Journal of Integer Sequences, Vol. 19 (2016),
Article 16.1.8

Carlitz-Type and Other Bernoulli Identities

Claudio de J. Pita Ruiz V.
Universidad Panamericana
Mexico City
Mexico
cpita@up.edu.mx

Abstract

By using an explicit formula for Bernoulli polynomials we obtained in a recent work (in which $B_n(x)$ is written as a linear combination of the polynomials $(x-r)^n$, $r=1,\ldots,K+1$, where $K \geq n$), it is possible to obtain Bernoulli polynomial identities from polynomial-combinatorial identities. Using this approach, we obtain some generalizations and new demonstrations of the 1971 Carlitz identity involving Bernoulli numbers, and we also obtain some new identities involving Bernoulli polynomials.

1 Introduction

Bernoulli numbers have been in the interest of mathematicians since they were discovered (by Jakob Bernoulli in the 17th century). It turns out that Bernoulli numbers are related to many important mathematical objects appearing in different fields of mathematics, as described by Mazur [15]. A popular way to introduce these numbers is through the generating function of Bernoulli polynomials $B_n(x)$, namely

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1}$$

The Bernoulli numbers are then $B_n = B_n(0)$. Many different explicit formulas for Bernoulli numbers and polynomials, together with some of their generalizations, have appeared throughout the years: Gould [10] gives a good review of this. We also mention the remarkable work of Dilcher and Slavutskii [5], containing hundreds of references related to Bernoulli numbers and polynomials.

Recently we found [20] yet another explicit formula for Bernoulli polynomials: for non-negative integers $K \geq n$, we have

$$B_n(x) = \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} (x - I + J - 1)^n . \tag{2}$$

(Since we will be dealing with sums and multiple sums troughout the work, we will use capital indices I and J and the parameter K for formula (2), and only for this formula.) It turns out [20, lemma 3] that the coefficients of the right-hand side of (2) are such that

$$\frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} = 1. \tag{3}$$

That is, formula (2) gives us the Bernoulli polynomial $B_n(x)$ expressed as an *affine* combination of the polynomials $(x-r)^n$, $r = 1, \ldots, K+1$. Note that the degree n of the Bernoulli polynomial $B_n(x)$ appears only in the term $(x - I + J - 1)^n$ of formula (2), and it is not related directly to the number $(K+1)$ of terms in the sum, except for the fact that K must be not lesser than n. For example, if we set $K = 4$ in (2), we obtain the formula

$$B_n(x) = \frac{137}{60}(x-1)^n - \frac{163}{60}(x-2)^n + \frac{137}{60}(x-3)^n - \frac{21}{20}(x-4)^n + \frac{1}{5}(x-5)^n , \tag{4}$$

which works for $n = 0, 1, 2, 3, 4$. That is, with $n = 0$ formula (4) gives us $B_0(x) = 1$, with $n = 1$ gives us $B_1(x) = x - \frac{1}{2}$, and so on $\cdots$, up to $n = 4$, corresponding to $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$.

Under a different point of view, for each $n \in \mathbb{N}$, formula (2) gives us infinitely many expressions for the n-th Bernoulli polynomial $B_n(x)$, namely, the right-hand side of (2) with $K \geq n$. For example, the Bernoulli polynomial $B_2(x) = x^2 - x + \frac{1}{6}$ can be written as

$$\begin{aligned}
B_2(x) &= \frac{11}{6}(x-1)^2 - \frac{7}{6}(x-2)^2 + \frac{1}{3}(x-3)^2 \\
&= \frac{25}{12}(x-1)^2 - \frac{23}{12}(x-2)^2 + \frac{13}{12}(x-3)^2 - \frac{1}{4}(x-4)^2 \\
&= \frac{137}{60}(x-1)^2 - \frac{163}{60}(x-2)^2 + \frac{137}{60}(x-3)^2 - \frac{21}{20}(x-4)^2 + \frac{1}{5}(x-5)^2 \\
&\vdots
\end{aligned}$$

corresponding to the right-hand side of (2) with $K = 2, 3, 4, \ldots$.

But formula (2) implies more interesting facts.

Theorem 1. *The polynomial identity*

$$\sum_{k=0}^{n} a_{n,k} (x+\alpha)^k = \sum_{k=0}^{n} b_{n,k} (x+\beta)^k , \tag{5}$$

implies the Bernoulli polynomial identity

$$\sum_{k=0}^{n} a_{n,k} B_k\left(x+\alpha\right) = \sum_{k=0}^{n} b_{n,k} B_k\left(x+\beta\right). \tag{6}$$

Proof. Take $K \geq n$. Beginning with the left-hand side of (6), and using first (2), using then (5), and finally using again (2), we obtain

$$\begin{aligned}
\sum_{k=0}^{n} a_{n,k} B_k\left(x+\alpha\right) &= \sum_{k=0}^{n} a_{n,k} \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} \left(-1\right)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} \left(x+\alpha-I+J-1\right)^k \\
&= \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} \left(-1\right)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} \sum_{k=0}^{n} a_{n,k} \left(x+\alpha-I+J-1\right)^k \\
&= \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} \left(-1\right)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} \sum_{k=0}^{n} b_{n,k} \left(x+\beta-I+J-1\right)^k \\
&= \sum_{k=0}^{n} b_{n,k} \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} \left(-1\right)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} \left(x+\beta-I+J-1\right)^k \\
&= \sum_{k=0}^{n} b_{n,k} B_k\left(x+\beta\right),
\end{aligned}$$

as desired. □

(The result established in Theorem 1 contains some of the spirit of umbral calculus [6, 23].)

For example, beginning with the expansion $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}$, and using Theorem 1, we obtain the following known property of Bernoulli polynomials:

$$B_n\left(x+y\right) = \sum_{k=0}^{n} \binom{n}{k} y^k B_{n-k}\left(x\right). \tag{7}$$

We will be using extensively the following property of Bernoulli polynomials:

$$B_n\left(1-x\right) = \left(-1\right)^n B_n\left(x\right). \tag{8}$$

Observe that (8), together with formula (2), give us the following identity:

$$\sum_{I=0}^{K} \sum_{J=0}^{I} \left(-1\right)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} \left(x+I-J\right)^n = \sum_{I=0}^{K} \sum_{J=0}^{I} \left(-1\right)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} \left(x-I+J-1\right)^n, \tag{9}$$

where $0 \leq n \leq K$. Moreover, we claim that the following identity is true

$$\begin{aligned}
&\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{I+J}\frac{\binom{K+1}{J}}{\binom{K}{I}}(x+I-J)^{n_1}(y-I+J)^{n_2} \qquad (10)\\
&\qquad =\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{I+J}\frac{\binom{K+1}{J}}{\binom{K}{I}}(x-I+J-1)^{n_1}(y+I-J+1)^{n_2},
\end{aligned}$$

where $0 \leq n_1 + n_2 \leq K$. In fact, if $n_2 = 0$, formula (10) is true by (9). If (10) is true for a given $n_2 \in \mathbb{N}$, then for $0 \leq n_1 + n_2 + 1 \leq K$ we have

$$\begin{aligned}
&\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{I+J}\frac{\binom{K+1}{J}}{\binom{K}{I}}(x+I-J)^{n_1}(y-I+J)^{n_2+1}\\
&\qquad =(y+x)\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{I+J}\frac{\binom{K+1}{J}}{\binom{K}{I}}(x+I-J)^{n_1}(y-I+J)^{n_2}\\
&\qquad\quad -\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{I+J}\frac{\binom{K+1}{J}}{\binom{K}{I}}(x+I-J)^{n_1+1}(y-I+J)^{n_2}\\
&\qquad =(y+x)\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{I+J}\frac{\binom{K+1}{J}}{\binom{K}{I}}(x-I+J-1)^{n_1}(y+I-J+1)^{n_2}\\
&\qquad\quad -\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{I+J}\frac{\binom{K+1}{J}}{\binom{K}{I}}(x-I+J-1)^{n_1+1}(y+I-J+1)^{n_2}\\
&\qquad =\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{I+J}\frac{\binom{K+1}{J}}{\binom{K}{I}}(x-I+J-1)^{n_1}(y+I-J+1)^{n_2+1},
\end{aligned}$$

proving our claim.

