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Parapermanents of Triangular Matrices and Some General Theorems on Number Sequences

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Abstract

Applying the apparatus of triangular matrices, this paper determines general relations between the terms of the sequences generated by linear homogeneous recurrence equations. We single out, in particular, the class of normal linear recurrence equations, for which the corresponding number sequences have some interesting number-theoretic properties.

1 Introduction

At present, quite a few general theorems have been proved for linear recurrence relations. In fact, their content is similar to corresponding theorems from the theory of linear homogeneous and nonhomogeneous systems of equations with constant coefficients and the theory of homogenous and nonhomogeneous ordinary differential equations with constant coefficients. But so far there are no general methods of solution even for linear homogenous recurrence equations with variable coefficients.

Linear recurrence sequences with constant coefficients are efficiently used in different fields of mathematics (see, for example, [7, 25, 28, 30]). It is necessary, however, to find the exact roots of a corresponding characteristic equation, which can cause considerable difficulties.

The theoretical research tool for studying linear recurrent equations in this paper is parapermanents of triangular matrices [34, 35]. The paper aims at applying parapermanents of triangular matrices to solution of linear recurrence equations, determination of general relations between the terms of the sequences generated by these equations, and isolation of the class of normal linear recurrence equations, the number sequences of which have some general number-theoretic properties. The method makes it possible to investigate linear recurrence equations without solving corresponding characteristic equations. Here, the bijection is established between the number sequence defined by linear recurrence relation, its generating function and its n-th term given as a parapermanent of some n-th order triangular matrix.

2 Preliminaries and notations

A triangular table of numbers

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \cdots & \cdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{1}$$

is called *a triangular matrix*, and the number n is called its order [34, 35]. Note that a triangular matrix (1) is not a triangular matrix in the usual sense of this term as it is not a square matrix.

To every element a_{ij} of the triangular matrix (1), we correspond $(i-j+1)$ elements a_{ik}, $k = j, j+1, \dots, i$, which are called *derived elements*, generated by element a_{ij}.

The product of all derived elements of the matrix (1) generated by element a_{ij} is denote by $\{a_{ij}\}$ and called the *factorial product* of the element a_{ij}, i.e., $\{a_{ij}\} = a_{ij}\, a_{i,j+1} \cdots a_{ii}$.

To each element a_{ij} of a matrix (1) we associate the triangular table of elements of matrix A that has a_{ij} in the bottom left corner. We call this table a *corner* of the matrix and denote it by $R_{ij}(A)$. Corner $R_{ij}(A)$ is a triangular matrix of order $(i-j+1)$, and it contains only elements a_{rs} of matrix (1) whose indices satisfy the inequalities $j \leqslant s \leqslant r \leqslant i$.

The *parapermanent* $\operatorname{pper}(A)$ of a triangular matrix (1) is the number

$$\operatorname{pper}(A) = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \cdots & \cdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_n = \sum_{r=1}^{n} \sum_{p_1+\cdots+p_r=n} a_{p_1,1} \prod_{s=2}^{r} \{a_{p_1+\cdots+p_s,\, p_1+\cdots+p_{s-1}+1}\}, \tag{2}$$

where $p_1, p_2, \ldots, p_r$ are positive integers, $\{a_{ij}\}$ is the factorial product of the element a_{ij} of the matrix A. We shall also denote a parapermanent of the matrix (1) briefly by $[a_{ij}]_{1\le j\le i\le n}$.

The rectangular table $T(i)$ of elements of the triangular matrix (1) is *inscribed* in matrix (1) if one of its vertex coincides with the element a_{n1}, and its opposite one coincides with the element a_{ii}, $i = 1, 2, \ldots, n$. [37]

The *parapermanent algebraic complement* P_{ij} to the factorial product $\{a_{ij}\}$ of the element a_{ij} of the matrix (1) is the number [34]

$$P_{ij} = \operatorname{pper}(R_{j-1,1}) \cdot \operatorname{pper}(R_{n,i+1}), \tag{3}$$

where $R_{j-1,1}$ and $R_{n,i+1}$ are corners of the matrix (1), and $\operatorname{pper}(R_{01}) \equiv 1$, $\operatorname{pper}(R_{n,n+1}) \equiv 1$.

Theorem 1. [34] (Decomposition of a parapermanent by elements of an inscribed rectangular table). *Let A be a triangular matrix (1), and $T(i)$ be some inscribed rectangular table. Then*

$$\operatorname{pper}(A) = \sum_{s=1}^{i}\sum_{r=i}^{n} \{a_{rs}\}\, P_{rs}, \tag{4}$$

where P_{rs} is a parapermanent algebraic complement to the factorial product of element a_{rs}, which belongs to $T(i)$.

Corollary 2. *For $i = n$ we get decomposition of the parapermanent* $\operatorname{pper}(A)$ *by elements of the last row:*

$$\operatorname{pper}(A) = \sum_{s=1}^{n} \{a_{ns}\} \cdot \operatorname{pper}(R_{s-1,1}). \tag{5}$$

Lemma 3. [35] *The system of equations*

$$x_i = a_{ii}b_1x_{i-1} + a_{ii}a_{i,i-1}b_2x_{i-2} + \cdots + a_{ii}\cdots a_{i1}b_i, \quad i = 1, 2, \ldots, n,$$

has the solution $x_i = \left[a_{sr}\frac{b_{s-r+1}}{b_{s-r}}\right]_{1\leqslant r\leqslant s\leqslant i}$, where $b_0 \equiv 1$.

Lemma 4. [35] *The system of equations*

$$\left[\bigl(1 + \delta_{sr}(x_s - 1)\bigr)\frac{a_{s-r+1}}{a_{s-r}}\right]_{1\leqslant r\leqslant s\leqslant i} = b_i, \quad i = 1, 2, \ldots, n,$$

where δ_{sr} is the Kronecker symbol, $a_0 = 1$, has the solution

$$x_i = \frac{b_i}{a_ib_0 + a_{i-1}b_1 + a_{i-2}b_2 + \cdots + a_1b_{i-1}} \quad (b_0 \equiv 1).$$

3 Parapermanents and k-th order linear recurrence equations

Let us consider the linear recurrence equation of k-th order

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k}, \tag{6}$$

with initial conditions

$$u_0 = 1,\ \ u_1 = b_1,\ \ u_2 = b_2,\ \ldots,\ \ u_{k-1} = b_{k-1}. \tag{7}$$

If it is difficult to solve a characteristic equation of the recurrence equation (6) then the following proposition can be useful, as it significantly complements Stanley's theorem ([29, p. 202]):

Theorem 5. [36]. *For the sequence $(u_n)_{n\geq 0}$ the following three equations are equivalent:*

