Journal of Integer Sequences, Vol. 18 (2015),
Article 15.6.2

Some Arithmetic Properties of Certain Sequences

E. L. Roettger
Department of General Education
Mount Royal University
4825 Mount Royal Gate SW
Calgary, AB T3E 6K6
Canada
eroettger@mtroyal.ca

H. C. Williams[1]
Department of Mathematics and Statistics
University of Calgary
2500 University Drive NW
Calgary, AB T2N 1N4
Canada
williams@math.ucalgary.ca

Abstract

In an earlier paper it was argued that two sequences, denoted by $\{U_n\}$ and $\{W_n\}$, constitute the sextic analogues of the well-known Lucas sequences $\{u_n\}$ and $\{v_n\}$. While a number of the properties of $\{U_n\}$ and $\{W_n\}$ were presented, several arithmetic properties of these sequences were only mentioned in passing. In this paper we discuss the derived sequences $\{D_n\}$ and $\{E_n\}$, where $D_n = \gcd(W_n - 6R^n, U_n)$ and $E_n = \gcd(W_n, U_n)$, in greater detail and show that they possess many number theoretic properties analogous to those of $\{u_n\}$ and $\{v_n\}$, respectively.

[1]The second author is supported by NSERC of Canada.

1 Introduction

Let p, $q \in \mathbb{Z}$ be relatively prime and α, β be the zeros of

$$x^2 - px + q$$

with discriminant $\delta = (\alpha - \beta)^2 = p^2 - 4q$. The well-known Lucas sequences $\{u_n\}$ and $\{v_n\}$ are defined by

$$u_n = u_n(p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = v_n(p,q) = \alpha^n + \beta^n.$$

These sequences possess many interesting properties and have found applications in primality testing, integer factorization, solution of quadratic and cubic congruences, and cryptography (see [4]). We note here that both sequences are linear recurrence sequences of order 2 and that u_n, $v_n \in \mathbb{Z}$ whenever $n \geq 0$.

Lucas' problem of extending or generalizing his sequences has been well studied and we refer the reader to [2, Chapter 1] and [3, Section 1] for further information on this topic. One possible extension of the Lucas sequences, which involves cubic instead of quadratic irrationalities, was investigated in [2] (also see Müller, Roettger and Williams [1]). In this case we let P, Q, $R \in \mathbb{Z}$ be integers such that $\gcd(P, Q, R) = 1$ and let α, β, γ be the zeros of

$$h(x) = x^3 - Px^2 + Qx - R, \tag{1}$$

with discriminant

$$\Delta = (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = Q^2P^2 - 4Q^3 - 4RP^3 + 18PQR - 27R^2 \neq 0.$$

Roettger's sequences $\{c_n\}$ and $\{w_n\}$ are defined as

$$c_n = c_n(P,Q,R) = (\alpha^n - \beta^n)(\beta^n - \gamma^n)(\gamma^n - \alpha^n)/((\alpha - \beta)(\beta - \gamma)(\gamma - \alpha))$$

and

$$w_n = w_n(P,Q,R) = (\alpha^n + \beta^n)(\beta^n + \gamma^n)(\gamma^n + \alpha^n) - 2R^n.$$

Note here that if $n \geq 0$, we have c_n, $w_n \in \mathbb{Z}$ and $\{c_n\}$, $\{w_n\}$ are linear recurrence sequences of order 6.

In [2], it is pointed out that the sequences $\{c_n\}$ and $\{w_n\}$ have many properties analogous to those of $\{u_n\}$ and $\{v_n\}$, respectively. Recently, these sequences were extended further by Roettger, Williams and Guy [3]. If we put $\gamma_1 = \alpha/\beta$, $\gamma_2 = \beta/\gamma$, $\gamma_3 = \gamma/\alpha$, $\lambda = R$, then we can write

$$\begin{aligned} c_n &= \lambda^{n-1}(1 - \gamma_1^n)(1 - \gamma_2^n)(1 - \gamma_3^n)/((1 - \gamma_1)(1 - \gamma_2)(1 - \gamma_3)) \quad \text{and} \\ w_n &= v_n - 2R^n, \quad \text{where} \\ v_n &= \lambda^n(1 + \gamma_1^n)(1 + \gamma_2^n)(1 + \gamma_3^n). \end{aligned}$$

One of the most important properties of the Lucas sequence $\{u_n\}$ when $n \geq 0$ is that it is a divisibility sequence. An integer sequence $\{A_n\}$ is said to be a *divisibility sequence* if $A_n \mid A_m$ whenever $n \mid m$ and $A_n \neq 0$. For example, Roettger's sequence $\{c_n\}$ ($n \geq 0$) is a divisibility sequence. Suppose we define

$$U_n = \frac{\lambda^{n-1}(1-\gamma_1^n)(1-\gamma_2^n)(1-\gamma_3^n)}{(1-\gamma_1)(1-\gamma_2)(1-\gamma_3)}, \tag{2}$$

$$V_n = \lambda^n(1+\gamma_1^n)(1+\gamma_2^n)(1+\gamma_3^n), \tag{3}$$

where λ, γ_1, γ_2, $\gamma_3 \in \bar{\mathbb{Q}}$; γ_1, γ_2, $\gamma_3 \neq 1$; $\gamma_i \neq \gamma_j$ when $i \neq j$ and $\gamma_1\gamma_2\gamma_3 = 1$. In [3], it is shown that if U_n, $V_n \in \mathbb{Z}$ whenever $n \geq 0$, $\{U_n\}$ is a linear recurrence sequence and $\{U_n\}$ is also a divisibility sequence, then we must have $\lambda = R \in \mathbb{Z}$ and $\rho_i = R(\gamma_i + 1/\gamma_i)$ ($i = 1, 2, 3$) must be the zeros of a cubic polynomial

$$g(x) = x^3 - S_1x^2 + S_2x - S_3, \tag{4}$$

where

$$S_3 = RS_1^2 - 2RS_2 - 4R^3 \tag{5}$$

and S_1, $S_2 \in \mathbb{Z}$. The six zeros of

$$\begin{aligned} G(x) &= (x^2 - \rho_1x + R^2)(x^2 - \rho_2x + R^2)(x^2 - \rho_3x + R^2) \\ &= x^6 - S_1x^5 + (S_2 + 3R^2)x^4 - (S_3 + 2R^2S_1)x^3 + R^2(S_2 + 3R^2)x^2 - R^4S_1x + R^6 \end{aligned}$$

are $R\gamma_i$, R/γ_i ($i = 1, 2, 3$). If we define $W_n = V_n - 2R^n$, then both $\{U_n\}$ and $\{W_n\}$ are linear recurrence sequences with characteristic polynomial $G(x)$. Also, $U_0 = 0$, $U_1 = 1$, $U_2 = S_1 + 2R$, $U_3 = S_1^2 + RS_1 - S_2 - 3R^2$, $W_0 = 6$, $W_1 = S_1$, $W_2 = S_1^2 - 2S_2 - 6R^2$, $W_3 = S_1^3 - 3S_1S_2 + 3RS_1^2 - 6RS_2 - 3R^2S_1 - 12R^3$. Furthermore, we have $U_{-n} = -U_n/R^{2n}$, $W_{-n} = W_n/R^{2n}$; hence, U_n, $W_n \in \mathbb{Z}$ when $n \geq 0$. It is also the case that $\{U_n\}$ is a divisibility sequence.

It is shown in [3] that if

$$S_1 = PQ - 3R, \qquad S_2 = P^3R + Q^3 - 5PQR + 3R^2, \tag{6}$$

then $U_n(S_1, S_2, R) = c_n(P, Q, R)$, $W_n(S_1, S_2, R) = w_n(P, Q, R)$. If, in the expression (2), we define

$$\begin{aligned} \Delta &= \lambda^2(1-\gamma_1)^2(1-\gamma_2)^2(1-\gamma_3)^2 \\ &= R^2(\gamma_1 + \gamma_2 + \gamma_3 - 1/\gamma_1 - 1/\gamma_2 - 1/\gamma_3)^2, \end{aligned} \tag{7}$$

we find that

$$\Delta = S_1^2 - 4S_2 + 4RS_1 - 12R^2, \tag{8}$$

but this is the same as $Q^2P^2 - 4Q^3 - 4RP^3 + 18PQR - 27R^2$, the discriminant of $h(x)$, when S_1 and S_2 are given by (6). If d denotes the discriminant of $g(x)$, then, as shown in [3], we have $d = \Delta\Gamma$, where

$$\begin{aligned} \Gamma &= R^4(\gamma_1 - \gamma_2)^2(\gamma_2 - \gamma_3)^2(\gamma_3 - \gamma_1)^2 &(9)\\ &= S_2^2 + 10RS_1S_2 - 4RS_1^3 - 11R^2S_1^2 + 12R^3S_1 + 24R^2S_2 + 36R^4. &(10)\end{aligned}$$

The discriminant D of $G(x)$ is given by $D = Ed^2R^{12}$, where

$$E = R^2\Delta(S_1 + 2R)^2 = (\rho_1 - 4R^2)(\rho_2 - 4R^2)(\rho_3 - 4R^2).$$

If S_1 and S_2 are given by (6), then

$$\Gamma = (RP^3 - Q^3)^2. \tag{11}$$

We remark that the condition analogous to $\gcd(P, Q, R) = 1$ for Roettger's sequences is $\gcd(S_1, S_2, R) = 1$ for the more general $\{W_n\}$ and $\{U_n\}$ sequences.