The starting point in Section 2 is the identity

$$(-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{m+l}=(-1)^m\sum_{l=0}^{m}\binom{m}{l}B_{n+l},$$

where m and n are arbitrary non-negative integers, posed by Carlitz [3] in 1971. The fascinating symmetric beauty of this identity has produced generalizations, different demonstrations, rediscoveries, and the kind of mathematical honors bestowed on nice (and/or important) results. Here we join the group of fans of this identity, and use Theorem 1 to give one more demonstration and show some generalizations (that we call "Carlitz-type identities") of this Carlitz's identity. In Section 3 we consider the expression obtained when one performs integrations in a multinomial expansion $(x_1 + \cdots + x_k)^n$, with respect to two of its variables. The obtained expression is a polynomial identity that produces, by using Theorem 1, some

Bernoulli identities. We show some of them. In Section 4 we show a miscellany of Bernoulli polynomial identities. The common characteristic is that all of them come from already known results. These known results are polynomial identities (combinatorial identities) that can be "converted" to Bernoulli polynomial identities (by using Theorem 1). We collected some of the results appearing in the works of H. W. Gould [7, 8, 11], the Abel's sums, the extensions of A. Hurwitz [13] on Abel's sums, among others, to give the Bernoulli version of each of them. These Bernoulli identities are, as far as we know, new results.

2 Carlitz-type identities

In 1971, Carlitz [3] posed the following identity for Bernoulli numbers:

$$(-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{m+l} = (-1)^m \sum_{l=0}^{m} \binom{m}{l} B_{n+l}. \tag{11}$$

where m, n are arbitrary non-negative integers. Since then, identity (11) has captured the attention of many mathematicians, generating contrasting comments which go from "nearly trivial result" [6, p. 17], to "a remarkable identity" [25]. And the interest continues nowadays: we have found three recent works [9, 18, 21] concerning identity (11). Besides the several generalizations available for (11), in the history of this identity we find also some rediscovers of it, and very likely also rediscovers of some of its demonstrations. We comment in passing that the case $m = 0$ of (11) says that, for $n \in \mathbb{N}$, we have (the known recurrence for Bernoulli numbers):

$$\sum_{l=0}^{n} \binom{n+1}{l} B_l = 0. \tag{12}$$

The polynomial version of (11) is

$$(-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{m+l}(y) = (-1)^m \sum_{l=0}^{m} \binom{m}{l} B_{n+l}(-y), \tag{13}$$

which gives us (11) with $y = 0$. More generally, we have the polynomial identity

$$(-1)^n \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_{m+l}(y) = (-1)^m \sum_{l=0}^{m} \binom{m}{l} x^{m-l} B_{n+l}(1-x-y), \tag{14}$$

which gives us (13) with $x = 1$. (Formulas (13) and (14) appear in the work of Chen and Sun [4].) By taking the derivative with respect to y in both sides of (14) (and shifting n and

m to $n+1$ and $m+1$, respectively), we get

$$(-1)^{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}(m+1+l)\,x^{n+1-l}B_{m+l}(y) \tag{15}$$
$$=(-1)^{m}\sum_{l=0}^{m+1}\binom{m+1}{l}(n+1+l)\,x^{m+1-l}B_{n+l}(1-x-y).$$

In particular, by setting $x=1$ and $y=0$ in (15), we obtain that

$$(-1)^{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}(m+1+l)\,B_{m+l}=(-1)^{m}\sum_{l=0}^{m+1}\binom{m+1}{l}(n+1+l)\,B_{n+l}. \tag{16}$$

Using that, for $m+n>0$ we have $((-1)^m+(-1)^n)\,B_{n+m+1}=0$ (consequence of the fact that odd Bernoulli numbers $B_3, B_5, \ldots$ are equal to 0), we can write (16) (with $m+n>0$) as

$$(-1)^{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(m+1+l)\,B_{m+l}=(-1)^{m}\sum_{l=0}^{m}\binom{m+1}{l}(n+1+l)\,B_{n+l}. \tag{17}$$

In particular, if $m=n$ we have the identity:

$$\sum_{l=0}^{n}\binom{n+1}{l}(n+1+l)\,B_{n+l}=0, \tag{18}$$

where $n\in\mathbb{N}$. Identity (18) was obtained by Kaneko [14] in 1995. Its importance is that it gives a recurrence for Bernoulli numbers that, comparing with (12), requires only "the half" of terms to obtain the n-th Bernoulli number B_n in terms of the previous ones (for example, in the case $n=5$ identity (18) gives us $B_{10}=-\frac{1}{66}\left(\binom{6}{1}7B_6+\binom{6}{3}9B_8\right)$, in contrast with the 6 terms in $B_{10}=-\frac{1}{11}\sum_{l=0}^{8}\binom{11}{l}B_l$ of (12)). Identity (16), obtained by Momiyama [16] in 2001, is then a generalization of Kaneko's identity (18).

In this section we present a proof of (14), and consider also generalizations of this Bernoulli polynomial identity.

Proposition 2. *We have the Bernoulli polynomial identity (14).*

Proof. We begin with the trivial identity $y^m(x+y)^n=(x+y)^n(x+y-x)^m$, which we write as

$$\sum_{l=0}^{n}\binom{n}{l}x^{n-l}y^{m+l}=(-1)^m\sum_{l=0}^{m}\binom{m}{l}x^{m-l}(-1)^{m+l}(x+y)^{n+l}. \tag{19}$$

By seeing (19) as a polynomial identity between two y-polynomials, we use Theorem 1 to obtain

$$\sum_{l=0}^{n}\binom{n}{l}x^{n-l}B_{m+l}(y)=\sum_{l=0}^{m}\binom{m}{l}x^{m-l}(-1)^{m+l}B_{n+l}(x+y). \tag{20}$$

Finally, use (8) to obtain (14) from (20). □

By using formula (2), we can obtain an explicit expression for the polynomial involved in identity (14). In fact, if $K \geq m+n$, the left-hand side of (14) is

$$
\begin{aligned}
&(-1)^n \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_{m+l}(y) \\
&= (-1)^n \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} (y-I+J-1)^m \sum_{l=0}^{n} \binom{n}{l} x^{n-l} (y-I+J-1)^l \\
&= \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} (y-I+J-1)^m (-x-y+I-J+1)^n . \qquad (21)
\end{aligned}
$$

A similar calculation gives us that the right-hand side of (14) is (for $K \geq m+n$)

$$
\begin{aligned}
&(-1)^m \sum_{l=0}^{m} \binom{m}{l} x^{m-l} B_{n+l}(1-x-y) \\
&\qquad = \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} (-x-y-I+J)^n (y+I-J)^m . \qquad (22)
\end{aligned}
$$

The fact that (21) and (22) are equal, comes from (10). Thus, we can write identity (14) as

$$
\begin{aligned}
(-1)^n \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_{m+l}(y) &= (-1)^m \sum_{l=0}^{m} \binom{m}{l} x^{m-l} B_{n+l}(1-x-y) \qquad (23) \\
&= \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} (y-I+J-1)^m (-x-y+I-J+1)^n ,
\end{aligned}
$$

where $K \geq m+n$.

By playing the game of the "trivial identity" (together with the use of Theorem 1), it is possible to produce similar formulas to (20). For example, one can obtain the following identities, where the non-negative indices n, m, p, q can be interchanged:

$$
\begin{aligned}
&(-1)^n \sum_{l_1=0}^{n} \sum_{l_2=0}^{p} \binom{n}{l_1} \binom{p}{l_2} y^{n-l_1} B_{p-l_2}(x+y) B_{m+l_1+l_2}(z) \qquad (24) \\
&\qquad = (-1)^m \sum_{l_1=0}^{m} \sum_{l_2=0}^{p} \binom{m}{l_1} \binom{p}{l_2} (-1)^{l_2} y^{m-l_1} B_{p-l_2}(x) B_{n+l_1+l_2}(1-y-z) .
\end{aligned}
$$

$$\sum_{l_1=0}^{m}\sum_{l_2=0}^{p}\binom{m}{l_1}\binom{p}{l_2}z^{p-l_2}B_{n+l_1}(x)\,B_{m-l_1+l_2}(y) \tag{25}$$
$$=\sum_{l_1=0}^{m}\sum_{l_2=0}^{n}\binom{m}{l_1}\binom{n}{l_2}z^{n-l_2}B_{m-l_1+l_2}(x-z)\,B_{p+l_1}(y+z)\,.$$

$$\sum_{l_1=0}^{m}\sum_{l_2=0}^{p}\sum_{l_3=0}^{q}\binom{m}{l_1}\binom{p}{l_2}\binom{q}{l_3}u^{q-l_3}B_{n+l_1}(x)\,B_{m-l_1+l_2}(y)\,B_{p-l_2+l_3}(z) \tag{26}$$
$$=\sum_{l_1=0}^{p}\sum_{l_2=0}^{m}\sum_{l_3=0}^{n}\binom{p}{l_1}\binom{m}{l_2}\binom{n}{l_3}(-u)^{n-l_3}B_{q+l_1}(z+u)\,B_{p-l_1+l_2}(y-u)\,B_{m-l_2+l_3}(x+u)\,.$$

In the following proposition we consider a first natural generalization of (14).