(1) *the linear recurrence equation* (6) *with initial conditions* (7)*;*

(2)

$$u_n = \begin{bmatrix} a_1c_1 & & & & & & & \\ \frac{a_2}{a_1} & a_1c_2 & & & & & & \\ \cdots & \cdots & \ddots & & & & & \\ \frac{a_{k-1}}{a_{k-2}} & \frac{a_{k-2}}{a_{k-3}} & \cdots & a_1c_{k-1} & & & & \\ \frac{a_k}{a_{k-1}} & \frac{a_{k-1}}{a_{k-2}} & \cdots & \frac{a_2}{a_1} & a_1 & & & \\ 0 & \frac{a_k}{a_{k-1}} & \cdots & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \\ 0 & 0 & \cdots & 0 & \frac{a_k}{a_{k-1}} & \cdots & \frac{a_2}{a_1} & a_1 \end{bmatrix}_n \tag{8}$$

with $u_0 = 1$, $a_i \neq 0$ for $i = 1, 2, \ldots, k-1$, where

$$c_i = \frac{b_i}{a_i b_0 + a_{i-1} b_1 + \cdots + a_1 b_{i-1}}, \quad i = 1, 2, \ldots, k-1, \tag{9}$$

with $b_0 = 1$;

(3)

$$1 + \sum_{i=1}^{\infty} u_i z^i = \frac{1 + b_1\left(1 - \frac{1}{c_1}\right) z + b_2\left(1 - \frac{1}{c_2}\right) z^2 + \cdots + b_{k-1}\left(1 - \frac{1}{c_{k-1}}\right) z^{k-1}}{1 - a_1 z - a_2 z^2 - \cdots - a_k z^k}, \tag{10}$$

where $c_i \neq 0$ for $i = 1, 2, \ldots, k-1$.

The theorem is established in [36]; we just give the proof for convenience of the reader.

Proof. Equivalence of the recurrence equation (6), (7) and the equation (8) follows from the fact that numbers $c_1, c_2, \ldots, c_{k-1}$, according to Lemma 4, are solutions of the system of equations

$$\left[\left(1+\delta_{sr}(c_s-1)\right)\frac{a_{s-r+1}}{a_{s-r}}\right]_{1\leqslant r\leqslant s\leqslant i} = b_i, \quad i=1,2,\ldots,k-1,$$

and the decomposition of the parapermanent in (8) for $n \geq k$ by elements of the last row (see Corollary 5).

Let us now prove the equivalence of the recurrence equation (6) with initial conditions (7) and the equation (10). To do this, we use the equality

$$\begin{gathered}\left(1-\sum_{i=1}^{k} a_i x^i\right)\left(1+\sum_{j=1}^{\infty} u_j x^j\right) = \\ = 1+(u_1-a_1)x+(u_2-a_1u_1-a_2)x^2+(u_3-a_1u_2-a_2u_1-a_3)x^3+\cdots \\ \cdots+(u_{k-1}-a_1u_{k-2}-a_2u_{k-3}-\cdots-a_{k-2}u_1-a_{k-1})x^{k-1}+ \\ +\sum_{i=k}^{\infty}(u_i-a_1u_{i-1}-a_2u_{i-2}-\cdots-a_ku_{i-k}).\end{gathered}$$

In the right hand of above equality, we shall replace u_i with $b_1, b_2, \ldots, b_{k-1}$, and factor out b_1 from the first parentheses, b_2 from the second parentheses, and so on. If we denote

$$c_1=\frac{b_1}{a_1}, \quad c_2=\frac{b_2}{a_2+a_1b_1}, \quad c_3=\frac{b_3}{a_3+a_2b_1+a_1b_2}, \ldots,$$

then

$$\begin{gathered}\left(1-\sum_{i=1}^{k} a_i x^i\right)\left(1+\sum_{j=1}^{\infty} u_j x^j\right) \\ = 1+b_1\left(1-\tfrac{1}{c_1}\right)x+b_2\left(1-\tfrac{1}{c_2}\right)x^2+\cdots+b_{k-1}\left(1-\tfrac{1}{c_{k-1}}\right)x^{k-1}+ \\ +\sum_{i=k}^{\infty}\left(u_i-a_1u_{i-1}-a_2u_{i-2}-\cdots-a_ku_{i-k}\right).\end{gathered}$$

□

Note that Theorem 5 is also true for the recurrence equations of the form (6) with variable coefficients.

The right hand of the equality (10) is called the *generating function* of the sequence (6).

Example 6. For the Tribonacci sequence $u_n = u_{n-1}+u_{n-2}+u_{n-3}$, $u_0=u_1=u_2=1$ (A000213) from (9) we shall find the numbers $c_1=\frac{b_1}{a_1}=1$, $c_2=\frac{b_2}{a_2+a_1b_1}=\frac{1}{2}$, from (8) — the

n-th term of the sequence written with help of the parapermanent of the triangular matrix

$$u_n = \begin{bmatrix} 1 & & & & & \\ 1 & \frac{1}{2} & & & & \\ 1 & 1 & 1 & & & \\ 0 & 1 & 1 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \ddots & \\ 0 & \cdots & 0 & 1 & 1 & 1 \end{bmatrix}_n, \tag{11}$$

and from (10) — the generating function $F(x) = \frac{1-x^2}{1-x-x^2-x^3}$.

Let us consider the case when in (9) $c_1 = c_2 = \cdots = c_{k-1}$. Then from (9) we get the system

$$b_i = a_1 b_{i-1} + a_2 b_{i-2} + \cdots + a_{i-1} b_1 + a_i, \quad i = 1, 2, \ldots, k-1,$$

which, according to Lemma 3, has the solution

$$b_i = \begin{bmatrix} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \cdots & \cdots & \ddots & \\ \frac{a_i}{a_{i-1}} & \frac{a_{i-1}}{a_{i-2}} & \cdots & a_1 \end{bmatrix}_i, \quad i = 1, 2, \ldots, k, \tag{12}$$

and the initial conditions (7) for the recurrence equation (6) becomes

$$u_0 = 1, \quad u_i = \begin{bmatrix} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \cdots & \cdots & \ddots & \\ \frac{a_i}{a_{i-1}} & \frac{a_{i-1}}{a_{i-2}} & \cdots & a_1 \end{bmatrix}_i, \quad i = 1, 2, \ldots, k-1. \tag{13}$$

The initial conditions (13) for the recurrence equation (6) are called *normal initial conditions*, and the sequence generated by normal initial conditions is called *a normal sequence.*

Corollary 7. *For the sequence $(u_n)_{n\geq 0}$ the following three equations are equivalent:*

(1) *the linear recurrence equation* (6) *with the conditions* (7)*, where numbers b_i are defined by* (12);

(2)

$$u_n = \begin{bmatrix} a_1 & & & & & & \\ \frac{a_2}{a_1} & a_1 & & & & & \\ \cdots & \cdots & \ddots & & & & \\ \frac{a_k}{a_{k-1}} & \frac{a_{k-1}}{a_{k-2}} & \cdots & a_1 & & & \\ 0 & \frac{a_k}{a_{k-1}} & \cdots & \frac{a_2}{a_1} & a_1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \\ 0 & \cdots & 0 & \frac{a_k}{a_{k-1}} & \cdots & \frac{a_2}{a_1} & a_1 \end{bmatrix}_n \tag{14}$$

with $u_0 = 1$, $a_i \neq 0$ *for* $i = 1, 2, \dots, k-1$;