The duplication formulas are

$$2W_{2n} = W_n^2 + \Delta U_n^2 - 4R^nW_n, \qquad U_{2n} = U_n(W_n + 2R^n) \tag{12}$$

and the triplication formulas are

$$4W_{3n} = 3\Delta U_n^2(W_n + 2R^n) + W_n^2(W_n - 6R^n) + 24R^{2n}, \tag{13}$$

$$4U_{3n} = U_n(3W_n^2 + \Delta U_n^2). \tag{14}$$

Since $\{U_n\}$ is a divisibility sequence, we must have $U_{3n}/U_n \in \mathbb{Z}$ $(n \geq 0)$ and by (14), this means that $4 \mid W_n^2 - \Delta U_n^2$. Thus, if $2 \mid U_n$, then $2 \mid W_n$ and we have proved Proposition 1.

Proposition 1. *If $n \geq 0$, then $2 \mid \gcd(W_n, U_n)$ if and only if $2 \mid U_n$.*

The general multiplication formulas for $\{W_n\}$ and $\{U_n\}$ are given as [3, (7.7) and (7.8)].

We observe here that in general for a given S_1, S_2 $R \in \mathbb{Z}$ there do not always exist, P, $Q \in \mathbb{Z}$ such that (6) holds. As a simple example consider $S_1 = -1$, $S_2 = -4$, and $R = 1$; it is not possible to find integers P, Q such that $PQ = 2$ and $P^3 + Q^3 = 3$. Thus, the sequences $\{W_n(S_1, S_2, R)\}$, $\{U_n(S_1, S_2, R)\}$ represent a non-trivial extension of Roettger's sequences $\{w_n\}$ and $\{c_n\}$.

In [3] it is mentioned that if we define

$$D_n = \gcd(W_n - 6R^n, U_n) \quad \text{and} \quad E_n = \gcd(W_n, U_n),$$

then the sequences $\{D_n\}$ and $\{E_n\}$ possess many number theoretic properties in common with $\{u_n\}$ and $\{v_n\}$, respectively. Indeed, some of these properties were presented in [3] without proof. The purpose of this paper is to supply these proofs or sketches thereof and to develop some new results concerning $\{D_n\}$ and $\{E_n\}$.

2 Some properties of $\{D_n\}$

In this section we will produce some results concerning $\{D_n\}$ that are similar to those possessed by $\{u_n\}$. We begin with two simple propositions that easily follow from Lemma 8.1 of [3] and results immediately following that lemma.

Proposition 2. *If* $\gcd(S_1, S_2, R) = 1$*, then for* $n \geq 0$ *we have*

$$\gcd(D_n, R) \mid 2.$$

Proposition 3. *If* $\gcd(S_1, S_2, R) = 1$*, then for any* $n \geq 0$*, we must have* $4 \nmid D_n$ *whenever* $2 \mid R$*.*

In the sequel we will assume that S_1, S_2, R have been selected such that $\gcd(S_1, S_2, R) = 1$.

If we define

$$F_n = \begin{cases} \Delta U_n^2, & \text{when } 2 \nmid \Delta U_n; \\ \Delta U_n^2/4, & \text{when } 2 \mid \Delta U_n \end{cases}$$

we see that since $4 \mid W_n^2 - \Delta U_n^2$, F_n must be an integer. If M is any divisor of F_n and $(M, R) = 1$, then we can use [3, (7.7) and (7.8)] to show that

$$U_{mn}/U_n \equiv R^{n(m-1)} K_m(W_n/2R^n) \pmod{M}, \tag{15}$$

$$W_{mn} \equiv 2R^{mn} L_m(W_n/2R^n) \pmod{M}, \tag{16}$$

where the polynomials $K_m(x)$ and $L_m(x)$ are respectively defined in [2, §4.3 and §5.1]. Also, from results in [2] it is easy to show that $L_m(3) = 3$ and $K_m(3) = m^3$. We next establish that like $\{u_n\}$, $\{D_n\}$ is a divisibility sequence.

Theorem 4. *If* n*,* $m \geq 1$*, then* $D_n \mid D_{mn}$*.*

Proof. Since $\{U_n\}$ is a divisibility sequence it suffices to show
$D_n \mid W_{mn} - 6R^{mn}$. We let $2^\lambda \parallel D_n$. If $\lambda = 0$ or $\lambda \geq 1$ and $2 \nmid R$, then $D_n \mid F_n$. By Proposition 2, we have $\gcd(D_n, R) = 1$ and by (16) we get

$$W_{mn} \equiv 2R^{mn} L_m(W_n/2R^n) \equiv 2R^{mn} L_m(3) \equiv 6R^{mn} \pmod{D_n}.$$

If $\lambda = 1$, then $\gcd(D_n/2, R) = 1$ and $D_n/2 \mid F_n$; hence,

$$W_{mn} \equiv 6R^{mn} \pmod{D_n/2}.$$

Also, since $2 \mid U_n$, we have $2 \mid U_{mn}$ and $2 \mid W_{mn}$ (Proposition 1). It follows that $W_{mn} \equiv 6R^{mn}$ $\pmod 2$ and since $\gcd(2, D_n/2) = 1$ we get

$$W_{mn} \equiv 6R^{mn} \pmod{D_n}.$$

There remains the case of $\lambda > 1$ and $2 \mid R$, but this is impossible by Proposition 3. ∎

Let p be any prime. We are next able to present the *law of repetition* for p in $\{D_n\}$. We denote by $\nu_p(x)$ $(x \in \mathbb{Z})$ that value of λ such that $p^\lambda \mid\mid x$.

Theorem 5. *Let p be any prime such that $p > 3$ and suppose that $\nu_p(D_n) \geq 1$.*

1. *If $\nu_p(U_n) > \nu_p(W_n - 6R^n)$, then $\nu_p(D_{pn}) = \nu_p(D_n) + 2$ and $\nu_p(W_{pn} - 6R^{pn}) < \nu_p(U_{pn})$.*
2. *If $\nu_p(U_n) = \nu_p(W_n - 6R^n)$ and $\nu_p(U_n) > 1$, then $\nu_p(D_{pn}) = \nu_p(D_n) + 2$ and $\nu_p(W_{pn} - 6R^{pn}) < \nu_p(U_{pn})$.*
3. *If $\nu_p(U_n) < \nu_p(W_n - 6R^n)$, then if $\nu_p(U_n) > 1$, $\nu_p(D_{pn}) = \nu_p(D_n) + 3$.*
4. *If $\lambda = 1$, then $\nu_p(D_{pn}) \geq 2$.*

Proof. These results can be established by making use of the techniques of [2, §5.2], together with the polynomial congruence

$$\begin{aligned} L_p(x) \equiv 3 + p^2(x-3) + (p^2(p^2-1)/12)(x-3)^2 \\ + (p^2(p^2-1)(p^2-4)/360)(x-3)^3 \pmod{(x-3)^4}, \end{aligned}$$

which holds for all primes $p \geq 5$. □

When $p = 3$, the law of repetition for 3 in $\{D_n\}$ is given below.

Theorem 6. *Let $\nu_3(D_n) \geq 1$.*

1. *If $\nu_3(U_n) \geq \nu_3(W_n - 6R^n) > 1$, then $\nu_3(D_{3n}) = \nu_3(D_n) + 2$.*
2. *If $\nu_3(U_n) \geq \nu_3(W_n - 6R^n) = 1$, then $\nu_3(D_{3n}) \geq \nu_3(D_n) + 2$.*
3. *If $\nu_3(U_n) < \nu_3(W_n - 6R^n)$, then*

$$\nu_3(D_{3n}) = \nu_3(D_n) + 3 \quad \textit{when} \quad \nu_3(D_n) > 1$$

or

$$\nu_3(D_{3n}) \geq \nu_3(D_n) + 3 \quad \textit{when} \quad \nu_3(D_n) = 1.$$

Proof. These results can be easily proved by making use of the the triplication formulas (13) and (14). □

In the case of $p = 2$, there exists a rather complicated law of repetition for p in $\{D_n\}$. We will not provide the complete law here, but we remark that if $\nu_2(D_n) > 1$, then the duplication formulas (12) can be used to show that $\nu_2(D_{2n}) \geq \nu_2(D_n) + 1$. The case of $\nu_2(D_n) = 1$, however, is more problematical. Certainly, if $2 \mid R$, there is no law of repetition for 2 in $\{D_n\}$ by Proposition 3. Thus, we need only consider the case of $2 \mid\mid D_n$ and $2 \nmid R$. In this case, we can use the duplication and triplication formulas to find that if

i) $4 \mid U_n$, $2 \mid\mid W_n - 6R^n$;

ii) $2 \mid\mid U_n$, $2 \mid\mid W_n - 6R^n$, $2 \mid \Delta$;

iii) $2 \mid\mid U_n$, $4 \mid W_n - 6R^n$, $2 \nmid \Delta$;

then $4 \mid D_{3n}$ and $4 \nmid D_{2n}$. In all other cases we have $4 \mid D_{2n}$.

We also have the following companion result to the law of repetition for any odd prime in $\{D_n\}$.

