Journal of Integer Sequences, Vol. 19 (2016),
Article 16.2.8

Minimal Polynomials of Algebraic Cosine Values at Rational Multiples of π

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Abstract

Lehmer proved that the values of the cosine function evaluated at rational multiples of π are algebraic numbers. We show how to determine explicit, closed form expressions for the minimal polynomials of these algebraic numbers.

1 Introduction

The following information concerning algebraic values of the cosine (and certain trigonometric) functions evaluated at rational multiples of π is well-known [2], [5, Chapter 3].

Let $n \in \mathbb{N},\ k \in \{1, 2, \ldots, n\}$ with $n > 2$ and $\gcd(k, n) = 1$. Then the value $2\cos(2k\pi/n)$ is an algebraic integer of degree $\varphi(n)/2$ whose minimal polynomial is $\psi_n(x) \in \mathbb{Z}[x]$, where

$$\psi_n(x + x^{-1}) = x^{-\varphi(n)/2}\Phi_n(x), \tag{1}$$

with Φ_n being the nth cyclotomic polynomial and φ denoting the Euler's totient function.

A simple proof of a special case of pure quadratic cosine values $\sqrt{r}$ is given in [10] and extended in [8], which also contains details of algebraic cosine values of small degrees.

It is natural to ask whether there is an explicit, closed form expression for $\psi_n(x)$. Since there is no explicit closed form expression for a general cyclotomic polynomial, $\Phi_n(x)$, such a task seems non-trivial. Recall that in order to find an expression for $\Phi_n(x)$, a usual approach is to use certain reduction formulas to write it as an algebraic expression involving known cyclotomic polynomials.

Our aim is to derive some reduction formulae, as with the case of cyclotomic polynomials, which will enable us to completely determine explicit forms of the polynomials $\psi_n(x)$. In addition, we mention two remarks concerning the values of the cosine functions evaluated at an algebraic irrational multiple of π and certain constructible values.

2 Minimal polynomials

Since $\cos(2k\pi/n) = \cos(2\pi(n-k)/n)$, as k runs those integers in $\{1, 2, \ldots, n\}$ which are relatively prime to n, there are precisely $\varphi(n)/2$ ($= \deg \psi_n(x)$) distinct values of $\cos(2k\pi/n)$ which shows that these cosine values are all the roots of $\psi_n(x)$ [9, Lemma, p. 473]. We base our investigation on the following explicit form of $\psi_p(x)$, p odd prime, due to Surowski and McCombs [7].

Proposition 1. *[7, Theorem 2.1] Let $p = 2s + 1$ be an odd prime. If $\psi_p(x) \in \mathbb{Z}[x]$ is the minimal polynomial of $2\cos(2\pi/p)$, then*

$$\psi_p(x) = \sum_{j=0}^{s} (-1)^j \sigma_j x^{s-j},$$

where

$$\begin{aligned}
\sigma_{2k} &= (-1)^k \binom{s-k}{k} && (k = 0, 1, \ldots, \lfloor s/2 \rfloor) \\
\sigma_{2k-1} &= (-1)^k \binom{s-k}{k-1} && (k = 1, \ldots, \lfloor (s+1)/2 \rfloor).
\end{aligned}$$

Incorporating this result with our preceding remark, we get

Theorem 2. *Let $p = 2s+1$ be an odd prime. The minimal polynomial of $2\cos(2k\pi/p)$, where $k \in \{1, 2, \ldots, p\}$, $\gcd(k,p) = 1$, is*

$$\psi_p(x) = \sum_{j=0}^{\lfloor s/2 \rfloor} (-1)^j \binom{s-j}{j} x^{s-2j} - \sum_{j=1}^{\lfloor (s+1)/2 \rfloor} (-1)^j \binom{s-j}{j-1} x^{s-(2j-1)}.$$

We also need some identities involving cyclotomic polynomials, [3, Chapter 2].

Lemma 3. *Let q be a prime, and let $m, e \in \mathbb{N}$. Then*

A. $\Phi_{q^e}(x) = \Phi_q\left(x^{q^{e-1}}\right) = 1 + x^{q^{e-1}} + x^{2q^{e-1}} + \cdots + x^{(q-1)q^{e-1}}$*;*

B. $\Phi_{mq^e}(x) = \Phi_{mq}\left(x^{q^{e-1}}\right)$*;*

C. $\Phi_{mq}(x) = \frac{\Phi_m(x^q)}{\Phi_m(x)}$ *provided that* $\gcd(m,q) = 1$.

In order to determine explicit forms of minimal polynomials for $2\cos(2k\pi/n)$ for other positive integers n, Lehmer's identity (1) indicates that we should first find explicit forms of $x^s + x^{-s}$ as a polynomial in $x + x^{-1}$, which is done in the next lemma.

