k^r sXl), \tag{49}$$
$$1 = a(k^r xl + k^r sXl). \tag{50}$$

By (39), (47)-(50) and the definitions of xl and Xl, we have

$$a(k^r)a(k^r xl - k^r) = 1, \tag{51}$$
$$a(lsk^r)a(sXlk^r - sk^r) = 1, \tag{52}$$
$$a(k^r(s+1))a(xlk^r - k^r)a(sXlk^r - sk^r) = 1. \tag{53}$$

From (51)-(53), we get

$$a(k^r(s+1)) = a(k^r)a(k^r s). \tag{54}$$

Put $h := a(k^{w(N(k))+w_{xl}(1)+1})$.

By (39), we have

$$a(sk^y) = t_{s,y}, \tag{55}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$.

By (43), (54), (55) and inductive computation, we get the relations

$$t_{s,w(N(k))+w_{xl}(1)+1+y} = h^{sk^y}, \tag{56}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$. From the assumption of $G_{A_\infty}(z)$, the proof is complete. □

Finally, we prove the main theorem in Section 3.

Theorem 15. *Let $A_\infty = (a(n))_{n=0}^{\infty}$ be an (L,k,κ)-TM sequence. The sequence $A_\infty = (a(n))_{n=0}^{\infty}$ is ultimately periodic if and only if there exists an integer A such that*

$$\kappa(s, A+y) \equiv \kappa(1, A)sk^y \pmod{L}, \tag{57}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$.

Moreover, if the (L,k,κ)-TM sequence is not ultimately periodic, then no arithmetical subsequence of (L,k,κ)-TM sequence is ultimately periodic.

Proof. We assume, without loss of generality, that $A_\infty = (a(n))_{n=0}^{\infty}$ is an (L,k,κ)-TM sequence with $a_j = \exp \frac{2\pi\sqrt{-1}j}{L}$ (for all $0 \leq j \leq L-1$). From this assumption and Lemma 11, $(a(n))_{n=0}^{\infty}$ satisfies the assumption of Proposition 14. Therefore, (57) is the necessary condition.

Now, we show the sufficient condition. Let $G_{A_\infty}(z)$ be the generating function of $(a(n))_{n=0}^{\infty}$. Notation is the same as for Proposition 14. If we assume that $(a(n))_{n=0}^{\infty}$ satisfies (57), then there exists a non-negative integer A such that

$$t_{s,A+y} = h^{sk^y} \qquad (\forall y \in \mathbb{N}). \tag{58}$$

Thus $G_{A_\infty}(z)$ has the infinite product expansion

$$G_{A_\infty}(z) = (\sum_{n=0}^{k^A-1} b(n)z^n)\prod_{y=0}^{\infty}(1+\sum_{s=1}^{k-1}(hz^{k^A})^{sk^y}). \tag{59}$$

Let $Z = hz^{k^A}$. As h is the L-th root of 1 and κ is a zero map in Lemma 11, we find

$$\prod_{y=0}^{\infty}(1+\sum_{s=1}^{k-1}Z^{sk^y}) = \sum_{n=0}^{\infty}Z^n \qquad \text{on } |Z|<1. \tag{60}$$

We put $G(z) = \sum_{n=0}^{k^A-1} a(n)z^n$. From (59) and (60),

$$G_{A_\infty}(z) = G(z)(\sum_{n=0}^{\infty}(hz^{k^A})^n). \tag{61}$$

As h is the L-th root of 1,

$$G_{A_\infty}(z) = (G(z)(\sum_{n=0}^{L-1}(hz^{k^A})^n))(1+\sum_{s=1}^{\infty} z^{sLk^A}) = \frac{G(z)(\sum_{n=0}^{L-1}(hz^{k^A})^n)}{1-z^{Lk^A}}. \tag{62}$$

As the degree of $G(z)$ is k^A-1, and using (62), we find that the sequence $(a(n))_{n=0}^{\infty}$ that satisfies (57) has a period Lk^A. Moreover, if the (L,k,κ)-TM sequence is not ultimately periodic, then no arithmetical sequence of (L,k,κ)-TM sequence is ultimately periodic by the above argument and by Proposition 14. □

If an (L,k,κ)-TM sequence satisfies

$$\kappa(s,y) = \kappa(s,y+1) \tag{63}$$

for all s with $1 \le s \le k-1$ and for all $y \in \mathbb{N}$, then $(\kappa(1),\kappa(2),\ldots,\kappa(k-1))$-$L$ will denote the (L,k,κ)-TM sequence.

