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Analytic Representations of the n-anacci Constants and Generalizations Thereof

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Abstract

We study generalizations of the sequence of the n-anacci constants that are constructed from the ratio limits generated by linear recurrences of an arbitrary order n with equal integer weights m. We derive the analytic representation of the class C^∞ of these ratio limits and prove that, for a fixed m, the ratio limits form a strictly

increasing sequence converging to $m+1$. We also show that the generalized n-anacci constants form a totally ordered set.

1 Introduction

We study properties of the ratio limits $\Phi^{(n)}(m)$, $m, n \in \mathbb{N}$, of the successive terms of the integer sequences $\left(F_k^{(n)}(m)\right)_{k=1}^{\infty}$ with the signatures $(m, \ldots, m)$ generated by the linear recurrences $F_k^{(n)}(m)$ of an arbitrary order n with equal weights m, i.e.,

$$F_k^{(n)}(m) \equiv m\left(F_{k-1}^{(n)}(m) + \cdots + F_{k-n}^{(n)}(m)\right),\ n \leq k, \quad \text{and} \quad F_k^{(n)}(m) = a_k \in \mathbb{N},\ 0 \leq k < n, \tag{1}$$

$$\Phi^{(n)}(m) \equiv \lim_{k\to\infty} F_{k+1}^{(n)}(m)/F_k^{(n)}(m),\ k > k_0, \tag{2}$$

where k_0 is the largest index for which $F_{k_0}^{(n)}(m) = 0$.

The linear recurrences (1) generate, in particular, the following sequences:

1. The n-step Fibonacci sequences, with the signatures $(1, \ldots, 1)$, introduced in 1960 by Miles [10] as the n-generalized Fibonacci sequences, and further investigated by Flores [6] and Dubeau [3], as well as more recently by Zhu and Grossman [17].

2. The n-step Lucas sequences, with the signatures $(1, \ldots, 1)$, introduced in 1967 by Fiedler [5], and recently studied by Catalani [2], Benjamin and Quin [1], as well as by Noe and Post [11].

3. The 2-step sequences with the signatures (m, m) that are a special case of the Horadam sequences [7, 8].

Sloane [14] and Khovanova [9] catalog numerous sequences with the signatures $(1, \ldots, 1)$, $2 \leq n \leq 13$, and a large variety of initial conditions; the Horadam sequences with the signatures (m, m), $2 \leq m \leq 10$, and various initial conditions; as well as several sequences with the signatures $(m, \ldots, m)$, $n > 2$, generated by the linear recurrences (1).

Following the terminology that refers to the elements of the sequence $\left(\Phi^{(n)}(1)\right)_{n=1}^{\infty}$ as the n-anacci constants [12], we call the elements of the set $\left\{\Phi^{(n)}(m) \mid m, n \in \mathbb{N}\right\}$ the (m, n)-anacci constants.

In 1966, Ostrowski showed that polynomials

$$P^{(n)}(\lambda) \equiv \lambda^n - b_1\lambda^{n-1} - \cdots - b_n, \quad b_i \in \mathbb{R}_+, \tag{3}$$

with the gcd of the indices i of coefficients $b_i \neq 0$ equal to 1, are asymptotically simple [13, Theorem 12.2]. Thus, for any given $p \in \mathbb{R}_+$ and n, the characteristic polynomial

$$P_p^{(n)}(\lambda) \equiv \lambda^n - p\,(\lambda^{n-1} + \cdots + 1) \tag{4}$$

of the linear recurrence

$$F_k^{(n)}(p) \equiv p\big(F_{k-1}^{(n)}(p) + \cdots + F_{k-n}^{(n)}(p)\big), \; n \leq k, \quad \text{and} \quad F_k^{(n)}(p) = a_k \in \mathbb{R}, \; 0 \leq k < n, \tag{5}$$

has the unique simple positive dominant zero $\lambda^{(n)}(p)$, i.e., all other zeros of polynomial (4) have moduli strictly smaller than $|\lambda^{(n)}(p)|$.