(3)

$$1 + \sum_{i=1}^{\infty} u_i x^i = \frac{1}{1 - a_1 x - a_2 x^2 - \cdots - a_k x^k}. \tag{15}$$

Proof. If the initial conditions of the recurrence equation (6) are normal, then it is obvious that all the numbers $c_1, c_2, \dots, c_n$ in the Theorem 5 are equal to one. So, the normal initial conditions (13) together with the equation (8) for $c_i = 1$, $i = 1, 2, \dots, k-1$, gives (14). □

Example 8. For the Jacobsthal sequence (A001045) $u_n = u_{n-1} + 2u_{n-2}$, $u_0 = u_1 = 1$, from (9) we shall find $c_1 = \frac{b_1}{a_1} = 1$, from (14) — the recurrence equation

$$u_n = \begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 0 & 2 & 1 & & & \\ 0 & 0 & 2 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \ddots & \\ 0 & \cdots & 0 & 0 & 2 & 1 \end{bmatrix}_n,$$

and from (15) — the generating function $F(x) = \frac{1}{1-x-2x^2}$.

Example 9. For Narayana's cows sequence (A000930) $u_n = u_{n-1} + u_{n-3}$, $u_0 = u_1 = u_2 = 1$, from (9) we shall find $c_1 = \frac{b_1}{a_1} = 1$, $c_2 = \frac{b_2}{a_2 + a_1 b_1} = 1$, and from (15) — the generating function $F(x) = \frac{1}{1-x-x^3}$.

Example 10. For Chebyshev polynomials of the second kind $u_n(x) = 2xu_n(x) - u_{n-1}(x)$, $u_0(x) = 1$, $u_1(x) = 2x$, similarly to the preceding example, we shall find the number $c_1 = 1$, the recurrence equation

$$u_n(x) = \begin{bmatrix} 2x & & & & & \\ -\frac{1}{2x} & 2x & & & & \\ 0 & -\frac{1}{2x} & 2x & & & \\ 0 & 0 & -\frac{1}{2x} & 2x & & \\ \cdots & \cdots & \cdots & \cdots & \ddots & \\ 0 & \cdots & 0 & 0 & -\frac{1}{2x} & 2x \end{bmatrix}_n,$$

and the generating function $\sum_{i=0}^{\infty} u_i(x) z^i = \frac{1}{1-2xz+z^2}$.

The next theorem makes it possible to reestablish an appropriate recurrence equation with initial conditions by using a known generating function.

Theorem 11. *Suppose we are given the generating function*

$$f(z) = \frac{1 + d_1 z + d_2 z^2 + \cdots + d_{k-1} z^{k-1}}{1 - a_1 z - a_2 z^2 - \cdots - a_k z^k} \tag{16}$$

of the sequence $(u_n)_{n\geq 0}$*. Then the sequence* $(u_n)_{n\geq 0}$ *satisfies the recurrence equation* (6) *with initial conditions*

$$u_0 = 1, \quad u_i = \begin{bmatrix} a_1 + d_1 & & & \\ \frac{a_2+d_2}{a_1} & a_1 & & \\ \cdots & \cdots & \ddots & \\ \frac{a_i+d_i}{a_{i-1}} & \frac{a_{i-1}}{a_{i-2}} & \cdots & a_1 \end{bmatrix}_i, \quad i = 1, 2, \ldots, k-1, \tag{17}$$

with $a_0 \neq 0$ *for* $i = 1, 2, \ldots, k-1$.

Proof. The generating function of the sequence $(u_n)_{n\geq 0}$, which satisfies the recurrence equation (6), follows from Theorem 5. According to Theorem 5, we also have the system $b_i\left(1 - \frac{1}{c_i}\right) = d_i$, $i = 1, 2, \ldots, k-1$, which, after considering (9), we shall reduce to the system

$$b_i = a_1 b_{i-1} + a_2 b_{i-2} + \cdots + a_{i-1} b_1 + a_i b_0 + d_i, \quad i = 1, 2, \ldots, k-1, \tag{18}$$

with $b_0 = 1$.

The proof of the theorem follows from the system (18) being decomposition of the right side of (17) by elements of the last row (see (5)). $\square$

4 Number-theoretic properties of sequences

Recurrences and sequences of numbers or polynomials generated by them arise in different fields of mathematics, and many problems in number theory and algebra come down to studying their properties [7, 25, 30]. But often, the properties of recurrence sequences are investigated regardless of the relevant equation, which complicates the process of studies.

Let us consider now general properties of sequences using linear recurrence equations generating them and the apparatus of triangular matrices.

Theorem 12. [35] *Let the sequences* $(u_n^*)_{n\geq 1}$ *and* $(u_n)_{n\geq 1}$ *satisfy the recurrence equations*

$$u_n^* = a_1 u_{n-1}^* + a_2 u_{n-2}^* + \cdots + a_k u_{n-k}^*,$$

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k},$$

with general initial conditions $u_1^* = b_1,\ u_2^* = b_2,\ \ldots,\ u_k^* = b_k$ *and normal initial conditions*

$$u_1 = 1, \quad u_i = \begin{bmatrix} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \cdots & \cdots & \ddots & \\ \frac{a_{i-1}}{a_{i-2}} & \frac{a_{i-2}}{a_{i-3}} & \cdots & a_1 \end{bmatrix}_{i-1}, \quad i = 2, 3, \ldots, k,$$

respectively, with $a_i \neq 0$ *for* $i = 1, 2, \ldots, k-1$. *Then*

$$u^*_{r+s} = \sum_{i=1}^{k} a_i \sum_{j=r-i+1}^{r} u^*_j u_{r+s-i-j+1}, \tag{19}$$

where $r + s > k$.

Proof. From Corollary 7 it follows that the parapermanent (14) is the solution of equation (6) with initial conditions (7). Let us decompose the parapermanent (14) for $n = r + s$ by elements of the inscribed rectangular table $T(r)$. If b_{ij} is some element of the table $T(r)$, then the first corner $R_{j-1,1}$ of its parapermanent algebraic complement

$$P_{ij} = \operatorname{pper}(R_{j-1,1}) \cdot \operatorname{pper}(R_{r+s-1,i+1}),$$

besides the coefficients of equation (6), will include the numbers $c_1, c_2, \ldots, c_k$, which are defined by (9). That is why the parapermanent of the corner $R_{j-1,1}$ is the j-th term of the sequence $(u^*_n)_{n\geq 1}$. In the second corner, all the numbers $c_i = 1$, and that is why its parapermanent is the $(r + s - i)$-th term of the sequence $(u_n)_{n\geq 1}$. It is obvious that $b_{ij} = 0$, if $k - 1 < i - j$ and $b_{ij} = a_{i-j+1}$, if $0 \leq i - j \leq k$. If $i - j = k - 1$, then $b_{ij} = a_k$, $i = j + k - 1$, and the number j gets the value from $r - k + 1$ to r. Thus, all the summands from the decomposition of the parapermanent u_{r+s} by elements of the table $T(r)$, which contain the coefficients a_k of the equation (6), will be part of the sum $a_k \cdot \sum\limits_{j=r-k+1}^{r} u^*_j u_{r+s-j-k+1}$.