Theorem 7. *If p is odd and $\nu_p(D_n) \geq 1$, then $\nu_p(D_{mn}) = \nu_p(D_n)$ whenever $p \nmid m$.*

Proof. Since $p \neq 2$, we have $p^{2\lambda} \mid F_n$ when $\lambda = \nu_p(D_n)$, $\gcd(p, R) = 1$ and $W_n \equiv 6R^n \pmod{p^\lambda}$. It follows from (16) that

$$W_{mn} \equiv 2R^{mn}L_m(W_n/2R^n) \equiv 2R^{mn}L_m(3) \equiv 6R^{mn} \pmod{p^\lambda}$$

and by (15) that

$$U_{mn}/U_n \equiv R^{n(m-1)}K_m(3) \equiv m^3R^{n(m-1)} \pmod{p^\lambda}.$$

Since $p \nmid m$, it follows that $p^\lambda \mid\mid U_{mn}$ and $p^\lambda \mid W_{mn} - 6R^{mn}$; hence $p^\lambda \mid\mid D_{mn}$. □

In the case of $p = 2$, Theorem 7 is not in general true when $\lambda = 1$ and $2 \nmid R$, as we have seen in the above remarks. Of course, we could eliminate this problem if we could impose additional restrictions on S_1, S_2, R such that none of i), ii) or iii) could occur. If $2 \mid\mid D_n$ and $2 \nmid R$, it is easy to show that cases i), ii) or iii) can occur if and only if $2 \mid \tilde{Q}_n$, where $\tilde{Q}_n = (W_n^2 - \Delta U_n^2)/4$. In a later section we will discuss the parity of $\tilde{Q}_n$ when $2 \mid D_n$. Note that if $4 \mid D_n$, then $2 \nmid R$ and
$\tilde{Q}_n \equiv 1 \pmod 2$. If $\lambda > 1$, then we certainly have $2^\lambda \mid D_{mn}$ by Theorem 4 and since $W_n/2R^n \equiv 3 \pmod{2^{\lambda-1}}$, we get

$$U_{mn}/U_n \equiv m^3R^{n(m-1)} \pmod{2^{\lambda-1}}.$$

Thus, if m is odd, then $2 \nmid U_{mn}/U_n$ and $2^\lambda \mid\mid D_{mn}$. Hence Theorem 7 is true when $p = 2$ and $\nu_2(D_n) > 1$.

We conclude this section with a result that is often useful.

Theorem 8. *If m, $n \geq 1$, then* $\gcd(U_{mn}/U_n, D_n) \mid 2m^3$.

Proof. It is easy to show this when $2 \nmid D_n$ because $D_n \mid F_n$ and $\gcd(D_n, R) = 1$. Suppose $2 \mid D_n$; then because $U_n/2 \mid F_n$, we have $D_n/2 \mid F_n$. Also, $\gcd(D_n/2, R) = 1$ by Propositions 2 and 3. Hence, by (15)

$$U_{mn}/U_n \equiv m^3R^{n(m-1)} \pmod{D_n/2}.$$

It follows that

$$\gcd(U_{mn}/U_n, D_n/2) \mid m^3$$

and

$$\gcd(U_{mn}/U_n, D_n) \mid 2m^3.$$

□

3 The law of apparition for m in $\{D_n\}$

In this section we deal with the problem of when $m \mid D_n$, when $m > 1$. We note that if p is an odd prime and $p \mid R$, then $p \nmid D_n$ $(n \geq 0)$ by Proposition 2. Thus, we may assume that if m is odd, then $\gcd(m, R) = 1$. We define $\omega = \omega(m)$, if it exists, to be the least positive value of n such that $m \mid D_n$. We call ω the *rank of apparition* of m in $\{D_n\}$.

We begin by examining the case where m is a prime p where $p \mid d$ and $p \nmid 2R$.

Theorem 9. *Let p by any prime such that $p \nmid 2R$ and $p \mid d$. There exists a rank of apparition ω of p in $\{D_n\}$ and if $p \mid D_n$ for some $n \geq 0$, then $\omega \mid n$. Also, $\omega = p$ or $\omega \mid p \pm 1$.*

Proof. By results in the early part of [3, §9], we know that if $p \mid S_1^2 - 3S_2$, then p has a simple rank of apparition r_1 in $\{U_n\}$. It is not difficult to show that $p \mid D_n$ if and only if $r_1 \mid n$; hence, $\omega = r_1$. If $p \nmid S_1^2 - 3S_2$, then p can have two ranks of apparition in $\{U_n\}$ when $p \mid \Delta$ and only one when $p \nmid \Delta$. In either case, it is a simple matter to show that there is a rank of apparition ω of p in $\{D_n\}$, that $\omega \neq p$ and that if $p \mid D_n$, then $\omega \mid n$. ☐

We next consider the case of $p = 3$ and $3 \nmid d$.

Lemma 10. *If $p = 3$ and $3 \nmid dR$, then $\omega = \omega(3)$ always exists in $\{D_n\}$ and if $3 \mid D_n$, then $\omega \mid n$.*

Proof. We see from [3, Table 2] that there is single rank of apparition r of 3 in $\{U_n\}$. From the duplication formulas we see that if $3 \mid U_n$ and $3 \nmid W_n$, then $3 \mid W_{2n}$ if and only if $W_n \equiv R^n \pmod 3$ and $3 \mid W_{4n}$ if and only if $W_n \equiv -R^n \pmod 3$. Thus, $\omega(3)$ always exists and $\omega = r$, $2r$ or $4r$. Furthermore, if $3 \mid D_n$, then $\omega \mid n$. ☐

There remains the case of odd p where $p \nmid 3dR$. We first need to establish a simple lemma in this case. Here and in the sequel we will denote by $\mathbb{K}_p$ the splitting field of $G(x) \in \mathbb{F}_p[x]$. We can denote the zeros of $G(x) \in \mathbb{F}_p[x]$ by $R\gamma_i$ and R/γ_i $(i = 1, 2, 3)$.

Lemma 11. *If $p \nmid 2\Delta R$, then $p \mid D_n$ if and only if $\gamma_1^n = \gamma_2^n = \gamma_3^n = 1$ in $\mathbb{K}_p$.*

Proof. Certainly, if $\gamma_1^n = \gamma_2^n = \gamma_3^n = 1$ in $\mathbb{K}_p$, then $p \mid W_n - 6R^n$ and $p \mid U_n$ by (2) and (3); hence, $p \mid D_n$. If $p \mid D_n$, then since $p \mid U_n$ and $p \nmid \Delta$, we may assume without loss of generality that $\gamma_1^n = 1$. By [3, (8.4)], we have $\gamma_2^n - 1 = 0$ and therefore $\gamma_3^n = 1/(\gamma_1^n\gamma_2^n) = 1$. ☐

Corollary 12. *If $p \nmid 2\Delta R$ and $\omega = \omega(p)$ exists for p in $\{D_n\}$, then $p \mid D_n$ if and only if $\omega \mid n$.*

Proof. Certainly $p \mid D_n$ when $\omega \mid n$ because $\{D_n\}$ is a divisibility sequence. Suppose next that $\omega \nmid n$ and $p \mid D_n$. In this case we have $n = qw + r$, where $0 < r < \omega$. Also, by the lemma we must have $\gamma_1^n = \gamma_2^n = \gamma_3^n = 1$, $\gamma_1^\omega = \gamma_2^\omega = \gamma_3^\omega = 1 \in \mathbb{K}_p$. It follows that $\gamma_1^r = \gamma_2^r = \gamma_3^r = 1$ in $\mathbb{K}_p$ and $p \mid D_r$, which contradicts the definition of ω. ☐

We now deal with the case of $p \nmid 6dR$. Under this condition, we say that p is an S-prime, Q-prime or I-prime if the splitting field of $g(x) \in \mathbb{F}_p[x]$ is $\mathbb{F}_p$, $\mathbb{F}_{p^2}$ or $\mathbb{F}_{p^3}$, respectively. The following theorem follows easily from Lemma 11 and results in [3, §9].

Theorem 13. *If p is a prime, $p \nmid 6dR$ and $\epsilon = (\Delta/p)$, then*

$$\begin{aligned} &p \mid D_{p-\epsilon} \text{ when } p \text{ is an S-prime,}\\ &p \mid D_{p^2-1} \text{ when } p \text{ is an Q-prime,}\\ &p \mid D_{p^2+\epsilon p+1} \text{ when } p \text{ is an I-prime.}\end{aligned}$$

We can now assemble the above results in the following theorem.