Proposition 3. *We have the following Bernoulli polynomial identity*

$$(-1)^n\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\ldots,l_k}x_2^{l_2}\cdots x_k^{l_k}B_{m+l_1}(x_1+y) \tag{27}$$
$$=(-1)^m\sum_{l_1+\cdots+l_k=m}\binom{m}{l_1,\ldots,l_k}x_2^{l_2}\cdots x_k^{l_k}B_{n+l_k}(1-x_1-\cdots-x_k-y)\,.$$

Proof. Consider the trivial identity

$$(x_1+y)^m\,(x_2+\cdots+x_k+(x_1+y))^n$$
$$=(x_1+\cdots+x_k+y)^n\,(-x_2-\cdots-x_k+(x_1+\cdots+x_k+y))^m\,,$$

or

$$\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\ldots,l_k}x_2^{l_2}\cdots x_k^{l_k}(x_1+y)^{m+l_1} \tag{28}$$
$$=\sum_{l_1+\cdots+l_k=m}\binom{m}{l_1,\ldots,l_k}(-x_2)^{l_2}\cdots(-x_k)^{l_k}(x_1+\cdots+x_k+y)^{n+l_1}\,.$$

By considering (28) as an identity between two y-polynomials, use Theorem 1 to obtain that

$$\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\ldots,l_k}x_2^{l_2}\cdots x_k^{l_k}B_{m+l_1}(x_1+y) \tag{29}$$
$$=\sum_{l_1+\cdots+l_k=m}\binom{m}{l_1,\ldots,l_k}(-x_2)^{l_2}\cdots(-x_k)^{l_k}B_{n+l_1}(x_1+\cdots+x_k+y)\,.$$

Finally, use (8) to obtain (27). □

For example, if we distribute an adequate amount of 1's and -1's in the variables $x_1,\ldots,x_k$, we see from (27) that the following identities hold for $k\geq 1$,

$$(-1)^n \sum_{\sum_{j=1}^{2k} l_j=n} \binom{n}{l_1,\ldots,l_{2k}} (-1)^{\sum_{j=2}^{k} l_j} B_{m+l_1}(x)$$

$$= (-1)^m \sum_{\sum_{j=1}^{2k} l_j=m} \binom{m}{l_1,\ldots,l_{2k}} (-1)^{\sum_{j=2}^{k} l_j} B_{n+l_1}(-x),$$

$$\sum_{\sum_{j=1}^{2k+1} l_j=n} \binom{n}{l_1,\ldots,l_{2k+1}} (-1)^{\sum_{j=1}^{k} l_j} B_{m+l_{2k+1}}(x)$$

$$= \sum_{\sum_{j=1}^{2k+1} l_j=m} \binom{m}{l_1,\ldots,l_{2k+1}} (-1)^{\sum_{j=1}^{k} l_j} B_{n+l_{2k+1}}(x).$$

If in (27) we set $y=\frac{1}{2}\left(1-\sum_{i=2}^k x_i\right)-x_1$, we obtain that

$$(-1)^n \sum_{\sum_{i=1}^k l_i=n} \binom{n}{l_1,\ldots,l_k} x_2^{l_2}\cdots x_k^{l_k} B_{m+l_1}\left(\frac{1-\sum_{i=2}^k x_i}{2}\right)$$
$$= (-1)^m \sum_{\sum_{i=1}^k l_i=m} \binom{m}{l_1,\ldots,l_k} x_2^{l_2}\cdots x_k^{l_k} B_{n+l_1}\left(\frac{1-\sum_{i=2}^k x_i}{2}\right). \tag{30}$$

Moreover, by using (2) we can easily see that both sides of (30) are equal to

$$\frac{1}{K+1}\sum_{I=0}^{K}\sum_{J=0}^{I}(-1)^{J+I}\frac{\binom{K+1}{J}}{\binom{K}{I}}\left(-\frac{1+\sum_{i=2}^k x_i}{2}-I+J\right)^n\left(\frac{1-\sum_{i=2}^k x_i}{2}+I-J\right)^m, \tag{31}$$

where $K\geq m+n$.

If we take the variables $x_2,\ldots,x_k$ in (30) such that $\sum_{i=2}^k x_i=1$ or $\sum_{i=2}^k x_i=-1$, we obtain essentially identity (11), as the following corollary says.

Corollary 4.

(a) If $\sum_{i=2}^{k} x_i = 1$, we have the identity

$$\begin{aligned} (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{m+l} &= (-1)^n \sum_{\sum_{i=1}^{k} l_i = n} \binom{n}{l_1, \ldots, l_k} x_2^{l_2} \cdots x_k^{l_k} B_{m+l_1} \\ &= (-1)^m \sum_{\sum_{i=1}^{k} l_i = m} \binom{m}{l_1, \ldots, l_k} x_2^{l_2} \cdots x_k^{l_k} B_{n+l_1} \\ &= (-1)^m \sum_{l=0}^{m} \binom{m}{l} B_{n+l}. \end{aligned} \tag{32}$$

(b) If $\sum_{i=2}^{k} x_i = -1$, we have the identity

$$\begin{aligned} (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{m+l} &= \sum_{\sum_{i=1}^{k} l_i = n} \binom{n}{l_1, \ldots, l_k} (-1)^{l_1} x_2^{l_2} \cdots x_k^{l_k} B_{m+l_1} \\ &= \sum_{\sum_{i=1}^{k} l_i = m} \binom{m}{l_1, \ldots, l_k} (-1)^{l_1} x_2^{l_2} \cdots x_k^{l_k} B_{n+l_1} \\ &= (-1)^m \sum_{l=0}^{m} \binom{m}{l} B_{n+l}. \end{aligned} \tag{33}$$

Proof.

(a) From (30) and (31) we see that, if $\sum_{i=2}^{k} x_i = 1$, we have

$$\begin{aligned} &(-1)^n \sum_{\sum_{i=1}^{k} l_i = n} \binom{n}{l_1, \ldots, l_k} x_2^{l_2} \cdots x_k^{l_k} B_{m+l_1} \\ &\quad = (-1)^m \sum_{\sum_{i=1}^{k} l_i = m} \binom{m}{l_1, \ldots, l_k} x_2^{l_2} \cdots x_k^{l_k} B_{n+l_1} \\ &\quad = \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{J+I} \frac{\binom{K+1}{J}}{\binom{K}{I}} (-1-I+J)^m (I-J)^n, \end{aligned} \tag{34}$$

where $K \geq m+n$. On the other hand, from (23) with $x = 1$ and $y = 0$ we see that, for $K \geq m+n$, we have

$$\begin{aligned} (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{m+l} &= (-1)^m \sum_{l=0}^{m} \binom{m}{l} B_{n+l} \\ &= \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{I+J} \frac{\binom{K+1}{J}}{\binom{K}{I}} (-1-I+J)^m (I-J)^n. \end{aligned} \tag{35}$$

Identity (32) comes from (34) and (35).