If $i - j = k - 2$, then $b_{ij} = a_{k-1}$, $i = j + k - 2$, $j = r - k + 2, \ldots, r$, and the similar sum is $a_{k-1} \cdot \sum\limits_{j=r-k+2}^{r} u^*_j u_{r+s-j-k+2}$. Continuing the process of calculating the sums like these, in $(k-1)$ steps we get the equality $i = j$, which means we shall come to the sum $a_1 \cdot \sum\limits_{j=r}^{r} u^*_j u_{r+s-j}$.

Thus, the index i of coefficients a_i will get the value from 1 to k, and the decomposition of the parapermanent u^*_{r+s} by elements of the inscribed rectangular table $T(r)$ will be (19). ☐

Let us write the equality (19) for $k = 1$, $k = 2$, and $k = 3$ respectively:

$$\begin{aligned} u^*_{r+s} &= a_1 u^*_r u_s, \\ u^*_{r+s} &= a_1 u^*_r u_s + a_2(u^*_{r-1} u_s + u^*_r u_{s-1}), \\ u^*_{r+s} &= a_1 u^*_r u_s + a_2(u^*_{r-1} u_s + u^*_r u_{s-1}) + a_3(u^*_{r-2} u_s + u^*_{r-1} u_{s-1} + u^*_r u_{s-2}). \end{aligned} \tag{20}$$

Replacing in (20) r with $r + q$, we get the equality

$$u^*_{r+q+s} = u^*_{r+1} u_{q+1} u_s + a_2 u^*_r u_q u_s + a_2 u^*_{r+1} u_q u_{s-1} + a_2^2 u^*_r u_{q-1} u_{s-1}. \tag{21}$$

Corollary 13. *If the sequence* $(u_n)_{n\geq 1}$ *satisfies the recurrence equality* (6) *with normal initial conditions* (13), *then for its terms the following relations hold:*

$$u_{r+s} = \sum_{i=1}^{k} a_i \sum_{j=r-i+1}^{r} u_j u_{r+s-i-j+1}. \tag{22}$$

The validity of Corollary 13 is immediate from Theorem 12.

Let us write (22) for $k = 1$, $k = 2$, and $k = 3$ respectively:

$$
\begin{aligned}
u_{r+s} &= a_1 u_r u_s, \\
u_{r+s} &= a_1 u_r u_s + a_2(u_{r-1}u_s + u_r u_{s-1}), \\
u_{r+s} &= a_1 u_r u_s + a_2(u_{r-1}u_s + u_r u_{s-1}) + a_3(u_{r-2}u_s + u_{r-1}u_{s-1} + u_r u_{s-2}).
\end{aligned}
$$

The first ten terms of the normal number sequence, which are generated by linear second-order recurrence equation

$$u_{n+2} = a_1 u_{n+1} + a_2 u_n, \quad a_1, a_2 \in \mathbb{Z}\backslash\{0\}, \tag{23}$$

are as follows:

$$
\begin{aligned}
u_1 &= 1, \\
u_2 &= a_1, \\
u_3 &= a_1^2 + a_2, \\
u_4 &= a_1^3 + 2a_1a_2, \\
u_5 &= a_1^4 + 3a_1^2a_2 + a_2^2, \\
u_6 &= a_1^5 + 4a_1^3a_2 + 3a_1a_2^2, \\
u_7 &= a_1^6 + 5a_1^4a_2 + 6a_1^2a_2^2 + a_2^3, \\
u_8 &= a_1^7 + 6a_1^5a_2 + 10a_1^3a_2^2 + 4a_1a_2^3, \\
u_9 &= a_1^8 + 7a_1^6a_2 + 15a_1^4a_2^2 + 10a_1^2a_2^3 + a_2^4, \\
u_{10} &= a_1^9 + 8a_1^7a_2 + 21a_1^5a_2^2 + 20a_1^3a_2^3 + 5a_1a_2^4.
\end{aligned}
$$

For $n \geq 2$ we get

$$
u_n = \begin{bmatrix}
a_1 & & & & & \\
\frac{a_2}{a_1} & a_1 & & & & \\
0 & \frac{a_2}{a_1} & a_1 & & & \\
0 & 0 & \frac{a_2}{a_1} & a_1 & & \\
\cdots & \cdots & \cdots & \cdots & \ddots & \\
0 & \cdots & 0 & 0 & \frac{a_2}{a_1} & a_1
\end{bmatrix}_{n-1}
= \sum_{k_1+2k_2=n-1} \frac{(k_1+k_2)!}{k_1!\,k_2!} a_1^{k_1} a_2^{k_2}.
$$

The first ten terms of the normal number sequence, which are generated by third order linear recurrence equation

$$u_{n+3} = a_1 u_{n+2} + a_2 u_{n+1} + a_3 u_n, \quad a_1,\, a_2,\, a_3 \in \mathbb{Z}\backslash\{0\},$$

are as follows:

$$\begin{aligned}
u_1 &= 1,\\
u_2 &= a_1,\\
u_3 &= a_1^2 + a_2,\\
u_4 &= a_1^3 + 2a_1a_2 + a_3,\\
u_5 &= a_1^4 + 3a_1^2a_2 + a_2^2 + 2a_1a_3,\\
u_6 &= a_1^5 + 4a_1^3a_2 + 3a_1a_2^2 + 3a_1^2a_3 + 2a_2a_3,\\
u_7 &= a_1^6 + 5a_1^4a_2 + 6a_1^2a_2^2 + 4a_1^3a_3 + 6a_1a_2a_3 + a_2^3 + a_3^2,\\
u_8 &= a_1^7 + 6a_1^5a_2 + 5a_1^4a_3 + 10a_1^3a_2^2 + 12a_1^2a_2a_3 + 4a_1a_2^3 + 3a_1a_3^2 + 3a_2^2a_3,\\
u_9 &= a_1^8 + 7a_1^6a_2 + 6a_1^5a_3 + 15a_1^4a_2^2 + 20a_1^3a_2a_3 + 10a_1^2a_2^3 + 6a_1^2a_3^2 + 12a_1a_2^2a_3 + a_2^4 + 3a_2a_3^2,
\end{aligned}$$

and for $n \geq 2$ we get

$$u_n = \begin{bmatrix} a_1 & & & & & & \\ \frac{a_2}{a_1} & a_1 & & & & & \\ \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & & & \\ 0 & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & & \\ \vdots & \cdots & \cdots & \cdots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & a_1 & \\ 0 & 0 & 0 & 0 & \cdots & \frac{a_2}{a_1} & a_1 \end{bmatrix}_{n-1} = \sum_{k_1+2k_2+3k_3=n-1} \frac{(k_1+k_2+k_3)!}{k_1!\,k_2!\,k_3!} a_1^{k_1} a_2^{k_2} a_3^{k_3}.$$