In 1997, Dubeau et al. [4] proved that, if the characteristic polynomial of an arbitrary linear recurrence is asymptotically simple, the limit of the ratios of the successive terms of the (not necessarily integer) sequence generated by the recurrence exists for at least one set of the initial conditions $\{a_{n-1} = 1, a_k = 0 \,|\, 0 \leq k < n-1\}$. Moreover, they showed that, if this limit exists for a given set of initial conditions, the limit coincides with the dominant zero of the characteristic polynomial, i.e., for a given p, n, and $\{a_k \in \mathbb{R} \,|\, 0 \leq k < n\}$,

$$\Phi^{(n)}(p) \equiv \lim_{k\to\infty} F_{k+1}^{(n)}(p)/F_k^{(n)}(p) = \lambda^{(n)}(p), \quad k > k_0. \tag{6}$$

In particular, for a given m and n, the (m, n)-anacci constant $\Phi^{(n)}(m)$ is equal to the dominant zero $\lambda^{(n)}(m)$ of the polynomial $P_m^{(n)}(\lambda) \equiv \lambda^n - m(\lambda^{n-1} + \cdots + 1)$.

We derive the analytic representation of the set $\{\Phi^{(n)}(m) \,|\, m, n \in \mathbb{N}\}$ of the (m, n)-anacci constants by proving there exist a continuous function $\overline{\mathbb{R}}_+^2 \ni (p, q) \to \lambda(p, q) \in \overline{\mathbb{R}}_+$ such that:

a. for any p and n, $\lambda(p, n) = \lambda^{(n)}(p)$, i.e., in particular, $\lambda(m, n) = \Phi^{(n)}(m)$;

b. for any $(p, q) \in \mathbb{R}_+^2$ such that $p \cdot q \neq 1$, $\lambda(p, q)$ is of class C^∞;

c. the restriction $\lambda(p, q)|_\ell$ of $\lambda(p, q)$ to any half-line $\ell \subset \mathbb{R}_+^2$ is strictly increasing;

d. for any $p > 0$ and $q \geq 1$, $p \leq \lambda(p, q) < p+1$, in particular, $m \leq \Phi^{(n)}(m) < m+1$;

e. for any $p \in \mathbb{R}_+$, $\lim_{q\to\infty} \lambda(p, q) = p+1$, in particular, $\lim_{n\to\infty} \Phi^{(n)}(m) = m+1$.

The latter result generalizes the well-known fact regarding the limit of the n-anacci constants sequence: $\lim_{n\to\infty} \Phi^{(n)}(1) = 2$, cf., e.g., [3, 6].

Moreover, the result stated in (d) implies that the set of the (m, n)-anacci constants is totally ordered as follows: if $n_2 > n_1$, then $\Phi^{(n_2)}(m_2) > \Phi^{(n_1)}(m_1)$ for any m_2 and m_1, whereas $\Phi^{(n)}(m_2) > \Phi^{(n)}(m_1)$ if $m_2 > m_1$.

We have shown [15] how to construct geometric representations of the (m, n)-anacci constants correlated with this order using dilations of convex compact sets in n-dimensional Euclidean spaces.

2 Analytic representation of the (m,n)-anacci constants

The limits $\Phi^{(n)}(p)$, $p \in \mathbb{R}_+$, are also zeros of the polynomials

$$Q_p^{(n)}(\lambda) \equiv \lambda^{n+1} - (p+1)\lambda^n + p = (\lambda - 1)P_p^{(n)}(\lambda). \tag{7}$$

We derive the analytic representation of the set $\left\{\Phi^{(n)}(p) \,|\, p\in\mathbb{R}_+,\ n\in\mathbb{N}\right\}$ using the function

$$Q(\lambda,p,q)\equiv\lambda^{q+1}-(p+1)\lambda^{q}+p, \quad \lambda,p,q\in\mathbb{R}_+. \tag{8}$$

The function $Q(\lambda,p,q)$ equals 0 at the plane $\lambda=1$ and at the zeros $\lambda^{(n)}(p)$. In particular, the restriction $Q(\lambda,1,q)$ of $Q(\lambda,p,q)$ to the plane $p=1$ includes the sequence of the n-anacci constants $\left(\Phi^{(n)}(1)\right)_{n=1}^{\infty}$.