Lemma 4. *For $t \in \mathbb{N}$, let $X_t := x^t + x^{-t}$, $X := X_1 = x + x^{-1}$. Then*

$$\begin{aligned} X_{2t} = X^{2t} &- \left\{ \binom{2t-1}{1} + \binom{2t-2}{0} \right\} X^{2t-2} + \left\{ \binom{2t-2}{2} + \binom{2t-3}{1} \right\} X^{2t-4} \\ &+ \cdots + (-1)^{t-1} \left\{ \binom{t+1}{t-1} + \binom{t}{t-2} \right\} X^2 + (-1)^t \cdot 2 \\ X_{2t+1} = X^{2t+1} &- \left\{ \binom{2t}{1} + \binom{2t-1}{0} \right\} X^{2t-1} + \left\{ \binom{2t-1}{2} + \binom{2t-2}{1} \right\} X^{2t-3} \\ &+ \cdots + (-1)^{t-1} \left\{ \binom{t+2}{t-1} + \binom{t+1}{t-2} \right\} X^3 + (-1)^t \left\{ \binom{t+1}{t} + \binom{t}{t-1} \right\} X^1, \end{aligned}$$

i.e., in general, for $s \in \mathbb{N}$, we have

$$\begin{aligned} X_s = X^s &- \left\{ \binom{s-1}{1} + \binom{s-2}{0} \right\} X^{s-2} + \left\{ \binom{s-2}{2} + \binom{s-3}{1} \right\} X^{s-4} + \cdots \\ &+ (-1)^{\lfloor s/2 \rfloor} \left\{ \binom{s-\lfloor s/2 \rfloor}{\lfloor s/2 \rfloor} + \binom{s-\lfloor s/2 \rfloor - 1}{\lfloor s/2 \rfloor - 1} \right\} X^{s-2\lfloor s/2 \rfloor} \\ &= \sum_{k=0}^{\lfloor s/2 \rfloor} (-1)^k \left\{ \binom{s-k}{k} + \binom{s-k-1}{k-1} \right\} X^{s-2k}, \end{aligned}$$

with the convention that $\binom{n}{r} = 0$ for negative r.

Proof. Taking $n = p = 2s+1$, an odd prime, Lehmer's identity (1) becomes

$$x^{-s}\Phi_p(x) = x^{-\frac{p-1}{2}}\Phi_p(x) = \psi_p\left(x + x^{-1}\right). \tag{2}$$

Equating the left hand side of (2) (using Lemma 3 A)

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = x^{2s} + x^{2s-1} + \cdots + x + 1,$$

with the right hand side (using Theorem 2), we get

$$\begin{aligned}
&\left(x^s + x^{-s}\right) + \left(x^{s-1} + x^{-(s-1)}\right) + \cdots + \left(x + x^{-1}\right) + 1 \\
&= \binom{s}{0}\left(x + x^{-1}\right)^s - \binom{s-1}{1}\left(x + x^{-1}\right)^{s-2} + \binom{s-2}{2}\left(x + x^{-1}\right)^{s-4} - \cdots \\
&\quad + (-1)^{\lfloor s/2 \rfloor}\binom{s - \lfloor s/2 \rfloor}{\lfloor s/2 \rfloor}\left(x + x^{-1}\right)^{s-2\lfloor s/2 \rfloor} \\
&+ \binom{s-1}{0}\left(x + x^{-1}\right)^{s-1} - \binom{s-2}{1}\left(x + x^{-1}\right)^{s-3} + \binom{s-3}{2}\left(x + x^{-1}\right)^{s-5} - \cdots \\
&\quad + (-1)^{\lfloor (s+1)/2 + 1 \rfloor}\binom{s - \lfloor (s+1)/2 \rfloor}{\lfloor (s+1)/2 \rfloor - 1}\left(x + x^{-1}\right)^{s-(2\lfloor (s+1)/2 \rfloor - 1)}.
\end{aligned} \tag{3}$$