(b) From (30) and (31) we see that, if $\sum_{i=2}^{k} x_i = -1$, we have

$$\begin{aligned}
&\sum_{\sum_{i=1}^{k} l_i = n} \binom{n}{l_1, \ldots, l_k} (-1)^{l_1} x_2^{l_2} \cdots x_k^{l_k} B_{m+l_1} \\
&\qquad = \sum_{\sum_{i=1}^{k} l_i = m} \binom{m}{l_1, \ldots, l_k} (-1)^{l_1} x_2^{l_2} \cdots x_k^{l_k} B_{n+l_1} \\
&\qquad = \frac{1}{K+1} \sum_{I=0}^{K} \sum_{J=0}^{I} (-1)^{J+I} \frac{\binom{K+1}{J}}{\binom{K}{I}} (I-J)^n (-1-I+J)^m ,
\end{aligned} \tag{36}$$

where $K \geq m+n$. Identity (33) comes from (36) and (35).

□

For example, we have the identities (for $k \geq 2$)

$$\begin{aligned}
(-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{m+l} &= (-1)^n \sum_{\sum_{i=1}^{k} l_i = n} \binom{n}{l_1, \ldots, l_k} \frac{B_{m+l_1}}{(k-1)^{n-l_1}} \\
&= (-1)^m \sum_{\sum_{i=1}^{k} l_i = m} \binom{m}{l_1, \ldots, l_k} \frac{B_{n+l_1}}{(k-1)^{m-l_1}} \\
&= (-1)^m \sum_{l=0}^{m} \binom{m}{l} B_{n+l},
\end{aligned} \tag{37}$$

and (for $k \geq 1$)

$$\begin{aligned}
(-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{m+l} &= \sum_{\sum_{i=0}^{k} l_i = n} \binom{n}{l_0, l_1, \ldots, l_k} (-1)^{l_0} \omega_1^{l_1} \cdots \omega_k^{l_k} B_{m+l_0} \\
&= \sum_{\sum_{i=0}^{k} l_i = m} \binom{m}{l_0, l_1, \ldots, l_k} (-1)^{l_0} \omega_1^{l_1} \cdots \omega_k^{l_k} B_{n+l_0} \\
&= (-1)^m \sum_{l=0}^{m} \binom{m}{l} B_{n+l},
\end{aligned} \tag{38}$$

where $1, \omega_1, \ldots, \omega_k \in \mathbb{C}$ are the $(k+1)$-th roots of 1. In particular, we have the identities

(([37]) and ([38]) together, with trinomial and quatranomial coefficients)

$$\begin{aligned}
(-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{m+l} &= (-1)^n \sum_{l_1+l_2+l_3=n} \binom{n}{l_1, l_2, l_3} \frac{B_{m+l_1}}{2^{n-l_1}} \\
&= (-1)^m \sum_{l_1+l_2+l_3=m} \binom{m}{l_1, l_2, l_3} \frac{B_{n+l_1}}{2^{m-l_1}} \\
&= \sum_{l_1+l_2+l_3=n} \binom{n}{l_1, l_2, l_3} (-1)^{l_1} \left(\tfrac{-1+i\sqrt{3}}{2}\right)^{l_2+2l_3} B_{m+l_1} \\
&= \sum_{l_1+l_2+l_3=m} \binom{m}{l_1, l_2, l_3} (-1)^{l_1} \left(\tfrac{-1+i\sqrt{3}}{2}\right)^{l_2+2l_3} B_{n+l_1} \\
&= (-1)^n \sum_{l_1+l_2+l_3+l_4=n} \binom{n}{l_1, l_2, l_3, l_4} \frac{B_{m+l_1}}{3^{n-l_1}} \\
&= (-1)^m \sum_{l_1+l_2+l_3+l_4=m} \binom{m}{l_1, l_2, l_3, l_4} \frac{B_{n+l_1}}{3^{m-l_1}} \\
&= (-1)^n \sum_{l_1+l_2+l_3+l_4=n} \binom{n}{l_1, l_2, l_3, l_4} (-i)^{l_2} (i)^{l_4} B_{m+l_1} \\
&= (-1)^m \sum_{l_1+l_2+l_3+l_4=m} \binom{m}{l_1, l_2, l_3, l_4} (-i)^{l_2} (i)^{l_4} B_{n+l_1} \\
&= (-1)^m \sum_{l=0}^{m} \binom{m}{l} B_{n+l}.
\end{aligned}$$

In the following corollary we collect some particular cases of ([30]). Instead of writing both sides of ([30]), we will say that "the left-hand side of this identity is invariant if we interchange n and m".

Corollary 5. *Let n, m, k be non-negative integers, with $k \geq 2$. The following expressions are invariant by interchanging n and m.*

(a)

$$(-1)^n \sum_{\sum_{i=1}^{k} l_i = n} \binom{n}{l_1, \ldots, l_k} B_{m+l_1} \left(1 - \frac{k}{2}\right). \tag{39}$$

(b) If $x \neq 1$,

$$(-1)^n \sum_{\sum_{i=1}^{k} l_i = n} \binom{n}{l_1, \ldots, l_k} x^{2l_2+\cdots+kl_k} B_{m+l_1} \left(\frac{x^2 + x - 1 - x^{k+1}}{2(x-1)}\right). \tag{40}$$

(c)
$$(-1)^n \sum_{\sum_{i=1}^k l_i=n} \binom{n}{l_1,\dots,l_k} 2^{l_2}\cdots k^{l_k} B_{m+l_1}\left(\frac{4-k-k^2}{4}\right). \tag{41}$$

(d)
$$(-1)^n \sum_{\sum_{i=1}^k l_i=n} \binom{n}{l_1,\dots,l_k} \frac{1}{2^{l_2}\cdots k^{l_k}} B_{m+l_1}\left(1-\frac{H_k}{2}\right), \tag{42}$$

where H_k is the k-th harmonic number.

(e)
$$(-1)^n \sum_{\sum_{i=1}^k l_i=n} \binom{n}{l_1,\dots,l_k} F_2^{l_2}\cdots F_k^{l_k} B_{m+l_1}\left(\frac{3-F_{k+2}}{2}\right), \tag{43}$$

where F_k is the k-th Fibonacci number.

(f)
$$(-1)^n \sum_{\sum_{i=1}^k l_i=n} \binom{n}{l_1,\dots,l_k} L_2^{l_2}\cdots L_k^{l_k} B_{m+l_1}\left(\frac{5-L_{k+2}}{2}\right), \tag{44}$$

where L_k is the k-th Lucas number.

(g)
$$(-1)^n \sum_{\sum_{i=1}^k l_i=n} \binom{n}{l_1,\dots,l_k} (\ln 2)^{l_2}\cdots(\ln k)^{l_k} B_{m+l_1}\left(\ln\sqrt{\frac{e}{k!}}\right). \tag{45}$$

Proof. The corresponding identity is obtained from (30) if, for $s=2,\dots,k$: (a) we set $x_s=1$. (b) we set $x_s=x^s$. (c) we set $x_s=s$. (d) we set $x_s=s^{-1}$. (e) we set $x_s=F_s$, and use that $\sum_{i=2}^k F_i=F_{k+2}-2$. (f) we set $x_s=L_s$, and use that $\sum_{i=2}^k L_i=L_{k+2}-4$. (g) we set $x_s=\ln s$. ☐

We can go further in the generalization of (14).

Proposition 6. *For integers $k,s\geq 2$, let α_{ij},β_{ij}, $i=1,\dots,k$, $j=1,\dots,s$, be given complex numbers such that*
$$\sum_{i=1}^k \alpha_{ij}=\beta_{kj} \quad \textit{and} \quad \sum_{i=1}^k \beta_{ij}=\alpha_{kj}, \tag{46}$$

for $j = 1, \ldots, s$. Then we have the following Bernoulli polynomial identity

$$\sum_{l_1+\cdots+l_k=n} \binom{n}{l_1, \ldots, l_k} \left(\prod_{i=1}^{k-1} \left(\sum_{j=1}^{s} \alpha_{ij} x_j \right)^{l_i} \right) B_{m+l_k} \left(\sum_{j=1}^{s} \alpha_{kj} x_j + y \right) \tag{47}$$
$$= \sum_{l_1+\cdots+l_k=m} \binom{m}{l_1, \ldots, l_k} \left(\prod_{i=1}^{k-1} \left(\sum_{j=1}^{s} \beta_{ij} x_j \right)^{l_i} \right) B_{n+l_k} \left(\sum_{j=1}^{s} \beta_{kj} x_j + y \right).$$