Theorem 14. [35] *Let the sequence $(u_n)_{n\geq 1}$ satisfy the second-order recurrence equation* (23) *with normal initial conditions $u_1 = 1$, $u_2 = a_1$. Then*

(1) *the following equalities hold:*

$$u_{r+s} = u_{r+1}u_s + a_2u_ru_{s-1}, \tag{24}$$

$$u_{sr} \equiv 0 \,(\mathrm{mod}\, u_r); \tag{25}$$

(2) *if in the equation* (23) *the coefficients are relatively prime, then*

$$\gcd(u_s, u_r) = u_{\gcd(s,r)}. \tag{26}$$

Proof. Let us first prove (24). To do this, we shall use the method of mathematical induction. We apply the Corollary 13 to the equation (23) with initial conditions $u_1 = 1$, $u_2 = a_1$. For $k = 2$ the equality (22) is

$$u_{r+s} = a_2u_{r-1}u_s + a_1u_ru_s + a_2u_ru_{s-1}. \tag{27}$$

From (27), considering (23), we get

$$u_{r+s} = u_s(a_1u_r + a_2u_{r-1}) + a_2u_ru_{s-1} = u_{r+1}u_s + a_2u_ru_{s-1}.$$

Let us prove (25). For $s = r$ the equality (24) is

$$u_{2r} = u_r(u_{r+1} + a_2 u_{r-1}), \tag{28}$$

that is (25) holds for $s = 1$ and $s = 2$. Let us assume (25) holds for $s = 1, 2, \ldots, m-1$, and prove its validity for $s = m$. As $u_{mr} = u_{(m-1)r+r} = u_{(m-1)r+1}u_r + a_2 u_{(m-1)r}u_{r-1}$, and $u_{(m-1)r} \equiv 0 \pmod{u_r}$, then $u_{mr} = u_{(m-1)r+1}u_r$, i.e., (25) holds for $s = m$.

In order to prove the equality (26), let us first prove that

$$\gcd(u_n, u_{n+1}) = 1, \tag{29}$$

$$\gcd(a_2, u_n) = 1. \tag{30}$$

For $n = 1$ the equality (29) is obvious. Let us assume that (29) holds for $n = k$ and prove it for $n = k+1$. Let, conversely, $\gcd(u_{k+1}, u_{k+2}) = d > 1$. Then from (23) for $n = k$ it follows that either $u_k \equiv 0 \pmod d$ or $a_2 \equiv 0 \pmod d$. In the first case, we come to a contradiction with the assumption that (29) holds for $n = k$, and in the second case from (23) for $n = k$ it follows that $a_1 \equiv 0 \pmod d$, and this contradicts the fact that the coefficients of equation (23) are relatively prime. Thus, the equality (29) holds for an arbitrary $n \in \mathbb{N}$.

The validity of the equality (30) follows from the fact that assumption of the contrary proposition together with (23) for $n = s-1$ leads to contradiction with (29).

Now let $s < r$. Then considering the equality (24), we have

$$\gcd(u_r, u_s) = \gcd(u_{r-s+s}, u_r) =$$
$$= \gcd(u_{r-s+1}u_s + a_2 u_{r-s}u_{s-1}, u_s) = \gcd(a_2 u_{r-s}u_{s-1}, u_s).$$

From (29), (30) it follows that $\gcd(a_2 u_{s-1}, u_s) = 1$. Therefore $\gcd(u_r, u_s) = \gcd(u_{r-s}, u_s)$, and we get the equality (26). $\square$

Note that Theorem 14 also holds when the coefficients of linear recurrence equation (23) are functions of some variables.

Example 15. Let us consider the normal sequence of polynomials $w_n(x) = \frac{x^n-1}{x-1}$, which is generated by the recurrence equation $w_{n+2}(x) = (x+1)w_{n+1}(x) - xw_n(x)$. Polynomials $w_n(x)$ do not have real roots, and for prime values $n > 3$, they are connected with the problem of circle division into n equal parts. Gauss showed that the roots of polynomials $w_n(x)$ are expressed as square radicals only when n is a Fermat prime, that is $n = 2^{2^k} + 1$ (A000215). Only for these values of n, a circle can be divided into n equal parts with the help of a ruler and compass. According to Theorem 14, $\gcd\big(w_n(x), w_m(x)\big) = \frac{x^{\gcd(n,m)}-1}{x-1}$.

Corollary 16. *If the sequence $(u_k)_{k\geq 1}$ satisfies the hypotheses of Theorem 14 and $u_k \neq 1$, $k \geq 2$, then the term u_s is a prime only when s is a prime.*

The proof follows from (25).

Corollary 17. *Let the sequence* $(u_k)_{k\geq 1}$ *satisfy the hypotheses of Theorem 14 and* p *is a prime. Then* u_p *has no common divisor with any of the preceding terms of sequence* $(u_k)_{k\geq 1}$.

The proof follows from (25).

Corollary 18. *Let the sequence* $(u_k)_{k\geq 1}$ *satisfy the hypotheses of Theorem 14 and* $a_2 = b^2$, *where* $b \in \mathbb{Z}$. *Then each term* u_{2m+1}, $m > 1$, *can be written as the sum of squares of two nonnegative integers, namely*

$$u_{2m+1} = u_{m+1}^2 + (bu_m)^2.$$

The validity of Corollary 18 follows from (24) for $r = n$ and $s = n + 1$.

Example 19. The recurrence equation $u_{n+2} = (k+1)u_{n+1} - ku_n$ with normal initial conditions $u_1 = 1$, $u_2 = k+1$, $k \geq 1$, generates the number sequence $\left(\frac{k^n-1}{k-1}\right)_{n\geq 1}$. The condition $k \geq 1$ satisfies the condition $u_s \neq 1$, $s \geq 2$, of Corollary 16. According to Corollary 16 the number $\frac{k^n-1}{k-1}$ is a prime, if n is a prime. For $k = 2$ we get a well known fact that Mersenne primes (A001348) are among the numbers of the form $2^p - 1$, where p is some prime. For $k = 10$ we get a famous proposition that repunits (primes which are written as $\underbrace{11\cdots 1}_{n}$ in the decimal system; see A002275), are also sought for prime n.

As the recurrence equation $u_{n+2} = u_{n+1} + u_n$ with normal initial conditions $u_1 = u_2 = 1$ generates the Fibonacci sequence, then Theorem 14 can be seen as generalization of some relations for Fibonacci numbers.