The weak version of the corollary that follows is given as Theorem 2 in Morton and Mourant [14]. See also Allouche and Shallit [5], Frid [13].

Corollary 16. *The sequence $(\kappa(1),\kappa(2),\ldots,\kappa(k-1))$-$L$ is periodic if and only if $\kappa(s)$ (for all s with $1 \le s \le k-1$) satisfies*

$$s\kappa(1) \equiv \kappa(s), \kappa(k-1) \equiv 0 \pmod{L}. \tag{64}$$

Moreover, if $(\kappa(1),\kappa(2),\ldots,\kappa(k-1))$-$L$ is not periodic, then no arithmetical subsequence of $(\kappa(1),\kappa(2),\ldots,\kappa(k-1)) - L$ is periodic.

Proof. By Theorem 15, the necessary-sufficient condition for the periodicity of $(\kappa(1),\kappa(2),\ldots,\kappa(k-1))$-$L$ comprises the following relations:

$$\kappa(1,A+1) \equiv \kappa(1,A)k \pmod{L}, \kappa(k-1) \equiv (k-1)\kappa(1) \equiv 0 \pmod{L}. \tag{65}$$

. □

4 Transcendence results of the generalized Thue-Morse sequences

Adamczewski, Bugeaud, and Luca [2] introduced a new class of sequences, as follows. For any positive number y, $\lfloor y \rfloor$ and $\lceil y \rceil$ are the floor and ceiling functions. Let W be a finite word on $\{a_0, a_1, \ldots, a_{L-1}\}$ and let $|W|$ be the length of W. For any positive number x, we let W^x defined the word $W^{\lfloor x \rfloor}W^{‘}$, where $W^{‘}$ is a prefix of W of length $\lceil (x-\lfloor x \rfloor)|W| \rceil$.

Definition 17. $(a(n))_{n=0}^{\infty}$ is called a *stammering sequence* if $(a(n))_{n=0}^{\infty}$ satisfies the following conditions:

(1) The sequence $(a(n))_{n=0}^{\infty}$ is a non-periodic sequence.

(2) There exist two sequences of finite words, $(U_m)_{m\geq 1}$ and $(V_m)_{m\geq 1}$, such that,

(A) there exists a real number $w > 1$ independent of n such that the word $U_m {V_m}^w$ is a prefix of the word $(a(n))_{n=0}^{\infty}$,

(B) $\lim_{m\to\infty} |U_m|/|V_m| < +\infty$, and

(C) $\lim_{m\to\infty} |V_m| = +\infty$.

Let $(a(n))_{n=0}^{\infty}$ be a sequence of positive integers. We define the *continued fraction* of $(a(n))_{n=0}^{\infty}$ as

$$[0, a(0), a(1), \ldots, a(n), \ldots] := \cfrac{1}{a(0) + \cfrac{1}{a(1) + \cfrac{1}{\cdots + \cfrac{1}{a(n) + \cfrac{1}{\ddots}}}}}. \tag{66}$$

Adamczewski, Bugeaud, Luca [2] and Bugeaud [9] proved the result that follows by the Schmidt subspace theorem.

Theorem 18 ([2, 9])**.** *If β is an integer greater than 1 and $(a(n))_{n=0}^{\infty}$ is a stammering sequence on $\{0, 1, \ldots, \beta-1\}$, then $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number. Moreover, if $(a(n))_{n=0}^{\infty}$ is a stammering sequence on bounded positive integers, then the continued fraction $[0, a(0), a(1) \ldots, a(n) \ldots]$ is also a transcendental number.*

We will prove the next theorem using Theorems and 2, 15 and 18.

Theorem 19. *Let $A_\infty = (a(n))_{n=0}^{\infty}$ be an (L, k, κ)-TM sequence and β be an integer greater than 1. We assume that $(a(n))_{n=0}^{\infty}$ takes its input from $\{0, 1, \ldots, \beta-1\}$. If there is no integer A such that*

$$\kappa(s, A + y) \equiv \kappa(1, A)sk^y \pmod{L} \tag{67}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is a transcendental number.