Proof. We have

$$\begin{aligned}
\sum_{l_1+\cdots+l_k=n} \binom{n}{l_1, \ldots, l_k} \left(\prod_{i=1}^{k-1} \left(\sum_{j=1}^{s} \alpha_{ij} x_j \right)^{l_i} \right) \left(\sum_{j=1}^{s} \alpha_{kj} x_j + y \right)^{m+l_k} \\
= \left(\sum_{j=1}^{s} \alpha_{kj} x_j + y \right)^{m} \left(\sum_{i=1}^{k} \sum_{j=1}^{s} \alpha_{ij} x_j + y \right)^{n} \\
= \left(\sum_{j=1}^{s} \alpha_{kj} x_j + y \right)^{m} \left(\sum_{j=1}^{s} \sum_{i=1}^{k} \alpha_{ij} x_j + y \right)^{n},
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
\sum_{l_1+\cdots+l_k=m} \binom{m}{l_1, \ldots, l_k} \left(\prod_{i=1}^{k-1} \left(\sum_{j=1}^{s} \beta_{ij} x_j \right)^{l_i} \right) \left(\sum_{j=1}^{s} \beta_{kj} x_j + y \right)^{n+l_k} \\
= \left(\sum_{j=1}^{s} \beta_{kj} x_j + y \right)^{n} \left(\sum_{i=1}^{k} \sum_{j=1}^{s} \beta_{ij} x_j + y \right)^{m} \\
= \left(\sum_{j=1}^{s} \beta_{kj} x_j + y \right)^{n} \left(\sum_{j=1}^{s} \sum_{i=1}^{k} \beta_{ij} x_j + y \right)^{m}.
\end{aligned} \tag{49}$$

Conditions (46) on the coefficients α_{ij} and β_{ij} give us, from (48) and (49), that

$$\sum_{l_1+\cdots+l_k=n} \binom{n}{l_1, \ldots, l_k} \left(\prod_{i=1}^{k-1} \left(\sum_{j=1}^{s} \alpha_{ij} x_j \right)^{l_i} \right) \left(\sum_{j=1}^{s} \alpha_{kj} x_j + y \right)^{m+l_k} \tag{50}$$
$$= \sum_{l_1+\cdots+l_k=m} \binom{m}{l_1, \ldots, l_k} \left(\prod_{i=1}^{k-1} \left(\sum_{j=1}^{s} \beta_{ij} x_j \right)^{l_i} \right) \left(\sum_{j=1}^{s} \beta_{kj} x_j + y \right)^{n+l_k}.$$

By seeing (50) as an identity between two y-polynomials, we use Theorem 1 to obtain the desired conclusion (formula (47)). $\square$

Some concrete examples of the result of Proposition 6 are the following:

$$(-1)^{n}\sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}(x+2y+3z)^{l_1}(2x+3y+4z)^{l_2}B_{m+l_3}(3x+4y+5z) \tag{51}$$
$$=(-1)^{m}\sum_{l_1+l_2+l_3=m}\binom{m}{l_1,l_2,l_3}2^{l_1}(x+2y+3z)^{l_1}(x+y+z)^{l_2}\times$$
$$\times B_{n+l_3}(1-3(2x+3y+4z)).$$

$$\sum_{l_1+l_2+l_3+l_4=n}\binom{n}{l_1,l_2,l_3,l_4}(x-1)^{l_1}(y-1)^{l_3}B_{m+l_4}(1-x-y) \tag{52}$$
$$=\sum_{l_1+l_2+l_3+l_4=m}\binom{m}{l_1,l_2,l_3,l_4}(-1)^{l_1+l_3}(x+y)^{l_1}2^{l_2}B_{n+l_4}.$$

$$\sum_{l_1+l_2+l_3+l_4=n}\binom{n}{l_1,l_2,l_3,l_4}(-1)^{l_4}(x+y)^{l_1}(z+x)^{l_2}(y+z)^{l_3}B_{m+l_4}(1+x+y+z) \tag{53}$$
$$=\sum_{l_1+l_2+l_3=m}\binom{m}{l_1,l_2,l_3}(-1)^{l_3}(x+y+z)^{l_1+l_2}B_{n+l_3}(x+y+z).$$

To end this section, we show five identities coming from (47), together with some additional known facts.

1. The binomial coefficient recurrence gives us that

$$\sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}(-1)^{l_1+l_2}\binom{p}{r}^{l_1}\binom{p}{r-1}^{l_2}B_{m+l_3}\left(\binom{p+1}{r}\right) \tag{54}$$
$$=\sum_{l_1+l_2+l_3=m}\binom{m}{l_1,l_2,l_3}(-1)^{l_1}\left(\binom{p+1}{r}+1\right)^{l_2}B_{n+l_3}.$$

2. The known fact $B_r(x+1)-B_r(x)=rx^{r-1}$, gives us the identity:

$$\sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}(-1)^{l_2}B_r^{l_1}(x+1)B_r^{l_2}(x)B_{m+l_3}\left(y-rx^{r-1}\right) \tag{55}$$
$$=\sum_{l_1+l_2+l_3=m}\binom{m}{l_1,l_2,l_3}\left(-\frac{r}{2}\right)^{l_1+l_2}x^{(r-1)(l_1+l_2)}B_{n+l_3}(y).$$

3. If $z_1, z_2, z_3 \in \mathbb{C}$ are vertices of an equilateral triangle in the complex plane (equivalently, if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_1 z_3 + z_2 z_3$), we have that

$$\begin{aligned}(-1)^n & \sum_{l_1+l_2+l_3+l_4=n} \binom{n}{l_1, l_2, l_3, l_4} z_1^{2l_1} z_2^{2l_2} z_3^{2l_3} B_{m+l_3} \\ &= (-1)^m \sum_{l_1+l_2+l_3+l_4=m} \binom{m}{l_1, l_2, l_3, l_4} (z_1 z_2)^{l_1} (z_1 z_3)^{l_2} (z_2 z_3)^{l_3} \times \\ &\qquad \times B_{n+l_4} (1 - z_1 z_2 - z_1 z_3 - z_2 z_3) .\end{aligned} \tag{56}$$

4. The known fact for harmonic numbers $\sum_{r=1}^{k-1} H_r = kH_k - k$, gives us that

$$\begin{aligned}\sum_{l_1+\cdots+l_k=n} & \binom{n}{l_1, \ldots, l_k} (-1)^{n+l_k} H_1^{l_1} \cdots H_{k-1}^{l_{k-1}} B_{m+l_k} (kH_k) \\ &= \sum_{l_1+\cdots+l_k=m} \binom{m}{l_1, \ldots, l_k} \frac{k^{m-l_k}}{2^{l_1} \cdots k^{l_{k-1}}} B_{n+l_k} (k) .\end{aligned} \tag{57}$$

5. If $(x_1, \ldots, x_k)$ is a Pythagorean k-tuple (that is, if $x_1^2 + \cdots + x_{k-1}^2 = x_k^2$), we have

$$\begin{aligned}\sum_{l_1+\cdots+l_k=n} & \binom{n}{l_1, \ldots, l_k} (-1)^{l_k} x_1^{2l_1} \cdots x_{k-1}^{2l_{k-1}} B_{m+l_k} \left(1 + x_k^2\right) \\ &= \sum_{l_1+\cdots+l_k=m} \binom{m}{l_1, \ldots, l_k} (-1)^{l_k} x_1^{2l_1} \cdots x_{k-1}^{2l_{k-1}} B_{n+l_k}.\end{aligned} \tag{58}$$

3 Some identities from integration

Beginning with the multinomial expansion of $(x_1 + \cdots + x_k)^n$, we fix two indices $1 \le j_1, j_2 \le k$. It is possible to show (we leave the details to the reader), that if we integrate m_1 times with respect to variable x_{j_1} (from 0 to x_{j_1}), and m_2 times with respect to variable x_{j_2} (from