Figure 1 depicts the restriction $Q(\lambda,1,q)$ and the zero function $O(\lambda,q)\equiv 0$. The functions intersect along the zero line $O(1,q)$ and the zero curve, say $\lambda_1(q)=0$, that is defined implicitly by the equation $Q(\lambda,1,q)=0$ and that includes the sequence $\left(\Phi^{(n)}(1)\right)_{k=1}^{\infty}$.

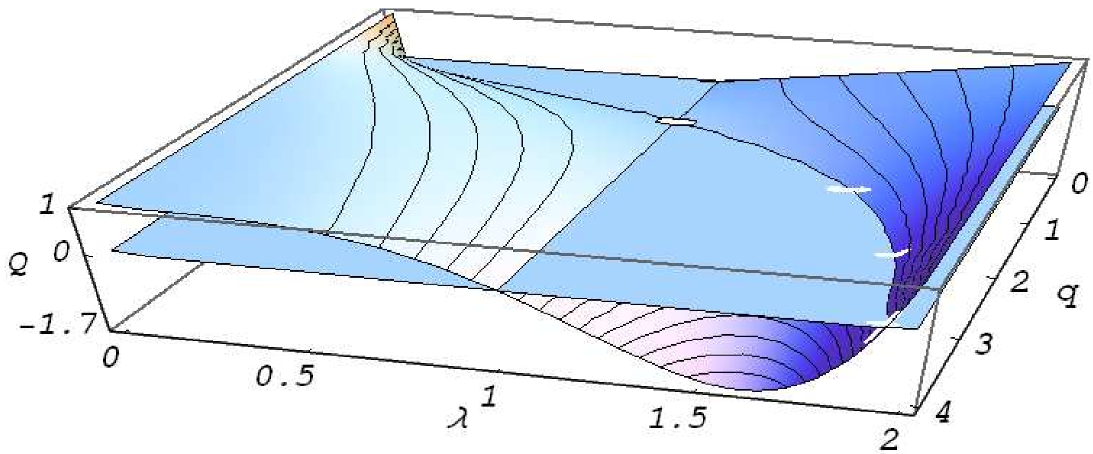

Figure 1: The restriction $Q(\lambda,1,q)$ of the function $Q(\lambda,p,q)$ and the function $O(\lambda,q)\equiv 0$, $0<\lambda\leq 2$ and $0<q\leq 4$, intersecting along the zero line $O(1,q)$ and the zero curve $\lambda_1(q)$. The locations of n-anacci constants $\Phi^{(n)}(1)$ with $1\leq n\leq 4$ are marked by white ovals.

The equation $Q(\lambda,a,q)=0$, $a\in\mathbb{R}_+$, defines the zero curve $\lambda_a(q)$. If $a=m\in\mathbb{N}$, the zero curve $\lambda_m(q)$ contains the sequence of the (m,n)-anacci constants $\left(\Phi^{(n)}(m)\right)_{n=1}^{\infty}$.

The next two theorems establish the analytic representation of the set of the (m,n)-anacci constants $\left\{\Phi^{(n)}(m) \,|\, m,n\in\mathbb{N}\right\}$ as well as of all roots of the function $Q(\lambda,p,q)$.

Theorem 1. *For any given $p,q\in\mathbb{R}_+$, $p\cdot q\neq 1$, the function $Q(\lambda,p,q)$ of one variable $\lambda\in\mathbb{R}_+$ has the unique root $\lambda(p,q)\neq 1$, whereas if $p\cdot q=1$, its unique root $\lambda(p,1/p)=1$. Moreover,*

$$1<(p+1)\,q/(q+1)<\lambda(p,q) \quad \textit{iff} \quad p\cdot q>1, \tag{9}$$

$$0<\lambda(p,q)<(p+1)\,q/(q+1)<1 \quad \textit{iff} \quad p\cdot q<1, \tag{10}$$

$$\textit{and} \quad \lambda(p,q)<p+1 \quad \textit{for any} \quad q\in\mathbb{R}_+. \tag{11}$$

Proof. The partial derivative of function (8) with respect to λ is given by

$$\partial Q(\lambda,p,q)/\partial\lambda = \lambda^{q-1}\big(\lambda(q+1)-(p+1)q\big). \tag{12}$$