Moreover, if we assume that $(a(n))_{n=0}^{\infty}$ takes its input from the positive integers, and if there is no integer A such that

$$\kappa(s, A + y) \equiv \kappa(1, A)sk^y \pmod{L} \tag{68}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $[0, a(N), a(N+s), \ldots, a(N+nl) \ldots]$ (for all $N \geq 0$ and for all $l > 0$) is a transcendental number.

Proof. Let N and $l > 0$ be positive integers. By Theorem 15, $(a(N+nl))_{n=0}^{\infty}$ is non-periodic. Therefore, we only have to prove that (L, k, κ)-TM satisfies the condition (2) of Definition 17.

We choose an integer M such that $k^M > 2(N+l)$, and assume that $m > M$. As f is a cyclic permutation of order L and by Definition 7, the $(Ll+1)k^m$ prefix word of $(a(n))_{n=0}^{\infty}$ is as follows

$$A_\infty = (a(n))_{n=0}^{\infty} = A_m f^{i_1}(A_m) \cdots f^{i_{Ll}}(A_m) \cdots , \tag{69}$$

where A_m is the k^m prefix word of $(a(n))_{n=0}^{\infty}$, $i_j (1 \leq j \leq Ll) \in \{0, \ldots, L-1\}$.

By (69), we have

$$f^{i_{tl}}(a(n)) = a(n + k^m tl) \tag{70}$$

for all $0 \leq n \leq k^m - 1$ and for all $1 \leq t \leq L$.

As f is a cyclic permutation of order L, by (69), (70) and the Dirichlet schubfachprinzip, we have

$$(a(N+nl))_{n=0}^{\infty} = W_{1,m} W_{2,m} W_{3,m} W_{2,m} \cdots , \tag{71}$$

where $W_{i,m}$ $(i \in \{1,2,3\})$ are finite words such that

$$|W_{1,m}| \leq ((Ll+1)k^m - N)/l + 1, \tag{72}$$

$$|W_{2,m}| \geq (k^m - N)/l - 1, \tag{73}$$

$$|W_{2,m}| + |W_{3,m}| \leq ((Ll+1)k^m - N)/l + 1. \tag{74}$$

We put $U_m := W_{1,m}$, $V_m := W_{2,m}W_{3,m}$ and $w := 1 + \frac{1}{2Ll+3}$.

By (72)-(74) and the assumption of m, we obtain

$$\begin{aligned} &\lceil (w-1)|V_m| \rceil = \lceil \tfrac{1}{2Ll+3}(|W_{2,m}| + |W_{3,m}|) \rceil \leq \\ &\tfrac{1}{2Ll+3}((Ll+1)k^m - N + l)/l \leq \tfrac{k^m}{2l} \ < |W_{2,m}|. \end{aligned} \tag{75}$$

From (75), $(a(n))_{n=0}^{\infty}$ satisfies Condition (A).

Furthermore,

$$\begin{aligned} |U_m|/|V_m| = |W_{1,m}|/|W_{2,m}W_{3,m}| \leq \\ ((Ll+1)k^m - N + l)/l \times l/(k^m - N - l) \leq 2Ll + 3. \end{aligned} \tag{76}$$

From (76), $(a(n))_{n=0}^{\infty}$ satisfies Condition (B).

It follows directly that $(V_m)_{m \geq 1}$ satisfies Condition (C). $\square$

Corollary 20. *Let $(a(n))_{n=0}^{\infty}$ be an (L, k, κ)-TM sequence and β be an integer greater than 1. If $(a(n))_{n=0}^{\infty}$ takes its input from $\{0, 1, \ldots, \beta - 1\}$ and there is no integer A such that*

$$\kappa(s, A + y) \equiv \kappa(1, A) s k^y \pmod{L} \tag{77}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then the generating function $f(z) := \sum_{n=0}^{\infty} \frac{a(N+nl)}{z^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is transcendental over $\mathbb{C}(z)$.

