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A New Lower Bound for the Distinct Distance Constant

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Abstract

The reciprocal sum of the Zhang sequence is not equal to the distinct distance constant. This note introduces a B_2-sequence with larger reciprocal sum, and provides a more precise estimate of the reciprocal sums of the Mian-Chowla sequence and the Zhang sequence.

1 Introduction

A *Sidon sequence*, also called a B_2*-sequence*, is a sequence of positive integers

$$a_1 < a_2 < a_3 < \ldots$$

such that all the sums $a_i + a_j$ $(i \leq j)$ are distinct.

The *distinct distance constant* (DDC) is the supremum of the set of the reciprocal sums of Sidon sequences. Levine [1] observed that

$$\mathrm{DDC} \leq \sum_{n=0}^{\infty} \frac{1}{1 + \frac{n(n+1)}{2}} = \frac{2\pi}{\sqrt{7}} \tanh\left(\frac{\sqrt{7}}{2}\pi\right) < 2.37366\,. \tag{1}$$

Let S_A be the reciprocal sum of the sequence A; that is, $S_A = \sum_{i=1}^{\infty} \frac{1}{a_i}$. It is an open problem to find, if it exists, a Sidon sequence U whose reciprocal sum is equal to the DDC [2, p. 351]. We only know from Taylor and Yovanof [3] that, if some Sidon sequence achieves the DDC, then it must begin with the values 1, 2, 4.

The B_2-sequence with the largest known reciprocal sum had been, for a long time, the one produced by the greedy algorithm, $G = \{1, 2, 4, 8, 13, 21, 31, 45, 66, 81, \dots\}$. The sequence G is named the *Mian-Chowla sequence*. Lewis found that

$$2.158435 \leq S_G \leq 2.158677\,, \tag{2}$$

where S_G is the reciprocal sum of G [4, pp. 164–165].

In 1991, Zhang [5] found a Sidon sequence with a reciprocal sum greater than S_A. Zhang's sequence Z is obtained by running the greedy algorithm for the first 14 terms, setting $z_{15} = 229$, and then continuing with the greedy algorithm. Zhang proved that

$$S_Z > 2.1597\,. \tag{3}$$

The aim of this note is to exhibit a Sidon sequence H such that $S_H > S_Z$.

2 Computation of reciprocal sums

2.1 Preliminary considerations

To estimate the reciprocal sum, we will use two basic properties of Sidon sequences: they are growing sequences, and their differences $a_i - a_j$ $(i \geq j)$ are all distinct. That is,

$$a_i \geq a_j + (i - j),\ \ i \geq j \tag{4}$$

$$a_i > \frac{i(i-1)}{2}\,. \tag{5}$$

Therefore, if we know the values of a_n for $1 \leq n \leq k$, we also know that

$$\sum_{n=1}^{k} \frac{1}{a_n} < S_A < \sum_{n=1}^{k} \frac{1}{a_n} + \sum_{n=k}^{\infty} \frac{1}{\max\left(a_k + n - k, \frac{n(n-1)}{2}\right)}\,. \tag{6}$$

Further assumptions could be made on a_i, but for large values of k they would not yield a significant improvement of the bounds.

2.2 The reciprocal sum of the Mian-Chowla sequence

Let G be the B_2-sequence constructed by the greedy algorithm. The values of g_n for $1 \leq n \leq 25000$, computed on my notebook, are listed in the accompanying file `MianChowla.txt`. We get the following bounds for S_G:

$$\sum_{n=1}^{25000} \frac{1}{g_n} < S_G < \sum_{n=1}^{25000} \frac{1}{g_n} + \sum_{n=25001}^{510096} \frac{1}{g_{25000} + n - 25000} + \sum_{n=510097}^{\infty} \frac{2}{n(n-1)}\,, \tag{7}$$

i.e.,

$$2.15845268 < S_G < 2.15846062\,. \tag{8}$$

2.3 The reciprocal sum of the Zhang sequence

The Zhang sequence Z [5] is, at the moment, the known Sidon sequence with the largest reciprocal sum. In the accompanying file `Zhang.txt` there are the first 25000 terms of Z. Their values allow us to compute the following bounds for S_Z:

$$\sum_{n=1}^{25000} \frac{1}{z_n} < S_Z < \sum_{n=1}^{25000} \frac{1}{z_n} + \sum_{n=25001}^{510290} \frac{1}{z_{25000} + n - 25000} + \sum_{n=510291}^{\infty} \frac{2}{n(n-1)} \tag{9}$$

$$S_G < 2.16007769 < S_Z < 2.16008532\,. \tag{10}$$

2.4 The sequence H and its reciprocal sum

Definition 1.

$$h_n = \begin{cases} 1, & \text{if } n = 1; \\ 229, & \text{if } n = 15; \\ 962, & \text{if } n = 27; \\ \min\{x | \forall i, j, k \le n, a_i + a_j \ne a_k + x\}, & \text{otherwise.} \end{cases}$$

The first 26 terms of the sequence H are the same of the Zhang seqence, the 27^{th} term is 962, and from there on the values are provided by the greedy algorithm.

We will now prove that the sequence H satisfies our aim, i.e., $S_H > S_Z$.

We can find the first values of h_n $(1 \le n \le 25000)$ in the accompanying file `H.txt`. Hence we get

$$\sum_{n=1}^{25000} \frac{1}{h_n} < S_H < \sum_{n=1}^{25000} \frac{1}{h_n} + \sum_{n=25001}^{510140} \frac{1}{h_{25000} + n - 25000} + \sum_{n=510141}^{\infty} \frac{2}{n(n-1)} \tag{11}$$

$$2.16027651 < S_H < 2.16028417\,. \tag{12}$$

In conclusion,

$$S_Z < 2.16027651 < S_H \le \mathrm{DDC}\,, \tag{13}$$

which was what we wanted to prove.

I found the sequence H using an algorithm which proceeds as follows: given a finite Sidon sequence $b_1, b_2, \ldots, b_n$, choose as b_{n+1} the value that would yield the largest reciprocal sum if the sequence were continued with the greedy algorithm; then repeat with the sequence $b_1, b_2, \ldots, b_n, b_{n+1}$. The program considered 20 candidate values in each step, and estimated the reciprocal sums with the terms up to 64000. Giving as input the first 13 terms of the Mian-Chowla sequence, I obtained the first 30 terms of H. I have experimented with a number of other starting sequences, without finding any other sequence with a reciprocal sum greater than S_Z.

The most likely place to find a B_2-sequence X achieving a reciprocal sum $S_X > S_H$, if it exists, is between the sequences having the first 27 terms in common with H, or with the Zhang sequence.

References

[1] Eugene Levine, An extremal result for sum-free sequences, *J. Number Theory* **12** (1980), 251–257.

[2] Richard K. Guy, *Unsolved Problems in Number Theory*, 3rd ed., Springer, 2004.

[3] H. Taylor and G. S. Yovanof, B_2-sequences and the distinct difference constant, *Comput. Math. Appl.* **39** (2000), 37–42.

[4] Steven R. Finch, *Mathematical Constants*, Cambridge University Press, 2003.

[5] Zhenxiang Zhang, A B_2-sequence with larger reciprocal sum, *Math. Comput.* **60** (1993), 835–839.

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