Journal of Integer Sequences, Vol. 19 (2016),
Article 16.3.2

A Sum of Gcd's over Friable Numbers

Julian Ransford[1]
Département de Mathématiques et de Statistique
Université Laval
Québec, PQ G1V 0A6
Canada
julian.ransford.1@ulaval.ca

Abstract

We study the function

$$g(n,y) := \sum_{\substack{i \le n \\ P(i) \le y}} \gcd(i,n),$$

where $P(n)$ denotes the largest prime factor of n, and we derive some estimates for its summatory function.

1 Introduction

Let $P(n)$ denote the largest prime factor of n, with the convention that $P(1) = 1$. We say that n is y-*friable* if $P(n) \le y$. Friable numbers have been studied in great detail by many authors, and a lot of attention has been devoted to estimates and properties of the function

$$\Psi(x,y) := \sum_{\substack{n \le x \\ P(n) \le y}} 1.$$

The article of Hildebrand–Tenenbaum [6] contains a survey on the subject, as well as numerous references. One of the most interesting results concerning this function is due to Dickman. He showed that, for any fixed u, we have

$$\Psi(x, x^{1/u}) \sim x\rho(u) \qquad (x \to \infty),$$

[1]Research supported by an NSERC USRA scholarship.

where $\rho(u)$ is the *Dickman function*, defined as the solution to the delayed differential equation

$$\begin{cases} \rho(u) = 1, & \text{if } 0 \le u \le 1; \\ u\rho'(u) + \rho(u-1) = 0, & \text{if } u > 1. \end{cases}$$

For more details, see for example De Koninck–Luca [7, Thm. 9.3, p. 134]. Hildebrand [5] showed that, in fact, the following stronger estimate holds.

Theorem 1. *Given any* $\epsilon > 0$*, uniformly for* $\exp((\log\log x)^{\frac{5}{3}+\epsilon}) \le y \le x$,

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right). \tag{1}$$

It turns out that improving the region of validity for this estimate improves our knowledge of the distribution of the zeros of the Riemann zeta function, and that (1) holds uniformly for $(\log x)^{2+\epsilon} \le y \le x$ if and only if the Riemann hypothesis is true; see Hildebrand [4].

On the other hand, the function $\Psi(x,y)$ behaves quite differently for very small values of y. Indeed, Ennola showed [6, Thm. 1.5] the following estimate.

Theorem 2. *Uniformly for* $2 \le y \le \sqrt{\log x}$,

$$\Psi(x,y) = \frac{1}{\pi(y)!}\prod_{p\le y}\left(\frac{\log x}{\log p}\right)\left(1 + O\left(\frac{y^2}{\log x \log y}\right)\right).$$

In particular, if y is fixed, this yields

$$\Psi(x,y) = \frac{1}{\pi(y)!}\prod_{p\le y}\left(\frac{1}{\log p}\right)(\log x)^{\pi(y)} + O((\log x)^{\pi(y)-1}). \tag{2}$$

In this paper, we introduce and study the function

$$g(n,y) := \sum_{\substack{i\le n\\ P(i)\le y}} \gcd(i,n). \tag{3}$$

This can be seen as a generalization of the function

$$g(n) := \sum_{i\le n} \gcd(i,n).$$

The latter was studied by Broughan [1], who showed that $g(n)$ is a multiplicative function. Indeed, we have that [1, Thm. 2.3]

$$g(n) = \sum_{d\mid n} \phi(d)\frac{n}{d}, \tag{4}$$

where $\phi(n)$ is Euler's totient function. Using well-known estimates for summatory functions related to $\phi(n)$, Broughan has shown that

$$\sum_{n\le x} g(n) = \frac{3x^2\log x}{\pi^2} + O(x^2) \qquad (x\to\infty).$$

Our main goal is to derive estimates for

$$G(x,y) := \sum_{n\le x} g(n,y)$$

in various ranges of $y \le x$.

2 Main results

Let $g(n,y)$ be defined as in (3). Also, here and in what follows, we set $u := \log x/\log y$.

Theorem 3. *For y fixed, we have, as $x\to\infty$*

$$G(x,y) = \left(\frac{1}{(2\pi(y))!}\prod_{p\le y}\frac{p-1}{p(\log p)^2}\right) x(\log x)^{2\pi(y)} + O(x(\log x)^{2\pi(y)-1}),$$

where $\pi(y)$ denotes the prime counting function.