Theorem 14. *If $p \nmid 2R$, there exists a rank of apparition ω $(\leq p^2+p+1)$ of p in $\{D_n\}$ and if $p \mid D_n$, then $\omega \mid n$.*

In [2, §4.6], S-, Q-, I-primes are discussed with respect to the polynomial $h(x) \in \mathbb{F}_p[x]$. We next show that if S_1, S_2 are given by (6), then the splitting fields of $h(x)$ and $g(x) \in \mathbb{F}_p[x]$ are the same whenever $p \nmid \Gamma$. We let $\mathbb{L}_1$ denote the splitting field of $h(x) \in \mathbb{F}_p[x]$, $\mathbb{L}_2$ denote the splitting field of $g(x) \in \mathbb{F}_p[x]$ and let α, β, γ denote the zeros of $h(x)$ in $\mathbb{L}_1$. Since the zeros of $g(x) \in \mathbb{F}_p[x]$ are given by

$$\rho_1 = \gamma(\alpha^2+\beta^2), \qquad \rho_2 = \alpha(\beta^2+\gamma^2), \qquad \rho_3 = \beta(\alpha^2+\gamma^2),$$

we see that ρ_1, ρ_2, $\rho_3 \in \mathbb{L}_1$. If $\mathbb{L}_1 = \mathbb{F}_p$, then clearly $\mathbb{L}_2 = \mathbb{F}_p = \mathbb{L}_1$. If $\mathbb{L}_1 = \mathbb{F}_{p^2}$, then $(\Delta/p) = -1$ and by (11), we get $(d/p) = (\Gamma\Delta/p) = (\Delta/p) = -1$; hence, $\mathbb{L}_2 = \mathbb{F}_{p^2} = \mathbb{L}_1$. If $\mathbb{L}_1 = \mathbb{F}_{p^3}$, then $(d/p) = 1$ and $\mathbb{L}_2 \neq \mathbb{F}_{p^2}$. Consider

$$\rho_1 = \gamma(P^2-2Q) - \gamma^3 \in \mathbb{L}_1.$$

We have

$$\rho_1^p = \gamma^p(P^2-2Q) - \gamma^{3p} = \alpha(P^2-2Q) - \alpha^3.$$

Thus, if $\rho_1 = \rho_1^p$, then since $\alpha \neq \gamma$ we must have

$$\alpha^2 + \alpha\gamma + \gamma^2 = P^2 - 2Q$$

and $\beta^2 = \alpha\gamma$ or $\beta^3 = R$. From (1), we get $P\beta - Q = 0$ and $P^3R - Q^3 = 0$, which is impossible because $p \nmid \Gamma$. Thus, $\rho_1 \neq \rho_1^p$, and therefore $\mathbb{L}_2 = \mathbb{F}_{p^3} = \mathbb{L}_1$.

We have not yet discussed the case of $p = 2$. The reason for this is easily seen in [3, Table 1]. We first observe that if $2 \mid R$, $2 \nmid S_1$ and $2 \mid S_2$, then $\omega(2)$ does not exist. Next, if $2 \mid S_1$ and $2 \nmid S_2R$, then $\omega(2) = 2$ by definition, but we also have $2 \mid D_3$ and $\omega(2) \nmid 3$. Thus to truly have a rank of apparition of 2 in the sense of the results given above we should eliminate the possibility that $2 \mid S_1$ and $2 \nmid S_2R$. When we do this, then by Proposition 2 we have $\omega(2)$ given by Table 1.

If $p \nmid 2R$, then p has a rank of apparition ω in $\{D_n\}$; we now deal with the case when $m = p^\alpha$ and $\alpha > 1$. By the law of repetition we know that $p^\alpha \mid D_n$ for some $n > 0$; hence $\omega(p^\alpha)$ must exist. If we put $\omega = \omega(p)$, then since $p \mid D_{\omega(p^\alpha)}$, we must have $\omega \mid \omega(p^\alpha)$ by Theorem 14. Put $s = \omega(p^\alpha)/\omega$ and let $p^\nu \mid\mid s$, then $s = p^\nu t$, where $p \nmid t$. If $p^\lambda \mid\mid D_{p^\nu\omega}$ and $\lambda < \alpha$, then $p^\lambda \mid\mid D_{p^\nu\omega t}$ by Theorem 7, which is a contradiction; thus $\omega(p^\alpha) = p^\nu\omega$. Notice that ν is the least positive integer such that $p^\alpha \mid D_{p^\nu\omega}$.

Next, suppose that $2 \nmid m$ and the prime power decomposition of m is

$$m = \prod_{i=1}^{k} p_i^{\alpha_i};$$

we must have

$$\omega(m) = \mathrm{lcm}(\omega(p_i^{\alpha_i}) : i = 1, 2, \ldots, k). \tag{17}$$

Thus, if $(m, 2R) = 1$, then $\omega(m)$ always exists and is given by (17).

4 The auxiliary sequences $\{U_n^*\}$ and $\{W_n^*\}$

In order to prove some results concerning $\{U_n\}$ and $\{W_n\}$, it is often useful to make use of the auxiliary sequences $\{U_n^*\}$ and $\{W_n^*\}$. We put $\gamma_1^* = \gamma_2/\gamma_1$, $\gamma_2^* = \gamma_3/\gamma_2$, $\gamma_3^* = \gamma_1/\gamma_3$, $R^* = R^2$ and define

$$\begin{aligned}
V_n^* &= R^{*n}(1+\gamma_1^{*n})(1+\gamma_2^{*n})(1+\gamma_3^{*n}),\\
U_n^* &= R^{*n-1}(1-\gamma_1^{*n})(1-\gamma_2^{*n})(1-\gamma_3^{*n})/((1-\gamma_1^*)(1-\gamma_2^*)(1-\gamma_3^*)),\\
W_n^* &= V_n^* - 2R^{*n},
\end{aligned}$$

where

$$\Delta^* = R^{*2}(1-\gamma_1^*)^2(1-\gamma_2^*)^2(1-\gamma_3^*)^2 = \Gamma \neq 0. \tag{18}$$

Notice also that

$$\begin{aligned}
\Gamma^* &= R^{*4}(\gamma_1^* - \gamma_2^*)^2(\gamma_2^* - \gamma_3^*)^2(\gamma_3^* - \gamma_1^*)^2\\
&= \Delta R^2 U_3^2.
\end{aligned}$$

If we put $\gamma_1^{**} = \gamma_2^*/\gamma_1^* = 1/\gamma_2^3$, then $\gamma_1^{**} = 1/\gamma_2^3$. We also have
$\gamma_2^{**} = \gamma_3^*/\gamma_2^* = 1/\gamma_3^3$, $\gamma_3^{**} = \gamma_1^*/\gamma_3^* = 1/\gamma_1^3$; hence,

$$W_n^{**} = R^n W_{3n}, \qquad U_n^{**} = R^{n-1}U_{3n}/U_3. \tag{19}$$

If we put $\rho_i^* = R^*(\gamma_i^* + 1/\gamma_i^*)$ $(i = 1, 2, 3)$, we get

$$S_1^* = \rho_1^* + \rho_2^* + \rho_3^* = S_2 - RS_1 \tag{20}$$

and

$$\begin{aligned} S_2^* &= \rho_1^*\rho_2^* + \rho_2^*\rho_3^* + \rho_3^*\rho_1^* = RW_3 + R^2S_1^* \\ &= RS_1^3 - 3RS_1S_2 + 3R^2S_1 - 5R^2S_2 - 4R^3S_1 - 12R^4. \end{aligned} \tag{21}$$

Also,

$$\begin{aligned} S_3^* &= \rho_1^*\rho_2^*\rho_2^* \\ &= R^*S_1^{*2} - 2R^*S_2^* - 4R^{*3}. \end{aligned}$$

It follows, then, from the results mentioned in §1, that if we compute the initial values of U_n^* and $W_n^*(= V_n^* - 2R^{*n})$ by replacing R, S_1, S_2 by R^*, S_1^*, S_2^*, respectively, then we have both $\{U_n^*\}$ and $\{W_n^*\}$ to be linear recurrence sequences of order 6 with characteristic polynomial $G^*(x)$ and $\{U_n^*\}$ is a divisibility sequence. It is easy to show as well that $W_{-n}^* = W_n^*/R^{*2n}$ and $U_{-n}^* = -U_n^*/R^{*2n}$. We observe further that $\gcd(S_1^*, S_2^*, S_3^*) = 1$ if and only if $\gcd(S_1, S_2, S_3) = 1$. Thus, the sequences $\{U_n^*\}$ and $\{W_n^*\}$ have the same properties as $\{U_n\}$ and $\{W_n\}$ with R, S_1, S_2, replaced by R^*, S_1^*, S_2^*, respectively.

We have shown how to relate the $\{U_n^{**}\}$ and $\{W_n^{**}\}$ sequences to $\{U_n\}$ and $\{W_n\}$ in (19); we can also relate the $\{U_n^*\}$ and $\{W_n^*\}$ sequences to $\{U_n\}$ and $\{W_n\}$. We define $\rho_i^{(n)} = R^n(\gamma_i^n + 1/\gamma_i^n)$ $(i = 1, 2, 3)$ and find that

$$S_1^{(n)} = \rho_1^{(n)} + \rho_2^{(n)} + \rho_3^{(n)} = W_n \tag{22}$$

and

$$S_2^{(n)} = \rho_1^{(n)}\rho_2^{(n)} + \rho_2^{(n)}\rho_3^{(n)} + \rho_1^{(n)}\rho_3^{(n)} = W_n^* + R^nW_n. \tag{23}$$

Since

$$\begin{aligned} \Delta U_n^2 &= R^{2n}(1-\gamma_1^n)^2(1-\gamma_2^n)^2(1-\gamma_3^n)^2 \\ &= S_1^{(n)2} - 4S_2^{(n)} + 4R^nS_1^{(n)} - 12R^{2n}, \end{aligned}$$

we get

$$\Delta U_n^2 = W_n^2 - 4W_n^* - 12R^{2n} \tag{24}$$

using (22) and (23). This formula, which generalizes (8), is similar to the well-known Lucas function identity

$$v_n^2 - \delta u_n^2 = 4q^n.$$

Note also that we get

$$\tilde{Q}_n = W_n^* + 3R^n \tag{25}$$

from (24) and

$$4W_n^* = W_n^2 - \Delta U_n^2 - 12R^{2n},$$

the relation connecting W_n^* to W_n and U_n. To relate U_n^* to W_n and U_n is somewhat more complicated. From (24), we have

$$\Delta^* U_n^{*2} = W_n^{*2} - 4W_n^{**} - 12R^{*2n}.$$

Hence, from (18), (19), and (24), we get

$$\Gamma U_n^{*2} = ((W_n^2 - \Delta U_n^2)/4 - 3R^{2n})^2 - 4R^n W_{3n} - 12R^{4n}.$$

From (13), we find that

$$\begin{aligned}16\Gamma U_n^{*2} = W_n^4 - 16R^n W_n^3 - 48R^n \Delta W_n U_n^2 + 72R^{2n} W_n^2 - 72R^{2n}\Delta U_n^2 \\ - 2\Delta W_n^2 U_n^2 + \Delta^2 U_n^4 - 432R^{4n}, \quad (26)\end{aligned}$$

a formula that generalizes (10).