0 to x_{j_2}), we obtain

$$\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\ldots,l_k}\frac{x_{j_1}^{l_{j_1}+m_1}x_{j_2}^{l_{j_2}+m_2}\prod_{i=1,i\neq j_1,j_2}^{k}x_i^{l_i}}{\prod_{i=1}^{m_1}(l_{j_1}+i)\prod_{i=1}^{m_2}(l_{j_2}+i)} \tag{59}$$

$$=\frac{\left(\sum_{i=1}^{k}x_i\right)^{n+m_1+m_2}}{\prod_{i=1}^{m_1+m_2}(n+i)}-\sum_{t=0}^{m_2-1}\frac{x_{j_2}^{t}\left(\sum_{i=1,i\neq j_2}^{k}x_i\right)^{n+m_1+m_2-t}}{t!\prod_{i=1}^{m_1+m_2-t}(n+i)}$$

$$-\sum_{s=0}^{m_1-1}\frac{x_{j_1}^{s}\left(\sum_{i=1,i\neq j_1}^{k}x_i\right)^{n+m_1+m_2-s}}{s!\prod_{i=1}^{m_1+m_2-s}(n+i)}$$

$$+\sum_{t=0}^{m_2-1}\sum_{s=0}^{m_1-1}\frac{x_{j_2}^{t}x_{j_1}^{s}\left(\sum_{i=1,i\neq j_1,j_2}^{k}x_i\right)^{n+m_1+m_2-s-t}}{s!t!\prod_{i=1}^{m_1+m_2-s-t}(n+i)}.$$

With this expression, together with Theorem 1, we obtain the following identity for Bernoulli polynomials:

$$\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\ldots,l_k}\frac{x_{j_2}^{l_{j_2}+m_2}\prod_{i=1,i\neq j_1,j_2}^{k}x_i^{l_i}}{\prod_{i=1}^{m_1}(l_{j_1}+i)\prod_{i=1}^{m_2}(l_{j_2}+i)}B_{l_{j_1}+m_1}(x_{j_1}) \tag{60}$$

$$=\frac{B_{n+m_1+m_2}\left(\sum_{i=1}^{k}x_i\right)}{\prod_{i=1}^{m_1+m_2}(n+i)}-\sum_{t=0}^{m_2-1}\frac{x_{j_2}^{t}B_{n+m_1+m_2-t}\left(\sum_{i=1,i\neq j_2}^{k}x_i\right)}{t!\prod_{i=1}^{m_1+m_2-t}(n+i)}$$

$$-\sum_{s=0}^{m_1-1}\frac{\left(\sum_{i=1,i\neq j_1}^{k}x_i\right)^{n+m_1+m_2-s}B_s(x_{j_1})}{s!\prod_{i=1}^{m_1+m_2-s}(n+i)}$$

$$+\sum_{t=0}^{m_2-1}\sum_{s=0}^{m_1-1}\frac{x_{j_2}^{t}\left(\sum_{i=1,i\neq j_1,j_2}^{k}x_i\right)^{n+m_1+m_2-s-t}B_s(x_{j_1})}{s!t!\prod_{i=1}^{m_1+m_2-s-t}(n+i)}.$$

(and a similar expression interchanging j_1 with j_2). If $j_3\neq j_1,j_2$, we obtain also from (59) and Theorem 1 that

$$\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\ldots,l_k}\frac{x_{j_1}^{l_{j_1}+m_1}x_{j_2}^{l_{j_2}+m_2}\prod_{i=1,i\neq j_1,j_2,j_3}^{k}x_i^{l_i}}{\prod_{i=1}^{m_1}(l_{j_1}+i)\prod_{i=1}^{m_2}(l_{j_2}+i)}B_{l_{j_3}}\left(x_{l_{j_3}}\right) \tag{61}$$

$$=\frac{B_{n+m_1+m_2}\left(\sum_{i=1}^{k}x_i\right)}{\prod_{i=1}^{m_1+m_2}(n+i)}-\sum_{t=0}^{m_2-1}\frac{x_{j_2}^{t}B_{n+m_1+m_2-t}\left(\sum_{i=1,i\neq j_2}^{k}x_i\right)}{t!\prod_{i=1}^{m_1+m_2-t}(n+i)}$$

$$-\sum_{s=0}^{m_1-1}\frac{x_{j_1}^{s}B_{n+m_1+m_2-s}\left(\sum_{i=1,i\neq j_1}^{k}x_i\right)}{s!\prod_{i=1}^{m_1+m_2-s}(n+i)}$$

$$+\sum_{t=0}^{m_2-1}\sum_{s=0}^{m_1-1}\frac{x_{j_2}^{t}x_{j_1}^{s}B_{n+m_1+m_2-s-t}\left(\sum_{i=1,i\neq j_1,j_2}^{k}x_i\right)}{s!t!\prod_{i=1}^{m_1+m_2-s-t}(n+i)}.$$

Some examples are the following:

$$\begin{aligned}\frac{B_{n+1}(x_1+x_2+x_3)}{n+1} &= \sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}\frac{x_2^{l_2}x_3^{l_3}}{l_1+1}B_{l_1+1}(x_1)+\frac{(x_2+x_3)^{n+1}}{n+1} \\ &= \sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}\frac{x_1^{l_1+1}x_2^{l_2}}{l_1+1}B_{l_3}(x_3)+\frac{B_{n+1}(x_2+x_3)}{n+1}.\end{aligned} \tag{62}$$

$$\begin{aligned}&\frac{B_{n+2}(x_1+x_2+x_3)}{(n+1)(n+2)} \\ =& \sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}\frac{x_2^{l_2+1}x_3^{l_3}B_{l_1+1}(x_1)}{(l_1+1)(l_2+1)}+\frac{B_{n+2}(x_1+x_3)+(x_2+x_3)^{n+2}-x_3^{n+2}}{(n+1)(n+2)} \\ =& \sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}\frac{x_1^{l_1+1}x_2^{l_2+1}B_{l_3}(x_3)}{(l_1+1)(l_2+1)}+\frac{B_{n+2}(x_1+x_3)+B_{n+2}(x_2+x_3)-B_{n+2}(x_3)}{(n+1)(n+2)} \\ =& \sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}\frac{x_2^{l_2}x_3^{l_3}B_{l_1+2}(x_1)}{(l_1+1)(l_1+2)}+\frac{(x_2+x_3)^{n+2}}{(n+1)(n+2)}+\frac{(x_2+x_3)^{n+1}}{n+1}B_1(x_1) \\ =& \sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}\frac{x_1^{l_1+2}x_2^{l_2}B_{l_3}(x_3)}{(l_1+1)(l_1+2)}+\frac{B_{n+2}(x_2+x_3)}{(n+1)(n+2)}+\frac{B_{n+1}(x_2+x_3)}{n+1}x_1\end{aligned} \tag{63}$$

$$\begin{aligned}\frac{B_{n+3}(4x)}{(n+1)(n+2)(n+3)} &= \sum_{l_1+l_2+l_3+l_4=n}\binom{n}{l_1,l_2,l_3,l_4}\frac{x^{n-l_1}B_{l_1+3}(x)}{(l_1+1)(l_1+2)(l_1+3)} \\ &\quad+\frac{(3x)^{n+3}}{(n+1)(n+2)(n+3)}+\frac{(3x)^{n+2}B_1(x)}{(n+1)(n+2)}+\frac{(3x)^{n+1}B_2(x)}{2(n+1)} \\ &= \sum_{l_1+l_2+l_3+l_4=n}\binom{n}{l_1,l_2,l_3,l_4}\frac{x^{n+3-l_3}B_{l_3}(x)}{(l_1+1)(l_1+2)(l_1+3)} \\ &\quad+\frac{B_{n+3}(3x)}{(n+1)(n+2)(n+3)}+\frac{xB_{n+2}(3x)}{(n+1)(n+2)}+\frac{x^2B_{n+1}(3x)}{2(n+1)}\end{aligned} \tag{64}$$

Some particular numerical cases of (60) are given in the following corollary.