There are various generalizations of Fibonacci numbers. Some generalizations are connected with preservation of the recurrence relation $F_{n+1} = F_n + F_{n-1}$ and replacement of initial conditions [13, 17, 33], others are based on a more general recurrence relation of the second order without changing the first two terms of the Fibonacci sequence [4, 5, 6, 10, 11, 26, 27, 32]. There are generalizations of the Fibonacci sequence when the recurrence relation has a more general form generalizing the initial conditions with that [1, 3, 9, 16, 14, 21, 22]. In particular, generalizations of the Fibonacci sequence for recurrence relations of higher orders were studied in [2, 8, 15, 19, 20, 30, 31].

Among the terms of number sequences, which satisfy one and the same recurrence equation but with different initial conditions, there are some interesting relations. Therefore studying the properties of the terms of one sequence, it is possible to draw some conclusions concerning the properties of the terms of another sequence.

As a consequence of Theorem 12 (when $k = 2$, $r = 1$, and $s = n - 1$), we have:

Theorem 20. *If the sequences* $(u_n)_{n\geq 1}$, $(u_n^*)_{n\geq 1}$ *satisfy the linear second-order recurrence equation*

$$u_{k+2} = a_1 u_{k+1} + a_2 u_k \tag{31}$$

with initial conditions

$$u_1 = 1,\ u_2 = a_1; \quad u_1^* = b_1,\ u_2^* = b_2, \tag{32}$$

respectively, then

$$u_n^* = b_2 u_{n-1} + a_2 b_1 u_{n-2}. \tag{33}$$

The next theorem singles out some classes of number sequences, whose odd-numbered terms can be written as the sum of several squares.

Theorem 21. [34] *Let the sequences $(u_n)_{n\geq 1}$, $(u_n^*)_{n\geq 1}$ satisfy the linear second-order recurrence equation* (31) *with initial conditions*

$$u_1 = 1,\ u_2 = a_1;\quad u_1^* = k,\ u_2^* = a_1. \tag{34}$$

Then

(1) *for $n \geq 3$ the following relation holds:*

$$u_n^* = u_n + (k-1)a_2 u_{n-2}; \tag{35}$$

(2) *if*

$$k = a_2 = s^2 + 1 \tag{36}$$

and $a_1 > 0$, then for $n \geq 3$ the number u_{2n-1}^* is the sum of three squares:

$$u_{2n-1}^* = u_n^2 + \big((s^2+1)u_{n-1}\big)^2 + \big((s^3+s)u_{n-2}\big)^2; \tag{37}$$

(3) *if*

$$k = s^2 + 1, \qquad a_2 = b^2, \tag{38}$$

then for $n \geq 2$ the number u_{2n+1}^* is the sum of four squares:

$$u_{2n+1}^* = u_{n+1}^2 + (bu_n)^2 + (sbu_n)^2 + (sb^2 u_{n-1})^2. \tag{39}$$

Proof. The equality (35) follows from (33):

$$\begin{gathered} u_n^* = b_2 u_{n-1} + a_2 b_1 u_{n-2} = a_1 u_{n-1} + a_2 k u_{n-2} = \\ = a_1 u_{n-1} + a_2 u_{n-2} + a_2(k-1)u_{n-2} = u_n + a_2(k-1)u_{n-2}. \end{gathered}$$

Let us prove (37). From (19) for $k = 2$ we have

$$u_{r+s}^* = a_1 u_r^* u_s + a_2(u_{r-1}^* u_s + u_r^* u_{s-1}) = u_{r+1}^* u_s + a_2 u_r^* u_{s-1}.$$

In the last equality we substitute $r = n-1$ and $s = n$. Then, using (35), we obtain

$$\begin{gathered} u_{2n-1}^* = (a_1 u_{n-1}^* + a_2 u_{n-2}^*)u_n + a_2 u_{n-1}^* u_{n-1} = u_n^* u_n + a_2 u_{n-1}^* u_{n-1} = \\ = (u_n + (k-1)a_2 u_{n-2})u_n + a_2(u_{n-1} + (k-1)a_2 u_{n-3})u_{n-1} = \\ = u_n^2 + (k-1)a_2 u_{n-2} u_n + a_2 u_{n-1}^2 + (k-1)a_2^2 u_{n-3} u_{n-1} = \\ = u_n^2 + (k-1)a_2 u_{n-2}(a_1 u_{n-1} + a_2 u_{n-2}) + a_2 u_{n-1}^2 + (k-1)a_2^2 u_{n-3} u_{n-1} = \\ = u_n^2 + a_2 u_{n-1}^2 + (k-1)a_2^2 u_{n-2}^2 + (k-1)a_2 u_{n-1}^2 = u_n^2 + k a_2 u_{n-1}^2 + (k-1)a_2^2 u_{n-2}^2. \end{gathered}$$

Hence, using (36), we get (37).

To prove (39) we shall write (35) for $n = 2m+1$:

$$u_{2m+1}^* = u_{2m+1} + (k-1)a_2 u_{2m-1}.$$

Hence, using (24) for $r = m$ and $s = m+1$ and the condition (38), after simple transformations we obtain (39). ∎

Example 22. If $s = 0$ we get $k = a_2 = 1$ and the equality (37) is written as

$$u_{2n-1} = u_n^2 + u_{n-1}^2. \tag{40}$$

Thus, all the terms of the normal number sequence with odd numbers, not less than 3, which are generated by the recurrence equation $u_{n+2} = au_{n+1} + u_n$, can be written as the sum of two squares, e.g.,

$$\begin{aligned}
u_3 &= a^2 + 1 = a^2 + 1^2, \\
u_5 &= a^4 + 3a^2 + 1 = (a^2+1)^2 + a^2, \\
u_7 &= a^6 + 5a^4 + 6a^2 + 1 = (a^3+2a)^2 + (a^2+1)^2, \\
u_9 &= a^8 + 7a^6 + 15a^4 + 10a^2 + 1 = (a^4+3a^2+1)^2 + (a^3+2a)^2, \\
u_{11} &= a^{10} + 9a^8 + 28a^6 + 35a^4 + 15a^2 + 1 = (a^5+4a^3+3a)^2 + (a^4+3a^2+1)^2.
\end{aligned}$$

In general,

$$u_{2n-1} = \sum_{i=2}^{n+1} \binom{2n-i}{2(n-i+1)} a^{2(n-i+1)} =$$

$$= \left(\sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{n-2i+1} a^{n-2i+1} \right)^2 + \left(\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i-1}{n-2i} a^{n-2i} \right)^2.$$

Note that for $a = 1$ the equality (40) is written as $F_{2n-1} = F_n^2 + F_{n-1}^2$, where F_n are Fibonacci numbers, and

$$F_n = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{n-2i+1}.$$

Example 23. In item 2 of Theorem 21, let $a_1 = 4$, $s = 2$. Then $u_1 = a_2 = k = 5$,

$$u_n^* = \frac{3 \cdot 5^{n-1} + 7\,(-1)^{n-1}}{2}, \quad u_n = \frac{5^n + (-1)^{n-1}}{6},$$

and the equality (37) becomes

$$\frac{3 \cdot 5^{2n-2} + 7}{2} = \left(\frac{5^n + (-1)^{n-1}}{6} \right)^2 + \left(\frac{5(5^{n-1} + (-1)^n)}{6} \right)^2 + \left(\frac{5(5^n + (-1)^{n-1})}{3} \right)^2.$$