As promised in §2 we will now investigate the parity of $\tilde{Q}_n$ when $2 \nmid R$ and $2 \mid D_n$. If $2 \nmid S_1$ and $2 \mid S_2$, then by (20) and (21), we have $2 \nmid S_1^*$ and $2 \mid S_2^*$. It follows that $2 \mid U_n^*$ if and only if $7 \mid n$ and $2 \mid W_n^*$ when $2 \mid D_n$. In this case we find from (25) that $2 \nmid \tilde{Q}_n$ whenever $2 \mid D_n$. If $2 \mid S_1$ and $2 \mid S_2$, then $2 \mid S_1^*$ and $2 \mid S_2^*$; hence, $2 \mid U_n^*$ if and only if $2 \mid n$ and we get $2 \mid W_n^*$, $\tilde{Q}_n \equiv 1 \pmod 2$ whenever $2 \mid D_n$. If $2 \nmid S_1$ and $2 \nmid S_2$, then $\Delta^* = \Gamma \equiv (S_2 + RS_1)^2 \equiv 0 \pmod 4$ from (10). Since $4 \mid W_n^{*2} - \Delta^* U_n^{*2}$, we get $2 \mid W_n^*$ and $\tilde{Q}_n \equiv 1 \pmod 2$.

The only remaining case is $2 \mid S_1$ and $2 \nmid S_2$. In this case $4 \mid \Delta$ and case (iii) can never occur. We get $U_2 \equiv S_1 + 2 \pmod 4$ and $W_2 - 6R^2 \equiv 2 \pmod 4$; thus, we see that cases (i) and (ii) can always occur, depending on the parity of $S_1/2$. In either of these cases, we get $4 \mid D_6$. It follows that if we eliminate the case of $2 \mid S_1$ and $2 \nmid S_2R$, then Thereom 7, will be true for all primes p. Also, we have already seen in §3 that if we eliminate this case, then we have a rank of apparition ω of 2 in $\{D_n\}$ and $2 \mid D_n$ if and only if $\omega \mid n$; indeed, if $\gcd(m, R) = 1$, there always exists a rank of apparition ω of m in $\{D_n\}$ given by (17) such that $m \mid D_n$ if and only if $\omega \mid n$. We remark here that if S_1 and S_2 are given by (6), then if $2 \nmid R$ and $2 \mid S_1$, we must have $2 \mid S_2$. Thus, for the sequences $\{c_n\}$ and $\{w_n\}$ we cannot have the case of $2 \mid S_1$ and $2 \nmid S_2R$.

If p is an I-prime and $p \equiv \epsilon = (\Delta/p) \pmod 3$, then $3 \mid p^2 + \epsilon p + 1$. Since we know in this case that $p \mid D_{p^2+\epsilon p+1}$, it is of some interest to determine a criterion for deciding whether or not $p \mid D_{(p^2+\epsilon p+1)/3}$. Roettger showed for the case of the $\{c_n\}$ and $\{w_n\}$ sequences that $p \mid D_{(p^2+p+1)/3}$ ($\epsilon = 1$ in this case if p is an I-prime) if and only if $R^{(p-1)/3} \equiv 1 \pmod p$ in [2, Theorem 5.14]. In what follows we will extend this result to the $\{U_n\}$ and $\{W_n\}$ sequences. We begin with three preliminary lemmas.

Lemma 15. *If* $3W_1^2 \equiv -\Delta \pmod p$*, then* p *cannot be an I-prime.*

Proof. We have $W_1 = S_1$ and by (8) we find that

$$S_2 \equiv RS_1^2 - 2RS_1 - 4R^3 \pmod p$$

and by (5)

$$S_3 \equiv -RS_1^2 - 2R^2S_1 + 2R^3 \pmod{p}.$$

Hence

$$g(x) \equiv (x+R)(x^2 - (S_1+R)x + S_1^2 + 2RS_1 - 2R^2) \pmod{p}.$$

Since $g(x)$ is reducible modulo p, p cannot be an I-prime. $\square$

Lemma 16. *Let p be an I-prime and let $\mathbb{K}_p$ be the splitting field of $G(x) \in \mathbb{F}[x]$. If ζ is a primitive cube root of unity in $\mathbb{K}_p$, then in $\mathbb{K}_p$ we can have*

$$\zeta^k(\gamma_1+\gamma_2+\gamma_3) + \zeta^{-k}(\gamma_1^{-1}+\gamma_2^{-1}+\gamma_3^{-1}) = \gamma_1+\gamma_2+\gamma_3+\gamma_1^{-1}+\gamma_2^{-1}+\gamma_3^{-1} \tag{27}$$

if and only if $3 \mid k$.

Proof. If $3 \mid k$ it is trivial that (27) must hold. If $3 \nmid k$, we first observe that $\zeta^k + \zeta^{-k} = -1$ and we have

$$\zeta^k + 1/2 = (\zeta^k - \zeta^{-k})/2, \qquad\qquad \zeta^{-k} + 1/2 = (\zeta^{-k} - \zeta^k)/2.$$

Thus (27) can hold only if

$$\frac{\zeta^k - \zeta^{-k}}{2}(\gamma_1+\gamma_2+\gamma_3-\gamma_1^{-1}-\gamma_2^{-1}-\gamma_3^{-1}) = \frac{3}{2}(\gamma_1+\gamma_2+\gamma_3+\gamma_1^{-1}+\gamma_2^{-1}+\gamma_3^{-1}).$$

On multiplying both sides by $2R$ and squaring we find that

$$3W_1^2 \equiv -\Delta \pmod{p},$$

which by the previous lemma is impossible. $\square$

Lemma 17. *If p is an I-prime and $p \mid U_n$, then $p \mid D_n$.*

Proof. Since $p \mid U_n$, we must have $\gamma_i^n = 1$ in $\mathbb{K}_p$ for some $i \in \{1,2,3\}$ by (2). We may assume that $\gamma_1^n = 1$. From the proof of [3, Theorem 9.8], we have $1 = \gamma_1^{pn} = \gamma_2^{\pm n}$; hence, $\gamma_2^n = 1$ and $\gamma_3^n = 1/(\gamma_1^n\gamma_2^n) = 1$. The result now follows by Lemma 11. $\square$

We are now able to derive our criterion for when $p \mid D_{(p^2+\epsilon p+1)/3}$.

Theorem 18. *If p is an I-prime and $p \equiv \epsilon \pmod 3$, then $p \mid D_{(p^2+\epsilon p+1)/3}$ if and only if*

$$W^*_{(p-\epsilon)/3} \equiv R^{2(p-\epsilon)/3-1}W_1 \pmod{p}.$$

Proof. We first note by Lemma 17 and 11 that $p \mid U_{(p^2+\epsilon p+1)/3}$ if and only if $\gamma_i^{(p^2+\epsilon p+1)/3} = 1$ in $\mathbb{K}_p$ for all $i \in \{1,2,3\}$. Since $\gamma_1^{p^2+\epsilon p+1} = 1$ in $\mathbb{K}_p$, we must have

$$\gamma_1^{\frac{p^2+\epsilon p+1}{3}} = \zeta^k,$$

where ζ is a primitive cube root of unity in $\mathbb{K}_p$. It follows that $p \mid D_{(p^2+\epsilon p+1)/3}$ if and only if $3 \mid k$. Now

$$(p^2 + \epsilon p + 1)/3 = (p - \epsilon)(p + 2\epsilon)/3 + 1.$$

Hence,

$$\zeta^k = \gamma_1^{(p^2+\epsilon p+1)/3} = (\gamma_1^{p+2\epsilon})^{(p-\epsilon)/3}\gamma_1.$$

Since $\gamma_1^p = \gamma_2^\epsilon$ (see the proof of [3, Theorem 9.8]), we get

$$\zeta^k = (\gamma_2\gamma_1^2)^{\epsilon(p-\epsilon)/3}\gamma_1 = \gamma_3^{*\epsilon(p-\epsilon)/3}\gamma_1$$

and

$$\gamma_3^{*(p-\epsilon)/3} = (\zeta^k/\gamma_1)^\epsilon.$$

Since $\gamma_3^{*p} = \gamma_1^p/\gamma_3^p = \gamma_2^\epsilon/\gamma_1^\epsilon = \gamma_1^{*\epsilon}$, we get

$$\gamma_1^{*\epsilon(p-\epsilon)/3} = (\zeta^{kp}/\gamma_1^p)^\epsilon = \zeta^k/\gamma_2$$

and

$$\gamma_1^{*(p-\epsilon)/3} = (\zeta^k/\gamma_2)^\epsilon.$$

Similarly $\gamma_2^{*(p-\epsilon)/3} = (\zeta^k/\gamma_3)^\epsilon$. It follows that

$$W^*_{(p-\epsilon)/3} = R^{*(p-\epsilon)/3}[\zeta^{-k\epsilon}(\gamma_1^\epsilon + \gamma_2^\epsilon + \gamma_3^\epsilon) + \zeta^{k\epsilon}(\gamma_1^{-\epsilon} + \gamma_2^{-\epsilon} + \gamma_3^{-\epsilon})].$$

By Lemma 16, we see that $3 \mid k$ if an only if

$$W^*_{(p-\epsilon)/3} \equiv R^{2(p-\epsilon)/3-1}W_1 \pmod p.$$

$\square$

This criterion can easily be converted to one that involves only the $\{U_n\}$ and $\{W_n\}$ sequences by using (24). At first glance, the criterion of Theorem 18 does not resemble the more elegant rule for $p \mid D_{(p^2+\epsilon p+1)/3}$ when dealing with Roettger's sequences. In this case we have $\gamma_1 = \alpha/\beta$, $\gamma_2 = \beta/\gamma$, $\gamma_3 = \gamma/\alpha$ and $R = \alpha\beta\gamma$. We can deduce Roettger's rule in the following corollary of Theorem 18.