Thus, for any $p,q\in\mathbb{R}_+$, the function $Q(\lambda,p,q)$ of the variable $\lambda\in\mathbb{R}_+$ has one local minimum at the point

$$\lambda_{min}(p,q) = (p+1)q/(q+1). \tag{13}$$

Formula (13) implies that the minimum is assumed at $\lambda_{min}=1$ iff $p\cdot q=1$. Because, function (8) equals zero at $\lambda=1$, therefore, if $p\cdot q=1$, 1 is the only root of the function $Q(\lambda,p,q)$ of the variable λ, cf. the left most white oval $\Phi^{(1)}(1)=1$ in Figure 1.

If $p\cdot q\neq 1$, there exists a second *positive* root $\lambda(p,q)$ of $Q(\lambda,p,q)$ besides 1 (if $\lambda_{min}<1$, the existence of the positive root is implied by the fact that $Q(0,p,q)=p>0$), cf. Figure 1.

Moreover, for any p and q, the following holds:

$$1<\lambda_{min}(p,q)<\lambda(p,q) \ \text{ iff } \ p\cdot q>1, \tag{14}$$

$$\text{and if } \ p\cdot q>1, \ Q(\lambda,p,q)<0 \ \text{ iff } \ 1<\lambda<\lambda(p,q); \tag{15}$$

$$0<\lambda(p,q)<\lambda_{min}(p,q)<1 \ \text{ iff } \ p\cdot q<1, \tag{16}$$

$$\text{and if } \ p\cdot q<1, \ Q(\lambda,p,q)<0 \ \text{ iff } \ \lambda(p,q)<\lambda<1; \tag{17}$$

$$\lambda(p,q)=\lambda_{min}(p,q)=1 \ \text{ iff } \ p\cdot q=1, \tag{18}$$

$$\text{and if } \ p\cdot q=1, \ Q(\lambda,p,q)>0 \ \text{ iff } \ \lambda\neq 1. \tag{19}$$

Since we have $Q(p+1,p,q)=p>0$, formulas (14)–(19) imply that $\lambda(p,q)<p+1$ for any $q\in\mathbb{R}_+$.

$\square$

Theorem 2. *(i) The assignment $\mathbb{R}^2_+\ni(p,q)\to\lambda(p,q)\in\mathbb{R}_+$ defines a continuous function such that, for any $(p,n)\in\mathbb{R}_+\times\mathbb{N}$, $\lambda(p,n)=\lambda^{(n)}(p)$ holds, i.e., $\lambda(m,n)=\Phi^{(n)}(m)$;*

(ii) if $p\cdot q\neq 1$, the function $\lambda(p,q)$ is of class C^∞;

(iii) the restriction $\lambda(p,q)|_\ell$ of $\lambda(p,q)$ to any half-line $\ell\subset\mathbb{R}^2_+$ is strictly increasing;

(iv) for any $p>0$ and $q\geq 1$, $p\leq\lambda(p,q)$, i.e., $m\leq\Phi^{(n)}(m)<m+1$;

(v) for any $p\in\mathbb{R}_+$, $\lim_{q\to\infty}\lambda(p,q)=p+1$, i.e., $\lim_{n\to\infty}\Phi^{(n)}(m)=m+1$;

(vi) the plane $\mathbb{R}^2_+ \ni (p,q) \to \mathcal{P}(p,q) \equiv p+1$ majorizes the function $\lambda(p,q)$ from above and is the asymptotic plane for $\lambda(p,q)$;

(vii) for any $p_0 \in \mathbb{R}_+$, $\lim_{(p,q)\to(p_0,0)} \lambda(p,q) = 0$, and for any $q_0 \in \mathbb{R}_+$, $\lim_{(p,q)\to(0,q_0)} \lambda(p,q) = 0$, i.e., the open domain $\mathbb{R}^2_+$ of $\lambda(p,q)$ can be extended to the closed domain $\overline{\mathbb{R}}^2_+$.