Putting $s = 2t$, equating terms with even powers, we get

$$\begin{aligned}
&X_{2t} + X_{2t-2} + \cdots + X_2 + 1 = \left(x^{2t} + x^{-2t}\right) + \left(x^{2t-2} + x^{-(2t-2)}\right) + \cdots + \left(x^2 + x^{-2}\right) + 1 \\
&= \binom{2t}{0}\left(x + x^{-1}\right)^{2t} - \binom{2t-1}{1}\left(x + x^{-1}\right)^{2t-2} + \binom{2t-2}{2}\left(x + x^{-1}\right)^{2t-4} + \cdots \\
&\quad + (-1)^{t-1}\binom{2t-t+1}{t-1}\left(x + x^{-1}\right)^2 + (-1)^t\binom{2t-t}{t} \\
&= X^{2t} - \binom{2t-1}{1}X^{2t-2} + \binom{2t-2}{2}X^{2t-4} + \cdots + (-1)^{t-1}\binom{t+1}{t-1}X^2 + (-1)^t.
\end{aligned} \tag{4}$$

Replacing t by $t-1$ in (4), we get

$$\begin{aligned}
&X_{2t-2} + X_{2t-4} + \cdots + X_2 + 1 \\
&= X^{2t-2} - \binom{2t-3}{1}X^{2t-4} + \binom{2t-4}{2}X^{2t-6} + \cdots + (-1)^{t-2}\binom{t}{t-2}X^2 + (-1)^{t-1}.
\end{aligned} \tag{5}$$

Subtracting (5) from (4), we get the first assertion. The second assertion follows from equating terms with odd exponents in (3) and proceed similarly. $\square$

Lemmas 4 and 3 enable us to find an explicit form of any minimal polynomial through the following reduction identities.

Theorem 5.
I. For odd prime p and $e \in \mathbb{N}$, the minimal polynomial of $2\cos(2k\pi/p^e)$, where $k \in \{1, 2, \ldots, p^e\}$, $\gcd(k,p) = 1$, is

$$\psi_{p^e}(x) = \psi_p\left(\sum_{k=0}^{\lfloor p^{e-1}/2 \rfloor} (-1)^k \left\{\binom{p^{e-1}-k}{k} + \binom{p^{e-1}-k-1}{k-1}\right\} x^{p^{e-1}-2k}\right).$$

II. If p is an odd prime, and e, m are positive integers with $m \geq 2$, $\gcd(m,p) = 1$, then the minimal polynomial of $2\cos(2k\pi/mp^e)$, where $k \in \{1, 2, \ldots, mp^e\}$, $\gcd(k, mp) = 1$, satisfies

$$\psi_{mp^e}(x) = \frac{\psi_m\left(\sum_{k=0}^{\lfloor p^e/2 \rfloor} (-1)^k \left\{\binom{p^e-k}{k} + \binom{p^e-k-1}{k-1}\right\} x^{p^e-2k}\right)}{\psi_m\left(\sum_{k=0}^{\lfloor p^{e-1}/2 \rfloor} (-1)^k \left\{\binom{p^{e-1}-k}{k} + \binom{p^{e-1}-k-1}{k-1}\right\} x^{p^{e-1}-2k}\right)}.$$

III. The minimal polynomial of $2\cos(2\pi/2)$ is

$$\psi_2(x) = x + 2.$$

IV. For $e \in \mathbb{N}$, $e \geq 2$, the minimal polynomial of $2\cos(2k\pi/2^e)$, where $k \in \{1, 2, \ldots, 2^e\}$, $\gcd(k, 2) = 1$, is

$$\psi_{2^e}(x) = \sum_{k=0}^{2^{e-2}} (-1)^k \left\{\binom{2^{e-1}-k}{k} + \binom{2^{e-1}-k-1}{k-1}\right\} x^{2^{e-1}-2k} + 2.$$

Proof.
I. Using Lehmer's identity (1) and Lemma 3 A, we have

$$\begin{aligned}
\psi_{p^e}(X) &= \psi_{p^e}\left(x + x^{-1}\right) = x^{-\varphi(p^e)/2}\Phi_{p^e}(x) = x^{-p^{e-1}(p-1)/2}\Phi_p(x^{p^{e-1}}) \\
&= \psi_p\left(x^{p^{e-1}} + x^{-p^{e-1}}\right) = \psi_p(X_{p^{e-1}}) \\
&= \psi_p\left(\sum_{k=0}^{\lfloor p^{e-1}/2 \rfloor} (-1)^k \left\{\binom{p^{e-1}-k}{k} + \binom{p^{e-1}-k-1}{k-1}\right\} X^{p^{e-1}-2k}\right).
\end{aligned}$$