Corollary 19. *Suppose $D_n = \gcd(w_n - 6R^n, c_n)$ and p is an I-prime with respect to $h(x) \in \mathbb{F}_p[x]$, then if $p \equiv 1 \pmod 3$, we have*

$$p \mid D_{(p^2+\epsilon p+1)/3} \Leftrightarrow R^{(p-1)/3} \equiv 1 \pmod p.$$

Proof. Suppose first that $p \nmid \Gamma$. In this case p is an I-prime with respect to $g(x) \in \mathbb{F}_p[x]$ and $1 = (d/p) = (\Gamma\Delta/p) = (\Delta/p) = \epsilon$. By Theorem 18 we have $p \mid D_{(p^2+\epsilon p+1)/3}$ if and only if $W^*_{(p-\epsilon)/3} \equiv R^{2(p-\epsilon)/3-1} W_1 \pmod p$. But in $\mathbb{K}_p$, we have $\gamma_1^* = \gamma_2/\gamma_1 = \beta^2/(\alpha\gamma) = \beta^3/R$; hence,

$$\gamma_1^{*\frac{p-1}{3}} = \beta^{p-1}/R^{(p-1)/3} = (\alpha/\beta)/R^{(p-1)/3} = \gamma_2^{-1}/R^{(p-1)/3}.$$

Similarly, $\gamma_2^{*\frac{p-1}{3}} = \gamma_3^{-1}/R^{(p-1)/3}$, $\gamma_3^{*\frac{p-1}{3}} = \gamma_1^{-1}/R^{(p-1)/3}$. It follows that

$$W^*_{\frac{p-1}{3}} = R^{*(p-1)/3}(R^{(p-1)/3}(\gamma_1+\gamma_2+\gamma_3) + R^{-(p-1)/3}(\gamma_1^{-1}+\gamma_2^{-1}+\gamma_3^{-1}))$$

and by Lemma 16 $W^*_{\frac{p-1}{3}} \equiv R^{2(p-1)/3-1}W_1 \pmod p$, if and only if $R^{(p-1)/3} = 1$ in $\mathbb{K}_p$.

Suppose next that $p \mid \Gamma$. In this case, p cannot be an I-prime with respect to $g(x)$. If $p \nmid P$, then by (11) we have $R \equiv (Q/P)^3 \pmod p$ and $h(Q/P) \equiv 0 \pmod p$. In this case p is not an I-prime with respect to $h(x)$, a contradiction. If $p \mid P$, then $p \mid Q$ and $\alpha^3 = \beta^3 = \gamma^3 = R$ in $\mathbb{L}_1$. We have $\alpha^{p-1} = \beta^{p-1} = \gamma^{p-1} = R^{(p-1)/3}$ and if $R^{(p-1)/3} \equiv 1 \pmod p$, we get $\alpha^p = \alpha$, and p is not an I-prime with respect to $h(x) \in \mathbb{F}_p[x]$, a contradiction. Now $p \mid D_3$ and since $3 \nmid (p^2+\epsilon p+1)/3$, we have $p \nmid D_{(p^2+\epsilon p+1)/3}$. Thus, if p is an I-prime with respect to $h(x) \in \mathbb{F}_p[x]$, then $R^{(p-1)/3} \not\equiv 1 \pmod p$ and $p \nmid D_{(p^2+\epsilon p+1)/3}$. ∎

We conclude this section with the following result concerning

$$D_n^* = \gcd(W_n^* - 6R^{*n}, U_n).$$

Theorem 20. *If p is an I-prime and $p \equiv \epsilon \pmod 3$, then $p \mid D^*_{(p^2+\epsilon p+1)/3}$.*

Proof. We observe as above that $\gamma_1^* = \gamma_2/\gamma_1$ and

$$(p^2+\epsilon p+1)/3 = (p-\epsilon)(p+2\epsilon)/3 + 1.$$

Hence

$$\gamma_1^{*(p^2+\epsilon p+1)/3} = (\gamma_2/\gamma_1)((\gamma_2/\gamma_1)^{p+2\epsilon})^{(p-\epsilon)/3}$$

in $\mathbb{K}_p$. Now $\gamma_2^p = \gamma_3^\epsilon$, $\gamma_1^p = \gamma_2^\epsilon$; hence,

$$(\gamma_2/\gamma_1)^{p+2\epsilon} = (\gamma_2\gamma_3/\gamma_1^2)^\epsilon = \gamma_1^{-3\epsilon}.$$

It follows that

$$((\gamma_2/\gamma_1)^{p+2\epsilon})^{(p-\epsilon)/3} = \gamma_1^{-\epsilon(p-\epsilon} = \gamma_1/\gamma_2$$

and

$$\gamma_1^{*(p^2+\epsilon p+1)/3} = 1.$$

Hence, $p \mid D^*_{(p^2+\epsilon p+1)/3}$. ∎

5 Some properties of $\{E_n\}$

We will devote the major portion of this section to the proof that if p (> 3) is a prime and $p \mid E_n$, then $p \equiv (\Gamma/p) \pmod 3$. This generalizes [2, Theorem 6.2]. We observe that by Proposition 2 we have $\gcd(E_n, R) = 2$. We now need some preliminary results.

Lemma 21. *Let p be any prime such that $p > 3$. If $p \mid E_n$, then in $\mathbb{K}_p$ we must have*

$$\gamma_i^n = 1, \qquad\qquad \gamma_j^{2n} + \gamma_j^n + 1 = 0,$$

where $i \in \{1, 2, 3\}$ and all $j \in \{1, 2, 3\}$ such that $j \neq i$.

Proof. If $p \nmid \Delta$ and $p \mid U_n$, we may assume with no loss of generality that $\gamma_1^n = 1$ in $\mathbb{K}_p$. If $p \mid \Delta$ we may assume with no loss of generality that $\gamma_1 = 1$ (and $\gamma_1^n = 1$) in $\mathbb{K}_p$. Now

$$\begin{aligned} W_n = V_n - 2R^n &= R^n(1+\gamma_1^n)(1+\gamma_2^n)(1+\gamma_3^n) - 2R^n \\ &= 2R^n(\gamma_2^n\gamma_3^n + \gamma_2^n + \gamma_3^n) \\ &= 2R^n(1 + \gamma_2^n + 1/\gamma_2^n) \\ &= 2R^n(1 + 1/\gamma_3^n + \gamma_3^n), \end{aligned}$$

the latter results following from $\gamma_1^n = 1$ and $\gamma_1^n\gamma_2^n\gamma_3^n = 1$. Since $W_n = 0$ in $\mathbb{K}_p$, we have $\gamma_2^{2n} + \gamma_2^n + 1 = \gamma_3^{2n} + \gamma_3^n + 1 = 0$. □

Lemma 22. *If p (> 3) is a prime, then $p \nmid (E_n, \Gamma)$.*

Proof. If $p \mid \Gamma$, then $\gamma_1 = \gamma_2$, $\gamma_2 = \gamma_3$ or $\gamma_3 = \gamma_1$ in $\mathbb{K}_p$ by (10). If $p \mid E_n$, then we may assume that $\gamma_1^n = 1$ and $\gamma_2^{2n} + \gamma_2^n + 1 = 0$ in $\mathbb{K}_p$ by Lemma 21. If $\gamma_1 = \gamma_2$, then $\gamma_2^n = 1$, which is impossible because $p > 3$. The same is true if $\gamma_2 = \gamma_3$ or $\gamma_3 = \gamma_1$. □

Lemma 23. *If p (> 3) is a prime, $p \mid \Delta$ and $p \mid E_n$, then*

$$p \equiv (\Gamma/p) \pmod 3.$$

Proof. Since $p \mid \Delta$, we may assume with no loss of generality that $\gamma_1 = 1$ and therefore $\gamma_2\gamma_3 = 1$ in $\mathbb{K}_p = \mathbb{F}_{p^2}$. Also, by Lemma 21 we may assume that if $p \mid E_n$, then

$$\gamma_2^{2n} + \gamma_2^n + 1 = 0$$

in $\mathbb{K}_p$. Hence, $\gamma_2^{3n} = 1$ and $\gamma_2^n \neq 1$ in $\mathbb{K}_p$. By Lemma 22, $p \nmid \Gamma$ and

$$\begin{aligned} \Gamma^{\frac{p-1}{2}} &= (\gamma_1 - \gamma_2)^{p-1}(\gamma_2 - \gamma_3)^{p-1}(\gamma_3 - \gamma_1)^{p-1} \\ &= \frac{(1-\gamma_2^p)(\gamma_2^p - \gamma_3^p)(\gamma_3^p - 1)}{(1-\gamma_2)(\gamma_2 - \gamma_3)(\gamma_3 - 1)}. \end{aligned} \tag{28}$$

If $\gamma_2 \in \mathbb{F}_p$, then $\Gamma^{\frac{p-1}{2}} = 1$. Also, from $\gamma_2^{pn} = \gamma_2^n$, we get $\gamma_2^{(p-1)n} = 1$, which, since $\gamma_2^n \neq 1$ means that $3 \mid p - 1$ and $p \equiv (\Gamma/p) \pmod 3$. If $\gamma_2 \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$, then $\gamma_2^p = \gamma_3$ and $\gamma_2^{(p-1)n} = -1$ by (28). Since $\gamma_2^{pn} = \gamma_3^n = 1/\gamma_2^n$ and $\gamma_2^{(p+1)n} = 1$, we see that $3 \mid p + 1$ and $p \equiv (\Gamma/p) \pmod 3$. □

We now show that if p is an I-prime, then $p \nmid E_n$.