For example, for $n = 16$ we have decomposition of the prime 3969838619232177773441 into the sum of three different squares of positive integers, two of which are sequential:

$$3969838619232177773441 = 25431315104^2 + 25431315105^2 + 10172526040^2.$$

Example 24. If in item 3 of Theorem 21 $a_1 = 3$, $a_2 = 4$, $b = 2$, $s = 1$, then

$$u_n^* = 4^{n-1} + (-1)^{n-1}, \quad u_n = \frac{4^n + (-1)^{n-1}}{5}$$

and the equality (39) becomes

$$2^{4m} + 1 = \left(\frac{4^{m+1} + (-1)^m}{5}\right)^2 + \left(\frac{2^{2m+1} + 2(-1)^{m-1}}{5}\right)^2 + \\ + \left(\frac{2^{2m+1} + 2(-1)^{m-1}}{5}\right)^2 + \left(\frac{2^{2m} + 4(-1)^m}{5}\right)^2.$$

For $m = 2^{n-2}$ we obtain Fermat numbers $\mathcal{F}_n = 2^{4m} + 1 = 2^{2^n} + 1$ (A000215). The last equality can now be written as

$$2^{2^n} + 1 = \left(\frac{2^{2^{n-1}+2} + 1}{5}\right)^2 + \left(\frac{2^{2^{n-1}+1} - 2}{5}\right)^2 + \left(\frac{2^{2^{n-1}+1} - 2}{5}\right)^2 + \left(\frac{2^{2^{n-1}} + 4}{5}\right)^2, \quad (41)$$

where $n \geqslant 3$. For example,

$$\mathcal{F}_6 = 2^{64} + 1 = 3435973837^2 + 1717986918^2 + 1717986918^2 + 858993460^2.$$

Using (41), with the help of simple transformations, Fermat numbers can be written (I. V. Fedak) as the sum of three squares:

$$\mathcal{F}_n = 2^{2^n} + 1 = a_n^2 + 1 = \left(\frac{2a_n - 2}{3}\right)^2 + \left(\frac{2a_n + 1}{3}\right)^2 + \left(\frac{a_n + 2}{3}\right)^2.$$

For example, $\mathcal{F}_6 = 2863311530^2 + 2863311531^2 + 1431655766^2$.

5 Generalization of the Cassini formula

Over 300 years have passed since the discovery of the Cassini formula for Fibonacci numbers

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

During this time, many studies have appeared presenting new similar formulae for other number sequences, which have become classical. But the methods proposed in the preceding sections make it possible to unify researches in this direction. It turns out that normal number sequences mostly have the same properties as Fibonacci numbers. It can be illustrated with the results of Theorem 14, Corollary 17, etc. In this section we shall prove some general theorems, which generalize the Cassini identity.

Consider normal number sequences of the second order

$$u_0 = 0, \;\; u_1 = 1, \;\; u_{n+2} = a_1 u_{n-1} + a_2 u_n. \tag{42}$$

For these sequences, the following equality holds [34]:

$$u_{r+s} = u_{r+1} u_s + a_2 u_r u_{s-1}, \tag{43}$$

where $r \geq 1$, $s \geq 2$. Let in (43) $r = n$, $s = n - 1$. Then

$$u_{2n-1} = u_{n+1} u_{n-1} + a_2 u_n u_{n-2}.$$

For $r = n - 1$, $s = n$ from (43) we have

$$u_{2n-1} = u_n^2 + a_2 u_{n-1}^2.$$

By subtracting two last equalities, we obtain the relation

$$u_{n+1} u_{n-1} - u_n^2 = -a_2 (u_n u_{n-2} - u_{n-1}^2).$$

Thus, we get the sequence of equalities

$$\begin{array}{rcl} (-a_2)^0 (u_{n+1} u_{n-1} - u_n^2) & = & (-a_2)^1 (u_n u_{n-2} - u_{n-1}^2), \\ (-a_2)^1 (u_{n+1} u_{n-1} - u_n^2) & = & (-a_2)^2 (u_n u_{n-2} - u_{n-1}^2), \\ & \vdots & \\ (-a_2)^{n-3} (u_4 u_2 - u_3^2) & = & (-a_2)^{n-2} (u_3 u_1 - u_2^2), \end{array}$$

by adding which, we get the generalized Cassini formula

$$u_{n+1} u_{n-1} - u_n^2 = (-1)^n a_2^{n-1}.$$

In the equality (43), let $r = n - 1$, $s = n - 3$. Then

$$u_{2n-4} = u_n u_{n-3} + a_2 u_{n-1} u_{n-4}.$$

If $r = n - 2$, $s = n - 2$, then from above equality we get

$$u_{2n-4} = u_{n-1} u_{n-2} + a_2 u_{n-2} u_{n-3}.$$

By subtracting two last equalities, we obtain

$$u_n u_{n-3} - u_{n-1} u_{n-2} = -a_2 (u_{n-1} u_{n-4} - u_{n-2} u_{n-3}).$$

Thus, we get the equalities

$$\begin{aligned}(-a_2)^0(u_nu_{n-3}-u_{n-1}u_{n-2}) &= (-a_2)^1(u_{n-1}u_{n-4}-u_{n-2}u_{n-3}),\\ (-a_2)^1(u_{n-1}u_{n-4}-u_{n-2}u_{n-3}) &= (-a_2)^2(u_{n-2}u_{n-5}-u_{n-3}u_{n-4}),\\ &\vdots\\ (-a_2)^{n-5}(u_5u_2-u_4u_3) &= (-a_2)^{n-4}(u_4u_1-u_3u_2).\end{aligned}$$

Hence, by adding all these equalities, we finally get the equality

$$u_nu_{n-3}-u_{n-1}u_{n-2}=(-1)^na_1a_2^{n-3}.$$

In this way, we prove the following proposition.

