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On the Binomial Identities of Frisch and Klamkin

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Abstract

In this paper we investigate two somewhat similar identities for sums of ratios of binomial coefficients. We give several proofs, and note that the identities all follow from a hypergeometric identity of Gauss. Inverse identities are also given.

1 Introduction

Klamkin [3] stated the following identity in a letter to Gould in 1966:

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{n+a}{k+b}} = \frac{a+1+n}{(a+1)\binom{a}{b}} \tag{1}$$

Identity (4.6) in Gould's book [2] is a special case of this, corresponding to $a = 2x$ and $b = x$.

Frisch [1], in his dissertation in 1926, gave the curious formula

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{\binom{b+k}{c}} = \frac{c}{n+c} \frac{1}{\binom{n+b}{b-c}}, \qquad b \geq c > 0. \tag{2}$$

This was cited and proved by Netto [4, pp. 337–338] and is tabulated as Formula (4.2) in Gould's book [2].

(It is interesting to note that Frisch's research laid the foundations for modern econometrics theory and micro- and macro-economics, work for which he later received the Nobel Prize.)

Klamkin's identity is actually a special case of the identity

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{x}{k+b}} = \frac{x+1}{(x-n+1)\binom{x-n}{b}}. \tag{3}$$

Here is how we obtain Eq. (3). Let $t_k = \frac{\binom{n}{k}}{\binom{x}{k+b}}$. Then

$$\frac{t_{k+1}}{t_k} = \frac{\binom{n}{k+1}\binom{x}{k+b}}{\binom{x}{k+1+b}\binom{n}{k}} = \frac{(n-k)(k+1+b)}{(k+1)(x-k-b)}.$$

Therefore,

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{x}{k+b}} = \frac{1}{\binom{x}{b}}\, {}_2F_1\left[\begin{matrix} 1+b, & -n \\ -x+b & \end{matrix} \,|1\right]. \tag{4}$$

Applying Gauss's ${}_2F_1$ formula, namely,

$${}_2F_1\left[\begin{matrix} a, & b \\ c & \end{matrix} \,|1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)},$$

we have

$$\begin{aligned}
\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{x}{k+b}} &= \frac{1}{\binom{x}{b}} \cdot \frac{\Gamma(-x+n-1)\Gamma(-x+b)}{\Gamma(-x-1)\Gamma(-x+b+n)} \\
&= \frac{b!}{x(x-1)\dots(x-b+1)} \cdot \frac{(x+1)x(x-1)\dots(x-n+3)(x-n+2)}{(x-b)(x-b-1)\dots(x-n-b+2)(x-n-b+1)} \\
&= \frac{b!(x+1)\prod_{j=0}^{n-2}(x-j)}{\prod_{j=0}^{n+b-1}(x-j)} \\
&= \frac{b!(x+1)}{\prod_{j=n-1}^{n+b-1}(x-j)} \\
&= \frac{x+1}{x-n+1} \cdot \frac{b!}{(x-n)(x-n-1)\dots(x-n-b+1)} \\
&= \frac{x+1}{x-n+1} \cdot \frac{1}{\binom{x-n}{b}}.
\end{aligned}$$

In this proof we assumed that $n \geq 2$. If $n = 0$, Eq. (1) is obviously true, while if $n = 1$, we obtain easily verified identity

$$\frac{1}{\binom{x}{b}} + \frac{1}{\binom{x}{b+1}} = \frac{x+1}{x\binom{x-1}{b}}.$$

2 Proof of Klamkin's and Frisch's formulas

We shall now prove Klamkin's formula directly from Formula (7.1) in Gould [2], namely

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{z}{k}}{\binom{y}{k}}=\frac{\binom{y-z}{n}}{\binom{y}{n}}, \tag{5}$$

and the easy binomial identity

$$\binom{x}{k+b}\binom{k+b}{k}=\binom{x}{b}\binom{x-b}{k} \tag{6}$$

Indeed, using these we have

$$\begin{aligned}\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{x}{k+b}} &= \frac{1}{\binom{x}{b}}\sum_{k=0}^{n}\binom{n}{k}\frac{\binom{k+b}{k}}{\binom{x-b}{k}} \\ &= \frac{1}{\binom{x}{b}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{-1-b}{k}}{\binom{x-b}{k}} \\ &= \frac{1}{\binom{x}{b}}\frac{\binom{x+1}{n}}{\binom{x-b}{n}}=\frac{x+1}{(x+1-n)\binom{x-n}{b}}\end{aligned}$$

by Formula (7.1), and our proof is complete. We remark that Formula (7.1) is equivalent to using the Gauss ${}_2F_1$ formula.

Simple binomial inversion yields the inverse Klamkin identity

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{x+1}{(x+1-k)\binom{x-k}{b}}=\frac{1}{\binom{x}{n+b}}. \tag{7}$$

We now proceed in a similar way to prove Frisch's identity. It is easy to verify the binomial identity

$$\frac{1}{\binom{k+b}{c}}=\frac{1}{\binom{b}{c}}\frac{\binom{c-b-1}{k}}{\binom{-b-1}{k}},$$

so that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{k+b}{c}}=\frac{1}{\binom{b}{c}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{c-b-1}{k}}{\binom{-b-1}{k}}.$$

Upon applying Eq. (5), we find the sum equals

$$\frac{1}{\binom{b}{c}}\frac{\binom{-c}{n}}{\binom{-b-1}{n}}=\frac{c}{n+c}\frac{1}{\binom{n+b}{b-c}},$$

as desired to show.

This new proof of Frisch's identity should be compared to Gould's original proof [5, Section 7.2], a two-page calculation involving an application of Melzak's formula.

An inverse Frisch identity then follows by simple binomial inversion, and we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{c}{k+c}\frac{1}{\binom{k+b}{b-c}}=\frac{1}{\binom{n+b}{c}}. \tag{8}$$

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