Proof. (i) It follows from Theorem 1 that the assignment defines a function. If, for (p_0, q_0) with $p_0 \cdot q_0 = 1$, $\lim_{(p,q)\to(p_0,q_0)} \lambda(p,q) \neq 1 = \lambda(p_0,q_0)$, then we have a contradiction with (19) due to the continuity of the function $Q(\lambda,p,q)$ and the fact that $Q\big(\lambda(p,q),p,q\big) = 0$.

Thus, $\lambda(p,q)$ is continuous at (p_0,q_0) with $p_0 \cdot q_0 = 1$. The continuity of $\lambda(p,q)$ at (p_0,q_0) with $p_0 \cdot q_0 \neq 1$ is implied by part (ii). The definition of $\lambda(p,q)$ assures that $\lambda(p,n) = \lambda^{(n)}(p)$.

(ii) The equation $Q\big(\lambda(p,q),p,q\big) = 0$ defines the function $\lambda(p,q)$ implicitly. It follows from formulas (12) and (13) that the partial derivative $\partial Q(\lambda,p,q)/\partial\lambda$ is continuous and equals 0 iff $\lambda(p,q) = \lambda_{min}(p,q)$, i.e., according to (19) iff $p \cdot q = 1$.

Thus, the implicit function theorem implies that, if $p \cdot q \neq 1$, the function $\lambda(p,q)$ is continuously differentiable and the following holds:

$$\frac{\partial \lambda(p,q)}{\partial p} = \frac{-1}{\frac{\partial Q\left(\lambda(p,q),p,q\right)}{\partial \lambda}} \cdot \frac{\partial Q\big(\lambda(p,q),p,q\big)}{\partial p} = \frac{1 - \big(\lambda(p,q)\big)^q}{[\lambda(p,q)(q+1) - (p+1)q]\,\big(\lambda(p,q)\big)^{q-1}}, \tag{20}$$

$$\frac{\partial \lambda(p,q)}{\partial q} = \frac{-1}{\frac{\partial Q\left(\lambda(p,q),p,q\right)}{\partial \lambda}} \cdot \frac{\partial Q\big(\lambda(p,q),p,q\big)}{\partial q} = \frac{[p+1-\lambda(p,q)]\,\big(\lambda(p,q)\big)^q \ln q}{[\lambda(p,q)(q+1) - (p+1)q]\,\big(\lambda(p,q)\big)^{q-1}}. \tag{21}$$

Since the function $\lambda(p,q)$ is continuously differentiable if $p \cdot q \neq 1$, it follows from formulas (20) and (21) that all partial derivatives of $\lambda(p,q)$ of an arbitrary order exist and are continuous. Consequently, the function $\lambda(p,q)$ is of class C^∞ if $p \cdot q \neq 1$.

(iii) If $p \cdot q > 1$ (respectively $p \cdot q < 1$), the denominator and, according to formulas (9)–(11), both numerators in (20) and (21) are positive (respectively negative). Thus, the directional derivative of $\lambda(p,q)$ along ℓ is positive, i.e., $\lambda(p,q)|_\ell$ is strictly increasing if $p \cdot q \neq 1$. It follows from Theorem 1 that $\lambda(p,q)|_\ell$ is also strictly increasing at $p \cdot q = 1$.

(iv) Formula (4) implies that $\lambda(p,1) = p$, which is smaller than $\lambda(p,q)$, $q > 1$, since for a fixed p, $\lambda(p,q)$ is strictly increasing.

(v) The convergence of $\lambda(p,q)$ to $p+1$ follows from (9) and (11).

(vi) The assertion follows from part (v).

(vii) The first limit equals 0 due to formulas (13) and (16). Definition (8) implies that the second limit is equal to either 0 or 1. The latter is impossible due to (13) and (16).

□

The lower bound $(p+1)\,q/(q+1)$ for $\lambda(p,q)$ in (9) predicts, e.g., that the golden ratio $\Phi \equiv \lambda(1,2)$ is just greater than $4/3$. In the case where $p=1$ and $q=n\in\mathbb{N}$, Wofram [16] showed that $2(1-1/2^n)<\lambda(1,n)$, which implies that the n-anacci constants $\Phi^{(n)}(1)=\lambda(1,n)$ are close to the limit 2 already when n is small.