Theorem 24. *If p is an I-prime, then $p \nmid E_n$.*

Proof. As noted above we know that if p is an I-prime, then $\gamma_1^p = \gamma_2^\epsilon$, $\gamma_2^p = \gamma_3^\epsilon$, $\gamma_3^p = \gamma_1^\epsilon$ in $\mathbb{K}_p$. If $p \mid E_n$, then by Lemma 21, we have $\gamma_1^n = 1$ and $\gamma_2^{2n} + \gamma_2^n + 1 = 0$. Now $\gamma_2^{p^2} = \gamma_3^{\epsilon p} = \gamma_1^{\epsilon^2} = \gamma_1$ and $\gamma_2^{p^2 n} = \gamma_1^n$. Hence,

$$0 = (\gamma_2^{2n} + \gamma_2^n + 1)^{p^2} = 3,$$

which is a contradiction. $\square$

We next deal with the case where $p \mid S_1 + 2R$.

Lemma 25. *If p (> 3) is a prime, $p \nmid d$, $p \mid S_1 + 2R$ and $p \mid E_n$, then*

$$p \equiv (\Gamma/p) \pmod 3.$$

Proof. Since $p \mid S_1 + 2R$ and $S_1 + 2R = R(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_3 + 1)$, we may assume in $\mathbb{K}_p$ that $\gamma_1 = -1$ and $\gamma_2\gamma_3 = -1$. We get

$$(\gamma_1 + \gamma_2)(\gamma_2 + \gamma_3)(\gamma_3 + \gamma_1) = -(\gamma_2^2 + 1/\gamma_2^2 - 2).$$

Since $S_1 \equiv -2R \pmod p$, we get $S_3 \equiv -2RS_2 \pmod p$ from (5) and

$$g(x) = (x + 2R)(x^2 + S_2) \in \mathbb{F}_p[x].$$

Since $\rho_1 = R(\gamma_1 + 1/\gamma_1) = -2R$, we get $\rho_2^2 = \rho_3^2 = -S_2$ and $\gamma_2^2 + 1/\gamma_2^2 = \rho_2^2/R^2 - 2 = -S_2/R^2 - 2 \in \mathbb{F}_p$. It follows that $(\gamma_1 + \gamma_2)(\gamma_2 + \gamma_3)(\gamma_3 + \gamma_1) \in \mathbb{F}_p$ and

$$\begin{aligned}((\gamma_1^2 - \gamma_2^2)(\gamma_2^2 - \gamma_3^2)(\gamma_3^2 - \gamma_1^2))^{p-1} &= ((\gamma_1 - \gamma_2)^2(\gamma_2 - \gamma_3)^2(\gamma_3 - \gamma_1)^2)^{\frac{p-1}{2}} \\ &= (\Gamma/p). \end{aligned} \tag{29}$$

As $\gamma_2^2 + 1/\gamma_2^2 \in \mathbb{F}_p$, we must have $\gamma_2^2, 1/\gamma_2^2 \in \mathbb{F}_{p^2}$ and $\gamma_2^{2p} = \gamma_2^2$ or $\gamma_2^{2p} = \gamma_3^2$. Since $p \nmid d$, we see from (29), that $(\Gamma/p) = 1$, when $\gamma_2^{2p} = \gamma_2^2$ and $(\Gamma/p) = -1$, when $\gamma_2^{2p} = \gamma_3^2$.

If $p \mid E_n$, then by Lemma 21, we have $\gamma_i^n = 1$ for some $i \in \{1, 2, 3\}$ and $\gamma_j^{2n} + \gamma_j^n + 1 = 0$ $(i \neq j)$. Since $\gamma_1 = -1$, we see that $i = 1$ and $2 \mid n$. If $(\Gamma/p) = 1$, then $\gamma_2^{np} = \gamma_2^n$ and $\gamma_2^{n(p-1)} = 1$. Since $\gamma_2^{3n} = 1$ and $\gamma_2^n \neq 1$, we see that $3 \mid p - 1$ and $p \equiv (\Gamma/p) \pmod 3$. If $(\Gamma/p) = -1$, then $\gamma_2^{np} = \gamma_3^n = 1/\gamma_2^n$ and $\gamma_2^{n(p+1)} = 1$; hence $3 \mid p + 1$ and $p \equiv (\Gamma/p) \pmod 3$. $\square$

We are now ready to prove our main result.

Theorem 26. *If p (> 3) is a prime divisor of E_n, then $p \equiv (\Gamma/p) \pmod 3$.*

Proof. We have already proved this result when $p \mid d$ and when $p \nmid d$ and $p \mid S_1 + 2R$. We may assume, then, that $p \nmid d$ and $p \nmid S_1 + 2R$. Since $p \mid E_n$, p can only be an S-prime or a Q-prime by Theorem 24. If p is an S-prime, then $1 = (d/p) = (\Delta/p)(\Gamma/p)$ and $(\Gamma/p) = \epsilon$; if p is an Q-prime, then $-1 = (d/p) = (\Delta/p)(\Gamma/p)$ and $(\Gamma/p) = -\epsilon$. Suppose p is an S-prime. By results in the proof of [3, Theorem 9.4], we have $\gamma_i^p = \gamma_i^\epsilon$ $(i = 1, 2, 3)$ in $\mathbb{K}_p$. By Lemma 21, we get $\gamma_2^{3n} = 1$, $\gamma_2^n \neq 1$; also, $\gamma_2^{np} = \gamma_2^{n\epsilon}$ means that $\gamma_2^{(p-\epsilon)n} = 1$ and $3 \mid p - \epsilon$. Similarly, if p is a Q-prime, then by the results in the proof of [3, Theorem 9.6], we have

$$\gamma_2^p = \gamma_3^\epsilon, \qquad \gamma_3^p = \gamma_2^\epsilon, \qquad \gamma_3^p = \gamma_1^\epsilon$$

in $\mathbb{K}_p$. In this case we get $\gamma_2^{pn} = \gamma_3^{\epsilon n} = (1/\gamma_2)^{\epsilon n}$ and $\gamma_2^{n(p+\epsilon)} = 1$, $\gamma_2^{3n} = 1$ and $\gamma_2^n \neq 1$. Hence $3 \mid p + \epsilon$ and in either case $p \equiv (\Gamma/p) \pmod 3$. □

In order to extend Theorem 26, we need to prove the following result.

Theorem 27. *For any $n > 0$, we have $E_n \mid D_{3n}$.*

Proof. We can rewrite (13) as

$$W_{3n} - 6R^{3n} = (W_n - 6R^n)\tilde{Q}_n + \Delta W_n U_n^2, \tag{30}$$

where $\tilde{Q}_n = (W_n^2 - \Delta U_n)/4$. Suppose p is any odd prime and $p^\lambda \mid\mid E_n$, where $\lambda \geq 1$. Since $p^\lambda \mid U_n$, we must have $p^\lambda \mid U_{3n}$. Also, $p^{2\lambda} \mid \tilde{Q}_n$ and $p^\lambda \mid W_{3n} - 6R^{3n}$ by (30). Next, suppose that $2^\lambda \mid\mid E_n$ and $\lambda \geq 1$. We have $2 \mid W_n - 6R^n$ and $2^{2\lambda-2} \mid \tilde{Q}_n$, $2^\lambda \mid U_n$. By (30) we see that $2^{2\lambda-1} \mid W_{3n} - 6R^{3n}$ and since $\lambda \geq 1$, we have $2\lambda - 1 \geq \lambda$ and $2^\lambda \mid D_{3n}$. Hence, $E_n \mid D_{3n}$. □

We next prove a result which is analogous to the theorem that states that if p is an odd prime and $p \mid v_n$, then $p \equiv \pm 1 \pmod{2^{\nu+1}}$, where $2^\nu \mid\mid n$. (See [2, Theorem 2.20]).