Proof. We assume $f(z)$ is algebraic over $\mathbb{C}(z)$. As $f(z)$ is algebraic over $\mathbb{Q}(z)$ if and only if $f(z)$ is algebraic over $\mathbb{C}(z)$ (see the Remark in Theorem 1.2 in Nishioka [16]), then $f(z)$ satisfies the equation

$$c_n(z)f^n(z) + c_{n-1}(z)f^{n-1}(z) + \cdots + c_0(z) = 0, \tag{78}$$

where $c_i(z) \in \mathbb{Q}[z]$ $(0 \leq i \leq n)$, $c_n(z)c_0(z) \neq 0$ and $c_i(z)$ $(0 \leq i \leq n)$ are coprime. From Theorem 19, $f(\frac{1}{\beta})$ is a transcendental number. From the above argument and by (78), $c_i(\frac{1}{\beta}) = 0$ (for all $0 \leq i \leq n$). This contradicts the assumption that $c_i(z)$ $(0 \leq i \leq n)$ are coprime. $\square$

5 k-Automatic generalized Thue-Morse sequences and some results

First, we introduce some definitions.

Definition 21. Let α be an irrational real number. The irrationality exponent $\mu(\alpha)$ of α is the supremum of the real numbers μ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\mu} \tag{79}$$

has infinitely many solutions in non-zero integers p and q.

Definition 22. The *k-kernel* of $(a(n))_{n=0}^\infty$ is the set of all subsequences of the form $(a(k^e n + j))_{n=0}^\infty$, where $e \geq 0$ and $0 \leq j \leq k^e - 1$.

Definition 23. The sequence$(a(n))_{n=0}^\infty$ is called a *k-automatic sequence* if the k-kernel of $(a(n))_{n=0}^\infty$ is the finite set.

Definition 24. The power series $\sum_{n=0}^\infty a(n)z^n \in \mathbb{C}[[x]]$ is called a *k-automatic power series* if $(a(n))_{n=0}^\infty$ is a k-automatic sequence.

Definition 25. An (L, k, κ)-TM sequence is called *y-periodic* if there exist non-negative integers N and $t(0 < t)$ such that

$$\kappa(s, y) = \kappa(s, y + t), \tag{80}$$

for all s with $1 \leq s \leq k-1$ and for all $y \geq N$.

Now we introduce two results.

Theorem 26 ([3])**.** *If β is an integer greater than 1 and $(a(n))_{n=0}^\infty$ is a non-periodic k-automatic sequence on $\{0, 1, \ldots, \beta - 1\}$, then $\mu(\sum_{n=0}^\infty \frac{a(n)}{\beta^{n+1}})$ is finite.*

Theorem 27 ([7]). *If $f(z) \in \mathbb{Q}[[z]] \setminus \mathbb{Q}(z)$ is a k-automatic power series and $0 < R < 1$, then $f(\alpha)$ is transcendental for all but finitely many algebraic numbers α with $|\alpha| \leq R$.*

Now we consider the necessary-sufficient condition that an (L, k, κ)-TM sequence is a k-automatic sequence.

Proposition 28. *An (L, k, κ)-TM sequence is y-periodic if and only if it is a k-automatic sequence.*

Proof. We assume, without loss of generality, that $A_\infty = (a(n))_{n=0}^{\infty}$ is an (L, k, κ)-TM sequence with $a_j = \exp \frac{2\pi\sqrt{-1}j}{L}$ (for all $0 \leq\ j \leq L-1$).

Let us assume that $(a(n))_{n=0}^{\infty}$ is a k-automatic sequence. As the k-kernel of $(a(n))_{n=0}^{\infty}$ is a finite set, there exist integers e for $0 < t$ such that

$$a(k^e n) = a(k^{e+t} n) \qquad (\forall n \geq 0). \tag{81}$$

Let s be any integer in $\{1, 2, \ldots, k-1\}$, and let y be any integer in $\mathbb{N}$. By Lemma 11 with (39) and (81), and substituting sk^y for n, we have

$$\exp \frac{2\pi\sqrt{-1}\kappa(s, e+y)}{L} = a(k^e sk^y) = a(k^{e+t} sk^y) = \exp \frac{2\pi\sqrt{-1}\kappa(s, e+y+t)}{L}. \tag{82}$$

By the definition of the (L, k, κ)-TM sequence and (82), $(a(n))_{n=0}^{\infty}$ is y-periodic.