The next lemma provides a lower bound for $\lambda(p,q)$ that is independent of q, which shows that, for progressively larger values of p, the roots $\lambda(p,q)$ are closer and closer to the limit $p+1$. Consequently, when the weight m increases, the (m,n)-anacci constants $\Phi^{(n)}(m)$ become closer and closer to the limit $m+1$.

Lemma 3. *For any $q\geq 2$ and $p>1/\Phi$, the following holds:*

$$p+1-1/(p+1)<\lambda(p,q). \tag{22}$$

Moreover, the lower bounds (9) and (22) satisfy

$$(p+1)\,q/(q+1)\leq p+1-1/(p+1) \quad \textit{iff} \quad q\leq(p+1)^2-1. \tag{23}$$

Proof. Since $Q_p^{(2)}\big(p+1-1/(p+1)\big)=-p\,(p^2+p-1)/(p+1)^2<0$ if $p>1/\Phi$, formula (14) implies that $p+1-1/(p+1)<\lambda(p,2)$. For $q>2$, $\lambda(p,2)<\lambda(p,q)$ since $\lambda(p,q)$ is strictly increasing for a fixed p, cf. part (iii) of Theorem 2. Formula (23) follows from a simple calculation. $\square$

Figure 2 shows the restrictions $\lambda(a,q)$ of $\lambda(p,q)$ to the lines $p=a$, which are the same as the zero curves $\lambda_a(q)$ introduced above. Each restriction starts at $\lambda(a,1)=a$, increases asymptotically to $a+1$, and is less than $1/(a+1)$ from $a+1$ in the region to the right of the line $q=2$. The restrictions $\lambda(m,q)$, $m\in\mathbb{N}$, include the (m,n)-anacci constants $\Phi^{(n)}(m)$.

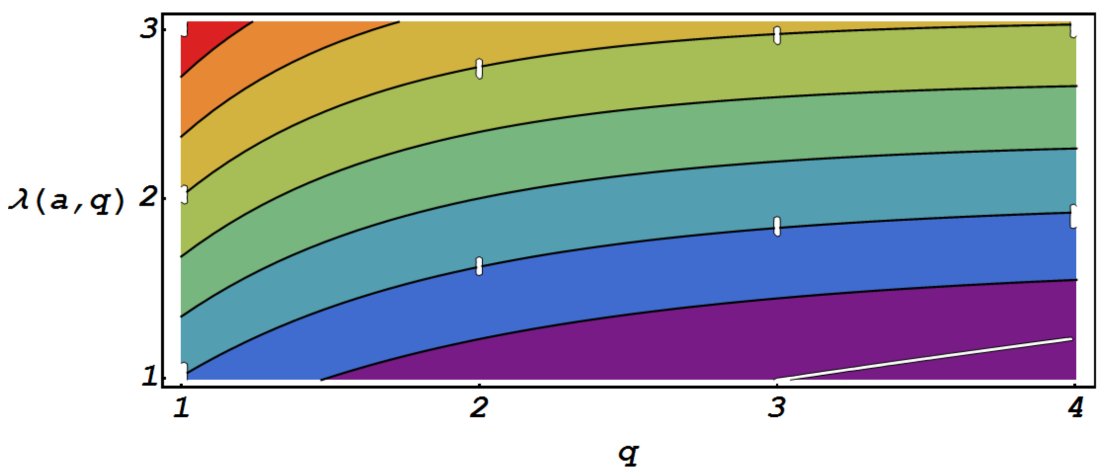

Figure 2: The restrictions $\lambda(a,q)$, $a=\frac{2}{3},1,\ldots,2\frac{2}{3}$, $1\leq q\leq 4$, of the function $\lambda(p,q)$. The locations of the (m,n)-anacci constants $\Phi^{(n)}(m)$, $m=1,2$, $1\leq n\leq 4$, are marked by white ovals. The lower bound (22) exceeds the lower bound (9) in the region above the white curve $q=(p+1)^2-1$ that goes through the points $(3,1)$ and $(4,2/\Phi)$.