II. Using Lehmer's identity (1), $\gcd(m,p) = 1$, Lemma 3 B and C, we get

$$\begin{aligned}
\psi_{mp^e}(X) &= \psi_{mp^e}\left(x + x^{-1}\right) = x^{-\varphi(mp^e)/2}\Phi_{mp^e}(x) = x^{-\varphi(m)p^{e-1}(p-1)/2}\Phi_{mp}(x^{p^{e-1}}) \\
&= \frac{\left(x^{p^e}\right)^{-\varphi(m)/2}\Phi_m(x^{p^e})}{\left(x^{p^{e-1}}\right)^{-\varphi(m)/2}\Phi_m(x^{p^{e-1}})} = \frac{\psi_m(x^{p^e} + x^{-p^e})}{\psi_m(x^{p^{e-1}} + x^{-p^{e-1}})} = \frac{\psi_m(X_{p^e})}{\psi_m(X_{p^{e-1}})}.
\end{aligned}$$

III. This is obvious from $2\cos(2\pi/2) = -2$.

IV. Observe from Lehmer's identity (1), Lemma 3 A, and Lemma 4 that

$$\psi_{2^e}(X) = \psi_{2^e}\left(x + x^{-1}\right) = x^{-\varphi(2^e)/2}\Phi_{2^e}(x) = x^{-2^{e-2}}\Phi_2(x^{2^{e-1}}) = \psi_2\left(x^{2^{e-1}} + x^{-2^{e-1}}\right)$$
$$= \psi_2(X_{2^{e-1}}) = \psi_2\left(\sum_{k=0}^{2^{e-2}}(-1)^k\left\{\binom{2^{e-1}-k}{k} + \binom{2^{e-1}-k-1}{k-1}\right\}X^{2^{e-1}-2k}\right),$$

and the desired assertion follows from part III. □

3 Two more remarks about values of the cosine function

We end this paper with two further related observations.

3.1 Values at irrational multiples of π

Since all the cosine values evaluated at rational multiples of π are algebraic, it is natural to ask about the values at irrational multiples of π. We give an answer to this question using the following result of Robinson [6, Theorem 3(a)], that is a nice consequence of the Lindeman-Weierstrass theorem and the Gelfond-Schneider theorem.

Proposition 6. *If α, β are algebraic with αi ($i = \sqrt{-1}$) irrational, then $\cos(\alpha \log \beta)$, $\sin(\alpha \log \beta)$ and $\tan(\alpha \log \beta)$ for $\beta \neq 0, 1$ are transcendental numbers regardless of the branch of the natural logarithm.*

The answer to our question is given in the following theorem.

Theorem 7. *If γ is an algebraic irrational number, then $\cos(\gamma\pi)$, $\sin(\gamma\pi)$, and $\tan(\gamma\pi)$ are transcendental numbers.*

Proof. Taking $\beta = \exp(\pi i) = -1$ and α as an algebraic number such that $\alpha i = \gamma$ is an algebraic irrational number, Proposition 6 shows that $\cos(\gamma\pi) = \cos(\alpha i \pi) = \cos(\alpha \log \beta)$ is transcendental, and similarly for the sine and tangent functions. □

3.2 Constructible values

Since $\cos(2k\pi/n)$ $(n > 2,\ k \in \{1, \ldots, n\},\ \gcd(k, n) = 1)$ is an algebraic number of degree $\varphi(n)/2$, another natural question is whether this algebraic number is constructible [1, Section 7.11]. The answer is an immediate consequence of the following result whose proof can be found in [4].

Theorem 8. *Let k/n $(n > 2)$ be a rational number with* $\gcd(k, n) = 1$. *Then the algebraic integers* $2\cos(2\pi k/n)$ *are constructible if and only if $\varphi(n)$ is a power of 2, i.e., if and only if $n = 2^\alpha p_1 p_2 \cdots p_r$, where α is a non-negative integer, $p_1, \ldots, p_r$ are distinct odd primes of the form $2^\beta + 1$.*

We are grateful to the referee for the following information. Theorem 8 is a well-known ancient result. Gauss proved in his *Disquisitiones arithmeticae* that the numbers $2\cos(2\pi k/n)$ are constructible (of course, it is enough to take $k = 1$); in particular, he stated the construction of the 17th-gon, that is a famous fact in the life of Gauss. The converse part is usually attributed to Wantzel (1838).

4 Acknowledgments

We thank the referee for several useful comments and information. The first author is supported by Phranakhon Rajabhat University. The second author is supported by the Faculty of Science, Kasetsart University, through the grant number RFG1-13.

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2010 *Mathematics Classification*: Primary 11R04; Secondary 33B10, 11J72.
Keywords: cosine function, algebraic value, multiple of π.

Received February 8 2016; revised version received March 15 2016. Published in *Journal of Integer Sequences*, March 19 2016.