Now we show the converse. If an (L, k, κ)-TM sequence $A_\infty = (a(n))_{n=0}^{\infty}$ is y-periodic, then there exist non-negative integers e for $0 < t$ such that

$$\kappa(s, e+y) = \kappa(s, e+y+t), \tag{83}$$

for all y being any integer in $\mathbb{N}$ and for all s with $1 \leq s \leq k-1$. Let l be any integer greater than $t-1$ and let $(a(k^{e+l}n + j))_{n=0}^{\infty}$ (where $0 \leq j \leq k^{e+l} - 1$) be any sequence in the k-kernel of $(a(n))_{n=0}^{\infty}$.

Therefore, from Lemma 11 with (39), we get

$$a(k^{e+l}n + j) = a(k^{e+l}n)a(j). \tag{84}$$

As $(a(n))_{n=0}^{\infty}$ takes on only finitely many values, then $a(j)$ also takes on only finitely many values.

Let the k-adic expansion of n be

$$n = \sum_{q=1}^{N(n)} s_{n,q} k^{w(j)} \qquad \textit{where } 1 \leq s_{n,q} \leq k-1, w(q+1) > w(q) \geq 0. \tag{85}$$

Let $l(t) \equiv l \pmod{t}$, where $0 \leq l(t) \leq t-1$. By Lemma 11 with (39), we have

$$a(k^{e+l}n) = a(\sum_{q=1}^{N(n)} s_q k^{w(q)+e+l}) = \prod_{q=1}^{N(n)} a(s_q k^{w(q)+e+l}). \tag{86}$$

From (85), (86), and Lemma 11 with (39), we get

$$a(k^{e+l}n) = \prod_{q=1}^{N(n)} a(s_q k^{w(q)+e+l}) = \prod_{q=1}^{N(n)} a(s_q k^{w(q)+e+l(t)})$$
$$= a(\sum_{q=1}^{N(n)} s_q k^{w(q)+e+l(t)}) = a(k^{e+l(t)}n). \tag{87}$$

As $a(j)$ takes on only finitely many values, and by (84) and (87), it follows that the k-kernel of $(a(n))_{n=0}^{\infty}$ is a finite set. □

Theorem 29. *Let $(a(n))_{n=0}^{\infty}$ be an (L, k, κ)-TM and β be an integer greater than 1. If $(a(n))_{n=0}^{\infty}$ takes on the values $\{0, 1, \ldots, \beta - 1\}$, is y-periodic and there is no integer A such that*

$$\kappa(s, A + y) \equiv \kappa(1, A)sk^y \pmod L \tag{88}$$

for all s with $1 \leq s \leq k - 1$ and for all $y \in \mathbb{N}$, then $\mu(\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}})$ (for all $N \geq 0$ and for all $l > 0$) is finite.

Proof. By the previous proposition, $(a(n))_{n=0}^{\infty}$ is a k-automatic sequence. As the arithmetical subsequence of a k-automatic sequence is k-automatic, see Theorem 2.3 and Theorem 2.6 in Allouche and Shallit [4], and by Theorems 19 and 26, $\mu(\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}})$ is finite. □

Theorem 30. *Let $(a(n))_{n=0}^{\infty}$ be an (L, k, κ)-TM, β be an integer greater than 1, $f(z) := \sum_{n=0}^{\infty} \frac{a(N+nl)}{z^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$), and $0 < R < 1$. If $(a(n))_{n=0}^{\infty}$ takes on the values $\{0, 1, \ldots, \beta - 1\}$, is y-periodic and there is no integer A such that*

$$\kappa(s, A + y) \equiv \kappa(1, A)sk^y \pmod L \tag{89}$$

for all s with $1 \leq s \leq k - 1$ and for all $y \in \mathbb{N}$, then $f(\alpha)$ is a transcendental number for all but finitely many algebraic numbers α with $|\alpha| \leq R$.

Proof. By Corollary 20, $f(z)$ is transcendental over $\mathbb{Q}(z)$. From Proposition 28, $(a(N + nl))_{n=0}^{\infty}$ (for all $N \geq 0$ and for all $l > 0$) is a k-automatic sequence. Therefore, $f(z)$ is a k-automatic power series. Theorem 27 implies that $f(\alpha)$ is transcendental for all but finitely many algebraic numbers α with $|\alpha| \leq R$. □

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