Theorem 25. *For normal number sequences of the second order (42) the following equalities are true:*

$$u_{n+1}u_{n-1}-u_n^2=(-1)^na_2^{n-1}, \tag{44}$$
$$u_nu_{n-3}-u_{n-1}u_{n-2}=(-1)^na_1a_2^{n-3}. \tag{45}$$

For the sequences $(u_n)_{n\geq 1}, (u_n^*)_{n\geq 1}$, which are generated by the second-order recurrence relation with initial conditions $u_1=1,\ u_2=a_1\ \ u_1^*=b_1,\ u_2^*=b_2$ respectively, the following equality is true

$$u_n^*=b_2u_{n-1}+a_2b_1u_{n-2}. \tag{46}$$

Let us apply (46) to determine the Cassini formula for the sequence $(u_n^*)_{n\geq 1}$:

$$\begin{gathered}u_{n+1}^*u_{n-1}^*-(u_n^*)^2=\\ =(b_2u_n+a_2b_1u_{n-1})(b_2u_{n-2}+a_2b_1u_{n-3})-(b_2u_{n-1}+a_2b_1u_{n-2})^2=\\ =b_2^2(u_nu_{n-2}-u_{n-1}^2)+a_2^2b_1^2(u_{n-1}u_{n-3}-u_{n-2}^2)+a_2b_2b_1(u_nu_{n-3}-u_{n-1}u_{n-2}).\end{gathered}$$

Using (44), (45), we shall substitute relevant values instead of parentheses in the last equalities

$$\begin{gathered}u_{n+1}^*u_{n-1}^*-(u_n^*)^2=(-1)^{n-1}b_2^2a_2^{n-2}+(-1)^na_2^2b_1^2a_2^{n-3}+(-1)^na_2b_1b_2a_1a_2^{n-3}=\\ =(-1)^{n-1}a_2^{n-2}(b_2^2-a_2b_1^2-a_1b_1b_2).\end{gathered}$$

Theorem 26. *For normal number sequences*

$$u_1=1,\ u_2=a_1,\ u_n=a_1u_{n-1}+a_2u_{n-2} \tag{47}$$

the following equalities hold:

$$u_nu_{n-k}-u_{n-1}u_{n-(k-1)}=(-1)^{n+k(\mathrm{mod}\ 2)-1}a_2^{n-k}u_{k-1}, \tag{48}$$

where $k\geq 2$, $n\geq k+1$.

Proof. We shall prove the theorem by induction for k. For $k = 2$ the proposition of the theorem is true based on the equality (44). Assume that the proposition of the theorem is true for $n = k$ and we shall prove that the proposition of the theorem is also true for $n = k + 1$. We get

$$\begin{gathered}
u_n u_{n-(k+1)} - u_{n-1}u_{n-k} = (a_1 u_{n-1} + a_2 u_{n-2})u_{n-(k+1)} - (a_1 u_{n-2} + a_2 u_{n-3})u_{n-k} = \\
= a_1 \left(u_{n-1}u_{(n-1)-k} - u_{n-2}u_{(n-1)-(k-1)}\right) + a_2 \left(u_{n-2}u_{(n-2)-(k-1)} - u_{n-3}u_{(n-2)-(k-2)}\right) = \\
= a_1(-1)^{(n-1)+(k \bmod 2)-1}a_2^{(n-1)-k}u_{k-1} + a_2(-1)^{(n-2)+(k-1)(\bmod 2)-1}a_2^{(n-2)-(k-1)}u_{k-2} = \\
= (-1)^{n+(k \bmod 2)}a_2^{n-(k+1)}(a_1 u_{k-1} + a_2 u_{k-2}) = (-1)^{n+(k \bmod 2)}a_2^{n-(k+1)}u_k.
\end{gathered}$$

□

Theorem 27. *The terms of the normal number sequence (47) satisfy the relation*

$$u_n u_{n-k} - u_{n-s}u_{n-(k-s)} = (-1)^{n+(k \bmod 2)-1}a_2^{n-k}u_s u_{k-s}. \tag{49}$$

Proof. We shall prove by induction for s. For $s = 1$ (49) is written as the equality (48) from Theorem 26. We shall denote $U(n, k, s) = u_n u_{n-k} - u_{n-s}u_{n-(k-s)}$. Then

$$\begin{gathered}
U(n, k, s + 1) = u_n u_{n-k} - u_{n-(s+1)}u_{n-(k-(s+1))} = \\
= (a_1 u_{n-1} + a_2 u_{n-2})\, u_{n-k} - u_{n-(s+1)} \left(a_1 u_{(n-1)-((k-1)-s)} + a_2 u_{(n-2)-((k-1)-s)}\right) = \\
= a_1 U(n - 1, k - 1, s) + a_2 U(n - 2, k - 2, s - 1) = \\
= (-1)^{(n-1)+((k-1) \bmod 2)-1}a_1 a_2^{(n-1)-(k-1)}u_s u_{(k-1)-s} + \\
+(-1)^{(n-2)+((k-2) \bmod 2)-1}a_2 a_2^{(n-2)-(k-2)}u_{s-1}u_{(k-2)-(s-1)} = \\
= (-1)^{n+(k \bmod 2)-1}a_2^{n-k}u_{k-(s+1)}(a_1 u_s + a_2 u_{s-1}) = (-1)^{n+(k \bmod 2)-1}a_2^{n-k}u_{s+1}u_{k-(s+1)}.
\end{gathered}$$

□

Theorem 28. *For the sequences $(u_n)_{n\geq 1}$ and $(u_n^*)_{n\geq 1}$, which satisfy the linear recurrence equation (31) with initial conditions $u_1 = a_1$, $u_2 = a_2$ and $u_1^* = b_1$, $u_2^* = b_2$ respectively, the following identity is true:*

$$u_n^* u_{n-k}^* - u_{n-s}^* u_{n-(k-s)}^* = (-1)^{n+(k \bmod 2)}a_2^{n-k-1}\left(b_2^2 - a_2 b_1^2 - a_1 b_1 b_2\right) u_s u_{k-s}.$$

Proof. To prove this we shall use the identity (33):

$$\begin{aligned}
u_n^* u_{n-k}^* - u_{n-s}^* u_{n-(k-s)}^* = (b_2 u_{n-1} + a_2 b_1 u_{n-2})(b_2 u_{n-k-1} + a_2 b_1 u_{n-k-2}) - \\
-(b_2 u_{n-s-1} + a_2 b_1 u_{n-s-2}) \left(b_2 u_{n-(k-s)-1} + a_2 b_1 u_{n-(k-s)-2}\right) = \\
= b_2^2 U(n-1,k,s) + a_2^2 b_1^2 U(n-2,k,s) + \\
+ a_2 b_2 b_1 U(n-1,k+1,s) + a_2 b_2 b_1 U(n-2,k-1,s) = \\
= (-1)^{n-1+(kbmod2)-1} b_2^2 a_2^{n-1-k} u_s u_{k-s} + (-1)^{n-2+(k \bmod 2)-1} a_2^2 b_1^2 a_2^{n-2-k} u_s u_{k-s} + \\
+(-1)^{n-1+((k+1) \bmod 2)-1} a_2 b_2 b_1 a_2^{(n-1)-(k+1)} u_s u_{(k+1)-s} + \\
+(-1)^{(n-2)+((k-1) \bmod 2)-1} a_2 b_2 b_1 a_2^{(n-2)-(k-1)} u_s u_{(k-1)-s} = \\
= (-1)^{n+(k \bmod 2)} a_2^{n-k-2} u_s \left(a_2 b_2^2 u_{k-s} - a_2^2 b_1^2 u_{k-s} - a_2 b_1 b_2 u_{k-s+1} + a_2^2 b_1 b_2 u_{k-s-1}\right) = \\
= (-1)^{n+(k \bmod 2)} a_2^{n-k-1} \left(b_2^2 - a_2 b_1^2 - a_1 b_1 b_2\right) u_s u_{k-s}.
\end{aligned}$$

□

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