The function $\lambda(p,q)$ is also defined implicitly by the continuous function

$$p(\lambda,q)\equiv\lambda^q(\lambda-1)/(\lambda^q-1) \quad \text{if} \quad \lambda\neq 1 \quad \text{and} \tag{24}$$

$$p(\lambda,q)\equiv 1/q \quad \text{if} \quad \lambda=1, \tag{25}$$

which is of class C^∞ if $\lambda\neq 1$. For $q=n\in\mathbb{N}$, the function defined by (24)–(25) takes the form

$$p(\lambda,n)=\frac{\lambda^n}{\sum_{k=0}^{n-1}\lambda^k}. \tag{26}$$

Figure 3 depicts the asymptotic plane $\mathcal{P}(p,q)\equiv p+1$ and the function $\lambda(p,q)$ generated by formulas (24) and (25). The thick curves going up the graph of the function $\lambda(p,q)$ are the restrictions $\lambda(a,q)$ with $a=\frac{1}{3},\frac{2}{3}\ldots,2\frac{2}{3},3$. They include the (m,n)-anacci constants $\Phi^{(n)}(m)$, $1\leq m\leq 3$, $1\leq n\leq 4$. The thin curves increasing from left to right are the restrictions $\lambda(p,q)|_\ell$. The horizontal thin curves are the level curves $\lambda(p,q)=c$ with $c=\frac{1}{2},1\ldots,3\frac{1}{2},4$.

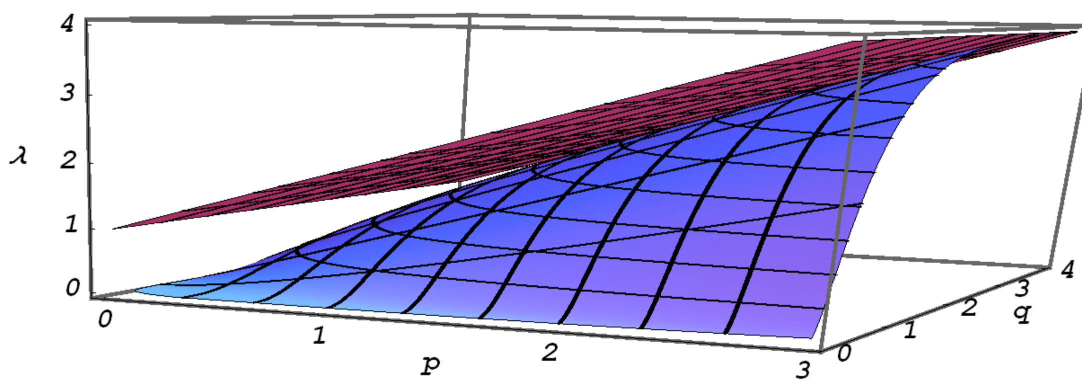

Figure 3: The function $\lambda(p,q)$ and the asymptotic plane $\mathcal{P}(p,q)$, $0\leq p\leq 3$, $0\leq q\leq 4.1$.

Theorems 1 and 2 together with formulas (22) and (26) imply the following properties of the (m,n)-anacci constants $\Phi^{(n)}(m)$.

Theorem 4. *(i) The sequences $\left(\Phi^{(n)}(m)\right)_{n=1}^\infty$ with a fixed $m\in\mathbb{N}$ and $\left(\Phi^{(n)}(m)\right)_{m=1}^\infty$ with a fixed $n\in\mathbb{N}$, as well as the sequences $\left(\Phi^{(n)}(kn)\right)_{n=1}^\infty$ and $\left(\Phi^{(km)}(m)\right)_{m=1}^\infty$ with a fixed $k\in\mathbb{N}$ are strictly increasing.*

(ii) For any $n\in\mathbb{N}$, the sequence $\left(\frac{m+1}{m}\,\Phi^{(n)}(m)\right)_{m=1}^\infty$ is strictly increasing.

(iii) If $n>1$, the sequence $\left(\frac{1}{m}\,\Phi^{(n)}(m)\right)_{m=1}^\infty$ is strictly decreasing to 1. If $n=1$, $\frac{1}{m}\,\Phi^{(1)}(m)=1$ for any m.