Corollary 7. *We have the identities*

$$\sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}\frac{(-1)^{l_2+1}B_{l_1+1}}{(l_1+1)(l_2+1)}=\frac{1}{(n+1)(n+2)}.$$

$$\sum_{l_1+l_2+l_3=n}\binom{n}{l_1,l_2,l_3}\frac{B_{l_3}}{(l_1+1)(l_2+1)}=\frac{1}{n+1}.$$

$$\sum_{l_1+l_2+l_3=2n}\binom{2n}{l_1,l_2,l_3}\frac{(-1)^{l_1}B_{l_3}}{(l_1+1)(l_2+1)(l_2+2)}=\frac{1}{2n+2}.$$

$$\sum_{l_1+l_2+l_3=2n+1}\binom{2n+1}{l_1,l_2,l_3}\frac{B_{l_3}}{(l_1+1)(l_1+2)(l_2+1)(l_2+2)}=\frac{1}{4(n+1)(n+2)(2n+3)}.$$

$$\sum_{l_1+l_2+l_3=2n}\binom{2n}{l_1,l_2,l_3}\left(\frac{B_{l_1+1}B_{l_2+1}}{(l_1+1)(l_2+1)}-\frac{B_{l_1+2}B_{l_2}}{(l_1+1)(l_1+2)}\right)=\frac{1-B_{2n+2}}{(2n+1)(2n+2)}.$$

$$\sum_{l_1+l_2+l_3+l_4=n}\binom{n}{l_1,l_2,l_3,l_4}\frac{(-1)^{l_4}B_{l_2+3}}{(l_1+1)(l_2+1)(l_2+2)(l_2+3)}=-\frac{n}{12(n+2)(n+3)(n+4)}.$$

$$\sum_{l_1+l_2+l_3+l_4=2n}\binom{2n}{l_1,l_2,l_3,l_4}\frac{(-1)^{l_4}B_{l_1+2}}{\prod_{i=1}^{2}(l_1+i)\prod_{j=1}^{3}(l_2+j)}=\frac{2n+3-2(2n+5)B_{2n+4}}{2\prod_{i=1}^{5}(2n+i)}.$$

4 New Bernoulli identities from old combinatorial identities

We form groups of identities, attending to the common theme behind them and/or the source where they come from. We begin with a group of identities coming from some of the Gould's works [7, 8, 11].

Proposition 8 (**Gould-Bernoulli identities**). *We have the following Bernoulli polynomial identities*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}B_{n+1}(x-k)=\frac{(n+1)!}{2}(2x-n-1). \tag{65}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{2(n+1)}{2k+1}\binom{n}{2k}B_{2k+1}(x)=B_{n+1}(x+1)+(-1)^n B_{n+1}(x-1). \tag{66}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{k-1}B_{n-k}(x+k)=B_n(x-1). \tag{67}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}(-1)^k B_k(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+m-k}{n}B_k(-x). \tag{68}$$

$$\sum_{k=0}^{n}\frac{(-1)^k}{4^k}\binom{n}{k}\binom{2k}{k}B_k(x)=\frac{1}{4^n}\sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{2k}{k}B_k(x-1). \tag{69}$$

$$\binom{n+r}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{k+r}{k}}B_{k+r}(x)=B_{n+r}(x+1)-\sum_{k=0}^{r-1}\binom{n+r}{k}B_k(x). \tag{70}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\sum_{j=0}^{n-2k}\binom{n}{2k}\binom{2k}{k}\binom{n+k}{k}\binom{n-2k}{j}B_{k+j}(x)B_{n-k-j}(y)=\sum_{k=0}^{n}\binom{n}{k}^3B_k(x)B_{n-k}(y). \tag{71}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}B_r(x-bk)=\begin{cases}0 & \text{if } r<n;\\ n!b^n & \text{if } r=n.\end{cases} \tag{72}$$

$$B_n(x+1)=\sum_{r=0}^{n}\sum_{k=r}^{n}\frac{(-1)^r(k-r)^k}{r+1}\binom{n}{k}\binom{k}{r}B_{n-k}(x-k). \tag{73}$$

$$y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{B_n(x-k)}{(y+k)^2}=B_n(x+y)\sum_{j=0}^{n}\frac{1}{y+j}-nB_{n-1}(x+y). \tag{74}$$

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{B_j\left(x\right)B_{k-j}\left(y\right)}{j!\left(k-j\right)!}z^{n-k}=\sum_{k=1}^{n}\frac{B_k\left(x+kz\right)B_{n-k}\left(y-kz\right)-B_k\left(x\right)B_{n-k}\left(y\right)}{k!\left(n-k\right)!}. \tag{75}$$

Proof. For (65), (66), (67), (68), (69), (70) and (71), see identities (1.14), (1.38), (1.117), (3.17), (3.84), (4.13) and (6.7) in Gould's book [7], respectively. For (72) see identity (10.6) in Gould's book [8, Vol. 4]. For (73) and (74), see identities (1.17) and (1.57), in Gould's book [8, Vol. 5], respectively. For (75), see formula (2.3) in [11]. □

Next we consider the Bernoulli version of some of the Abel's sums.

Proposition 9 (**Abel-Bernoulli sums**). *We have the following Bernoulli polynomial identities*

$$\sum_{l=0}^{n}\binom{n}{l}\left(x+l\right)^{l-1}B_{n-l}\left(y+n-l\right)=\frac{1}{x}B_n\left(y+x+n\right). \tag{76}$$

$$\sum_{l=0}^{n}\binom{n}{l}\left(x+l\right)^{l-2}B_{n-l}\left(y+n-l\right)=\frac{1}{x^2}B_n\left(x+y+n\right)-\frac{n}{x\left(x+1\right)}B_{n-1}\left(x+y+n\right). \tag{77}$$

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}\left(x+l\right)^{l-3}B_{n-l}\left(y+n-l\right) \qquad (78)\\ =\;&\frac{1}{x^3}B_n\left(x+y+n\right)-\frac{n\left(2x+1\right)}{x^2\left(x+1\right)^2}B_{n-1}\left(x+y+n\right)+\frac{n\left(n-1\right)}{x\left(x+1\right)\left(x+2\right)}B_{n-2}\left(x+y+n\right).\end{aligned}$$

$$\sum_{l=0}^{n}\binom{n}{l}\left(x+l\right)^{l}B_{n-l}\left(y+n-l\right)=\sum_{l=0}^{n}\binom{n}{l}l!B_{n-l}\left(x+y+n\right). \tag{79}$$

$$\sum_{l=0}^{n}\frac{\left(x+l\right)^{l+1}}{l!\left(n-l\right)!}B_{n-l}\left(y+n-l\right)=\sum_{l_1+l_2+l_3=n}\frac{x+l_1}{l_3!}B_{l_3}\left(x+y+n\right). \tag{80}$$

Proof. See the corresponding Abel's sums in Riordan's book [22, p. 23]. □

By using one of the Hurwitz multinomial extensions of Abel's identities [13], we have the following proposition:

Proposition 10 (**Abel-Hurwitz-Bernoulli identity**). *We have the following Bernoulli polynomial identities* *(a) For* $1 \leq r \leq m$

$$\sum_{l_1+\cdots+l_m=n} \binom{n}{l_1,\ldots,l_m} B_{l_r}(x_r+l_r) \prod_{j=1,j\neq r}^{m} (x_j+l_j)^{l_j} \tag{81}$$
$$= \sum_{l=0}^{n} \binom{n}{l} \frac{(m+l-2)!}{(m-2)!} B_{n-l}\left(\sum_{i=1}^{m} x_i + n\right).$$

(b)

$$\sum_{l_1+\cdots+l_m=n} \frac{1}{l_1!\cdots l_m!} \prod_{j=1}^{m} B_{l_j}(x_j+l_j) \tag{82}$$
$$= \sum_{l=0}^{n} \sum_{s_1+\cdots+s_{m+1}=n-l} \frac{n^{s_{m+1}}}{s_1!\cdots s_{m+1}!} \binom{m+l-2}{l} \prod_{j=1}^{m} B_{s_j}(x_j).$$

Proof. The polynomial identity behind Bernoulli identities (81) and (82) comes from Riordan's book [22, p. 25], formula (35). Identity (81) is obtained by seeing the mentioned formula as an identity between two x_r-polynomials and applying Theorem 1. Identity (82) is obtained expanding first the multinomial term $\left(\sum_{i=1}^{m} x_i + n\right)^{n-l}$ of the right-hand side of the mentioned formula, and then applying Theorem 1 m times (one for each variable x_i, $i = 1, \ldots, m$). $\square$