Theorem 28. *If p (> 3) is a prime and $p \mid E_n$, then $p \equiv (\Gamma/p) \pmod{3^{\nu+1}}$, where $3^\nu \mid\mid n$.*

Proof. Since $p \mid E_n$ and $p > 3$, we have $p \nmid D_n$, as $p \nmid 6R$. But, by Theorem 27, we know that $p \mid D_{3n}$. Thus, if ω is the rank of apparition of p in $\{D_n\}$, we have $\omega \mid 3n$ and $\omega \nmid n$. It follows that $3^{\nu+1} \mid \omega$. Also, since p is not an I-prime and $p \nmid 6R$, we must have $\omega = p$ or $\omega \mid p^2 - 1$ by results in §3. Since $3 \mid \omega$ we cannot have $\omega = p$ and therefore $\omega \mid p^2 - 1$ and $3^{\nu+1} \mid p^2 - 1$. Since $p \nmid \Gamma$, we have $p^2 - 1 = (p - (\Gamma/p))(p + (\Gamma/p))$ and $3 \mid p - (\Gamma/p)$. Hence $3^{\nu+1} \mid p - (\Gamma/p)$. □

6 Primality tests

In Williams [4], it is shown how Lucas used the properties of $\{u_n\}$ and $\{v_n\}$ to develop primality tests for certain families of integers. In this section we will indicate how the properties of $\{U_n\}$ and $\{W_n\}$ can be used to produce some primality tests. We begin with a simple result concerning integers of the form $A3^n + \eta$, where $\eta^2 = 1$.

Theorem 29. *Let $N = A3^n + \eta$, where $2 \mid A$, $n \geq 2$, $3 \nmid A$, $\eta \in \{1, -1\}$ and $A < 3^n$. If*

$$N \mid U_{N-\eta}/U_{(N-\eta)/3},$$

then N is a prime.

Proof. Let p be any prime divisor of N and put $m = (N - \eta)/3$. We note that $p \neq 2, 3$ and by (14)

$$4U_{3m}/U_m = 3W_m^2 + \Delta U_m^2.$$

Since $p \mid U_{3m}$, there must exist some rank of apparition r of p in $\{U_n\}$ such that $r \mid 3m$. If $p \mid U_m$ and $p \mid W_m$, then $p \mid E_m$ and $p \equiv (\Gamma/p) \pmod{3^n}$ by Theorem 28. If $p \nmid U_m$, then $r \nmid m$ and $r \mid 3m$ means that $3^n \mid r$. Suppose $p \nmid dR$. If p is an S-prime or a Q-prime, then by [3, Corollary 9.5 and Theorem 9.7] we must have $r \mid p - \epsilon$, where $\epsilon = (\Delta/p)$; hence $p \equiv (\Delta/p) \pmod{3^n}$. If p is an I-prime, then $r \mid p^2 + \epsilon p + 1$ by Theorem 9.9 of [3]. Since $9 \mid r$, this is impossible. If $p \mid dR$, then $r = 3, p$ or divides $p \pm 1$. Since $9 \mid r$, $r \neq 3$ and since $p \nmid N - \eta$, we cannot have $r = p$. Thus, in all possible cases, we find that $p \equiv \pm 1 \pmod{3^n}$ and since p is odd, we have $p \geq 2 \cdot 3^n - 1$. Since $(2 \cdot 3^n - 1)^2 > N$, N can only be a prime. $\square$

We also note that if N obeys the conditions in the first line of Theorem 29 and $N \mid E_{(N-\eta)/3}$, then N must be a prime.

By extending the results in [2, Chapter 7] it is possible to select the parameters of S_1, S_2 to make Theorem 29 both a necessary and sufficient test for the primality of N, but this test is much less efficient than one based on the Lucas Functions.

In [3, §9] several primality tests for N are presented. These tests can be easily proved by using the techniques in [2, Chapter 7], but to be usable they require that we know the complete factorization of

$$N^2 + N + 1 \quad \text{or} \quad N^2 - N + 1.$$

Of course, such a circumstance is very unlikely, but we might have a partial factorization of $N^2 \pm N + 1$. In what follows we will devise a test for the primality of N in this case. We first require a simple lemma.

Lemma 30. *If p and q are distinct primes, $p > 3$ and $p \mid D_{qn}$ and $p \mid U_{qn}/U_n$, then $q^{\lambda+1} \mid \omega$, where ω is the rank of apparition of p in $\{D_n\}$ and $q^\lambda \mid\mid n$.*

Proof. Suppose $p \mid D_n$. If $p \mid U_{qn}/U_n$, then by Theorem 8, we get $p \mid 2q^3$, which is impossible. Hence, $p \nmid D_n$. It follows that since $p \mid D_{qn}$ ($\{D_n\}$ is a divisibility sequence), we get $\omega \mid qn$ and $\omega \nmid n$, which means that $q^{\lambda+1} \mid \omega$. $\square$

We will also need the easily established technical lemma below.

Lemma 31. *If $x \geq 5$, then*

$$(x^2 + x + 1)^2 < 2(x^4 - x^2 + 1).$$

Theorem 32. *Let N be a positive integer such that $\gcd(N, 6) = 1$ and put $\eta = 1$ or -1. Let $T = N^2 + \eta N + 1$ and suppose that $T' \mid T$, where $\gcd(T', T/T') = 1$ and $T'^2 > 2T$. If $N \mid D_T$ and $N \mid U_T/U_{T/q}$ for all distinct primes q such that $q \mid T'$, then N is a prime.*

Proof. Let p be any prime divisor of N and q be any prime divisor of T'; then $p \geq 5$ and by Lemma 30 we have $q^\lambda \mid \omega(p)$, where $\omega(p)$ is the rank of apparition of p in $\{D_n\}$ and $q^\lambda \mid\mid T$. Since $\gcd(T', T/T') = 1$, we have $q^\lambda \mid\mid T'$; hence, $T' \mid \omega(p)$. Let ω denote the rank of apparition of T in $\{D_n\}$. We have $\omega \mid T$ and $\omega/q \nmid T$; hence, $q^\lambda \mid \omega$, where $q^\lambda \mid\mid T$ and therefore $T' \mid \omega$.

By (17), we have

$$\omega = \operatorname{lcm}(\omega(p_i^{\alpha_i}) : i = 1, 2, \ldots, j),$$

where

$$N = \prod_{i=1}^{j} p_i^{\alpha_i}$$

is the prime power factorization of N. Since $\omega(p_i^{\alpha_i}) = p_i^{\nu_i}\omega(p_i)$, we must have $\nu_i = 1$ because $p_i \nmid T$. We get

$$\omega = \operatorname{lcm}(\omega(p_i) : i = 1, 2, \ldots, j_| T' \prod_{i=1}^{j} \frac{\omega(p_i)}{T'}.$$

If we put $T = k\omega$, then

$$T \leq kT' \prod_{i=1}^{j} \frac{\omega(p_i)}{T'} \leq kT' \prod_{i=1}^{j} \frac{p_i^2 + p_i + 1}{T'}$$

by Theorem 13. Also, since

$$T = N^2 + \eta N + 1 > 2 \prod_{i=1}^{j} \frac{p_i^2 + p_i + 1}{2},$$

([3, Lemma 9.11], cf. [2, Lemma 7.1]) we get

$$kT' \prod_{i=1}^{j} \frac{p_i^2 + p_i + 1}{T'} > 2 \prod_{i=1}^{j} \frac{p_i^2 + p_i + 1}{2}$$

and

$$kT'2^j > 2(T')^j.$$

Hence,

$$k > (T'/2)^{j-1} \geq T'/2 \quad (\text{when } j \geq 2).$$

But since $T/T' = k\omega/T'$, we have $k \leq T/T' < T'/2$, a contradiction; consequently, we can only have $j = 1$ and $N = p^\alpha$. Since $\omega(N) = p^\nu\omega(p)$ and $\gcd(p, \omega(N)) = 1$, we get $\omega(p^\alpha) = \omega(p)$. It follows that

$$\omega(N) = \omega(p) \leq p^2 + p + 1.$$

Now $T' \mid \omega(p)$ means that $\omega(p) \geq T'$ and $p^2+p+1 \geq T'$. Since $T'^2 > 2T$, we have for $\alpha \geq 2$

$$(p^2+p+1)^2 > 2(p^{2\alpha}+\eta p^\alpha+1) \geq 2(p^{2\alpha}-p^\alpha+1) \geq 2(p^4-p^2+1)$$

which is impossible by Lemma 31. Hence we can only have $N = p$. $\square$

Many other primality tests can be devised by making use of the ideas in [2, Chapter 7], but the above should suffice to illustrate the kind of results that can be established.

7 Conclusions

In [3] we showed that the $\{U_n\}$ and $\{W_n\}$ sequences can be considered respectively as the sextic analogues of Lucas' $\{u_n\}$ and $\{v_n\}$ sequences. In this paper we have produced a number of results that are the number-theoretic analogues of well-known properties of the Lucas functions. Of course, there are many other properties of $\{D_n\}$ and $\{E_n\}$ that are similar to those of the $\{D_n\}$ and $\{E_n\}$ sequences discussed at some length in [2], and these can be proved by using the results presented here and the techniques of [2].

References

[1] S. Müller, H. C. Williams, and E. Roettger, A cubic extension of the Lucas functions, *Ann. Sci. Math. Québec* **33** (2009), 185–224.

[2] E. Roettger, *A Cubic Extension of the Lucas Functions*, PhD thesis, University of Calgary, 2009. Available online at http://people.ucalgary.ca/~williams/.

[3] E. L. Roettger, H. C. Williams, and R. K. Guy, Some extensions of the Lucas functions. In *Number Theory and Related Fields*, Springer, 2013, pp. 279–319.

[4] H. C. Williams, *Édouard Lucas and Primality Testing*, Wiley-Interscience, 1998.

2010 *Mathematics Subject Classification*: Primary 11B37; Secondary 11Y11, 11B50.
Keywords: linear recurrence, Lucas function, primality testing.

Received February 5 2015; revised version received May 11 2015; May 29 2015. Published in *Journal of Integer Sequences*, May 30 2015.

Return to Journal of Integer Sequences home page.