(iv) The triple $\big(\lambda(p,q),p,q\big)\in\mathbb{N}^3$ iff $\big(\lambda(p,q),p,q\big)=(m,m,1)$, $m\in\mathbb{N}$, i.e., the (m,n)-anacci constants $\Phi^{(n)}(m)$ are integer iff $n=1$.

(v) If the function $\lambda(p,n)=m\in\mathbb{N}$ for some $1<n\in\mathbb{N}$, then p is rational and $(m–1)<p<m$.

Proof. (i) The assertions follow from part (iii) of Theorem 2.

(ii) If $n=1$, the sequence $\left(\frac{m+1}{m}\,\Phi^{(1)}(m)\right)_{m=1}^{\infty}=m+1$ since $\Phi^{(1)}(m)=m$ according to definitions (1) and (2). If $n>1$, formulas (11) and (22) imply that

$$m+1-1/(m+1)<\Phi^{(n)}(m)<m+1. \tag{27}$$

Thus, if $n>1$,

$$\frac{m+1}{m}\Phi^{(n)}(m)<\frac{(m+1)^2}{m}\leq\frac{m+2}{m+1}\Big(m+2-\frac{1}{(m+2)}\Big)<\frac{m+2}{m+1}\,\Phi^{(n)}(m+1) \tag{28}$$

is true for any m, due to formula (27) and the fact that the middle inequality in formula (28) reduces to $m\geq 1$.

(iii) If $n>1$, the following inequalities hold due to formula (27):

$$\frac{m+2}{m+1}-\frac{1}{(m+2)(m+1)}<\frac{\Phi^{(n)}(m+1)}{m+1}<1+\frac{1}{m+1}<\frac{\Phi^{(n)}(m)}{m}<1+\frac{1}{m}. \tag{29}$$

(iv) The *only if* part of the assertion follows from formula (27).

(v) The assertion is implied by formula (26) and the fact that $p<\lambda(p,n)=m<p+1$ for $n>1$, due to formulas (11) and (22), cf. also Figure 2.

□

References

[1] A. T. Benjamin and J. J. Quinn, The Fibonacci numbers–exposed more discretely, *Math. Mag.* **76** (2003), 182–192.

[2] M. Catalani, Polymatrix and generalized polynacci numbers, published electronically at http://arxiv.org/abs/math/0210201.

[3] F. Dubeau, On r-generalized Fibonacci numbers, *Fibonacci Quart.* **27** (1989), 221–228.

[4] F. Dubeau et al., On weighted r-generalized Fibonacci sequences, *Fibonacci Quart.* **35** (1997), 102–110.

[5] D. C. Fielder, Certain Lucas-like sequences and their generation by partitions of numbers, *Fibonacci Quart.* **5** (1967), 319–324.

[6] I. Flores, Direct calculation of k-generalized Fibonacci numbers, *Fibonacci Quart.* **5** (1967), 259–266.

[7] A. F. Horadam, A generalized Fibonacci sequence, *Amer. Math. Monthly* **68** (1961), 455–459.

[8] A. F. Horadam, Generating functions for powers of a certain generalized sequence of numbers, *Duke Math J.* **32** (1965), 437–446.

[9] T. Khovanova, *Recursive sequences*, `http://www.tanyakhovanova.com/RecursiveSequences/RecursiveSequences.html`.

[10] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* **67** (1960), 745–752.

[11] T. D. Noe and J. V. Post, Primes in Fibonacci n-step and Lucas n-step sequences, *J. Integer Seq.* **8** (2005), Article 05.4.4.

[12] T. D. Noe et al., Fibonacci n-step number, published electronically at `http://mathworld.wolfram.com/Fibonaccin-StepNumber.html`.

[13] A. M. Ostrowski, *Solutions of Equations and Systems of Equations*, Academic Press, 1966.

[14] N. J. A. Sloane, Online Encyclopedia of Integer Sequences, `http://oeis.org`

[15] I. Szczyrba et al., Analytic and geometric representations of the generalized n-anacci constants, published electronically at `http://arxiv.org/abs/1409.0577`.

[16] D. A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* **36** (1998), 129–145.

[17] X. Zhu and G. Grossman, Limits of zeros of polynomial sequences, *J. Comput. Anal. Appl.* **11** (2009), 140–158.

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