By using a result concerning Abel-type sums, we have

Proposition 11. *We have the following Bernoulli polynomial identities*

$$z^n B_m(x+y) = \sum_{l=0}^{n} \sum_{k=0}^{m} \binom{n}{l} \binom{m}{k} x(x-lu)^{k-1} (-ku)^l (z+ku)^{n-l} B_{m-k}(y+lu). \tag{83}$$

$$(x+y)^m B_n(z) = \sum_{l=0}^{n} \sum_{k=0}^{m} \binom{n}{l} \binom{m}{k} x(x-lu)^{k-1} (-ku)^l (y+lu)^{m-k} B_{n-l}(z+ku). \tag{84}$$

$$B_m(x+y) B_n(z) = \sum_{l=0}^{n} \sum_{k=0}^{m} \binom{n}{l} \binom{m}{k} x(x-lu)^{k-1} (-ku)^l B_{m-k}(y+lu) B_{n-l}(z+ku). \tag{85}$$

Proof. All three identities are obtained applying Theorem 1 to formula (1) of the Huang and Liu paper [12] about Abel-type polynomial identities: identity (83) is obtained by seeing the mentioned formula (1) as an identity between two y-polynomials. Identity (84) is obtained by seeing that formula as an identity between two z-polynomials. And identity (85) is obtained by seeing (83) as an identity between two z-polynomials (or (84) as an identity between two y-polynomials). □

In 2001, Simons [24] found a "curious identity" (of two polynomials involving some products of binomial coefficients in their coefficients). Four years later, Munarini [17] generalized Simons' result. In the following proposition we show the Bernoulli version of each of these identities.

Proposition 12. *We have the following Bernoulli polynomial identities*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}B_k(x)=(-1)^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}B_k(-x) \tag{86}$$

$$\sum_{k=0}^{n}\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(-y)^{n-k}B_k(x+y)=\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}y^{n-k}B_k(x). \tag{87}$$

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\alpha}{k}\binom{2\alpha-2k}{n-2k}(-1)^k y^{2k}B_{n-2k}(x+y)=\sum_{k=0}^{n}\binom{\alpha}{k}\binom{2\alpha-k}{n-k}(2y)^k B_{n-k}(x). \tag{88}$$

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{2\beta+n+1}{n-2k}\binom{\beta+k}{k}x^{2k}B_{n-2k}(y-x)=\sum_{k=0}^{n}\binom{2\beta+n+1}{n-k}\binom{\beta+k}{k}(-2x)^k B_{n-k}(y). \tag{89}$$

Proof. Identity (86) comes from Theorem 1 and the "curious identity" of Simons [24]. Identities (87), (88) and (89) are obtained with Theorem 1, and the work of Munarini [17] concerning a generalization of Simons' identity, formulas (3), (8) —corrected—, and (9), respectively. □

A natural corollary from (86) is the following:

Corollary 13. *The n-th degree polynomial*

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}B_k(x), \tag{90}$$

is even (odd) if and only if n is even (odd, respectively). In particular, if n is odd, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} B_k = 0. \tag{91}$$

Proof. This is a straightforward consequence of (86). □

In [19] we used the following two-parameters generalization of Stirling numbers of the second kind, due to P. Blasiak [2]: for positive integers r, s, such that $r \geq s$, the (r, s)-Stirling numbers of the second kind $S_{r,s}(p, k)$ can be defined by the explicit formula

$$S_{r,s}(p,k) = \frac{(-1)^k}{k!} \sum_{i=s}^{k} (-1)^i \binom{k}{i} \prod_{l=1}^{p} (i + (l-1)(r-s))^{\underline{s}}. \tag{92}$$

The case $r = s = 1$ corresponds to the standard Stirling numbers of the second kind $S(p, k)$. It can be shown that $S_{r,s}(p, k) \neq 0$ for $r \leq k \leq rp$. In the case $r = s$ we proved the following identity [19, formula (58)]

$$\sum_{k=r}^{rp} k! S_{r,r}(p,k)\, x^{k-r} = \sum_{k=r}^{rp} (-1)^{k-rp}\, k! S_{r,r}(p,k)\,(x+1)^{k-r}, \tag{93}$$

Thus we have

Proposition 14. *We have the following Bernoulli polynomial identity*

$$\sum_{k=r}^{rp} k! S_{r,r}(p,k)\, B_{k-r}(-x) = (-1)^{r(p+1)} \sum_{k=r}^{rp} k! S_{r,r}(p,k)\, B_{k-r}(x). \tag{94}$$

Proof. Use Theorem 1 in (93), change x by $-x$ and use (8). □

Corollary 15. *The $r(p-1)$-th degree polynomial*

$$Q_{r,p}(x) = \sum_{k=r}^{rp} k! S_{r,r}(p,k)\, B_{k-r}(x), \tag{95}$$

is even if and only if r is even or p is odd, and is odd if and only if r is odd and p is even. In particular, if r is odd and p is even we have the identity

$$\sum_{k=r}^{rp} k! S_{r,r}(p,k)\, B_{k-r} = 0. \tag{96}$$

Proof. The affirmations are easy to see consequences from (94). □

In the particular case $r=1$ (where the standard Stirling numbers of the second kind are involved), we have: for $p \in \mathbb{N}$, the p-th degree polynomial $Q_p(x) = \sum_{k=1}^{p} k! S(p,k) B_{k-1}(x)$ is even (odd) if and only if p is odd (even, respectively). In particular, if p is even, we have the identity $\sum_{k=1}^{p} k! S(p,k) B_{k-1} = 0$.

By using the explicit formula for the (r,r)-Stirling numbers of the second kind ((92) with $s=r$), identity (96) acquires the following nice form (for r odd and p even)

$$\sum_{k=0}^{r(p-1)} \sum_{l=0}^{k} \binom{k+r}{l+r} \binom{l+r}{r}^{p} (-1)^{k+l} B_k = 0. \tag{97}$$

In the following proposition we show the Bernoulli polynomial version of some combinatorial identities obtained by (improper) integration of some more sophisticated expressions than those we considered in Section 4.

Proposition 16. *We have the following Bernoulli polynomial identities*

$$\sum_{l=0}^{n} \frac{1+(-1)^l}{l+1} B_{l+1}(x) = \sum_{l=0}^{n} \frac{(-1)^l}{l+1} \binom{n+1}{l+1} \left(B_{l+1}(x+1) + (-1)^l B_{l+1}(x-1) \right). \tag{98}$$

$$\sum_{l=1}^{n} \binom{n}{l} \frac{B_l(x-1)}{l} = \sum_{l=1}^{n} \frac{B_l(x)-1}{l}. \tag{99}$$

$$\sum_{l=1}^{k} (-1)^l \frac{B_l(x-1)}{l} = \sum_{l=1}^{k} \binom{k}{l} (-1)^l \frac{B_l(x)-1}{l}. \tag{100}$$

Proof. See [1, Cor. 2.1 and 2.3]. □

To end this section, we want to consider the famous Faulhaber formula (closely related to the origins of Bernoulli numbers)

$$\sum_{j=1}^{n} j^r = \frac{B_{r+1}(n+1) + (-1)^r B_{r+1}}{r+1}, \tag{101}$$

where r is a non-negative integer.

Proposition 17 (Faulhaber-type formula). *Let k, n be non-negative integers. We have the following Bernoulli polynomial identity*

$$\sum_{j=1}^{n} B_k(x+j) = \frac{1}{k+1} \sum_{r=0}^{k} \binom{k+1}{r+1} \left(B_{r+1}(n+1) + (-1)^r B_{r+1} \right) B_{k-r}(x). \tag{102}$$

Proof. We use Faulhaber formula (101) to write

$$\sum_{j=1}^{n} (x+j)^k = \sum_{r=0}^{k} \binom{k}{r} x^{k-r} \sum_{j=1}^{n} j^r = \sum_{r=0}^{k} \binom{k}{r} \frac{B_{r+1}(n+1) + (-1)^r B_{r+1}}{r+1} x^{k-r}. \tag{103}$$

By using Theorem 1 in (103) we obtain the desired conclusion (102). □

5 Acknowledgments

I thank the referee for his/her careful reading of the first version of this work and the corresponding (mathematical and non-mathematical) comments, which certainly helped me to prepare the final version of the paper.

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2010 *Mathematics Subject Classification*: Primary 11B68; Secondary 05A10.
Keywords: Bernoulli number, Bernoulli polynomial, Bernoulli identity.

Received June 8 2015; revised version received December 23 2015. Published in *Journal of Integer Sequences*, January 7 2016.

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