Theorem 4. *We have*

$$G(x,y) = \frac{3x^2\rho_2(u)\log y}{\pi^2}\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right),$$

uniformly for

$$\exp((\log\log x)^{\frac{5}{3}+\epsilon}) \le y \le x,$$

where $\rho_2(u)$ is the continuous solution to the delayed differential equation

$$\begin{cases} \rho_2(u) = u, & \text{if } 0\le u\le 1;\\ u\rho_2'(u) - \rho_2(u) + 2\rho_2(u-1) = 0, & \text{if } u>1. \end{cases}$$

3 Preliminary results

An important fact that we will need later is the value of $g(n)$ when n is a prime power.

Lemma 5. *If p is a prime, then*

$$g(p^k) = (k+1)p^k - kp^{k-1}.$$

Proof. Using (4), we have

$$g(p^k) = \sum_{d|p^k} d\phi\left(\frac{p^k}{d}\right) = \left(\sum_{j=0}^{k-1} p^j \cdot p^{k-j-1}(p-1)\right) + p^k$$
$$= p^{k-1}(p-1)\sum_{j=0}^{k-1} 1 + p^k = p^{k-1}(p-1)k + p^k = (k+1)p^k - kp^{k-1}. \qquad \square$$

The next lemma is the analog of (4) for friable numbers.

Lemma 6. *The function $g(n, y)$ satisfies the relation*

$$g(n, y) = \sum_{\substack{d|n \\ P(d)\le y}} d\phi\left(\frac{n}{d}, y\right), \tag{5}$$

where we set

$$\phi(n, y) := \sum_{\substack{k\le n \\ (n,k)=1 \\ P(k)\le y}} 1.$$

Proof. The proof is very similar to the proof of (4) given in Broughan [1]. It is clear that the gcd's in the sum are divisors of n and must be y-friable. How many times does a y-friable divisor of n appear in the sum (5)? We have that

$$\gcd(i, n) = d \Longleftrightarrow \gcd\left(\frac{n}{d}, \frac{i}{d}\right) = 1.$$

Since i/d is always y-friable we conclude that d is in the sum as often as there are integers less than or equal to n/d that are coprime with n/d and y-friable. This is precisely the same thing as $\phi(n/d, y)$, and therefore

$$g(n, y) = \sum_{\substack{d|n \\ P(d)\le y}} d\phi\left(\frac{n}{d}, y\right). \qquad \square$$

Using Lemma 6, we obtain that

$$\sum_{n\le x} g(n, y) = \sum_{n\le x} \sum_{\substack{d|n \\ P(d)\le y}} d\phi\left(\frac{n}{d}, y\right) = \sum_{\substack{d\le x \\ P(d)\le y}} \sum_{\substack{n\le x \\ d|n}} d\phi\left(\frac{n}{d}, y\right)$$
$$= \sum_{\substack{d\le x \\ P(d)\le y}} d \sum_{kd\le x} \phi\left(\frac{kd}{d}, y\right) = \sum_{\substack{d\le x \\ P(d)\le y}} d \sum_{k\le \frac{x}{d}} \phi(k, y)$$
$$= \sum_{\substack{d\le x \\ P(d)\le y}} d\Phi\left(\frac{x}{d}, y\right),$$

where we have set
$$\Phi(x,y) := \sum_{n\le x} \phi(n,y).$$

Lemma 7. *For fixed y, we have*
$$\Phi(x,y) = \frac{1}{\pi(y)!}\prod_{p\le y}\left(\frac{p-1}{p\log p}\right) x(\log x)^{\pi(y)} + O(x(\log x)^{\pi(y)-1}) \qquad (x\to\infty).$$

Proof. Let $P_y := \prod_{p\le y} p$. Thus
$$\sum_{n\le x}\phi(n,y) = \sum_{d\mid P_y}\ \sum_{\substack{n\le x\\ (n,P_y)=d}} \phi(n,y).$$

If $(n,P_y)=d$, then $\phi(n,y)$ will only count integers less than or equal to n that contain only prime factors of P_y that do not divide d. It is therefore clear that the smaller d is, the bigger $\phi(n,y)$ will be. Hence, it suffices to evaluate the sum
$$\sum_{\substack{n\le x\\ (n,P_y)=1}} \phi(n,y),$$

since all the other terms will be absorbed in its error term. But $\phi(n,y)=\Psi(n,y)$ when $(n,P_y)=1$. Therefore
$$\begin{aligned}
\sum_{\substack{n\le x\\ (n,P_y)=1}} \phi(n,y) &= \sum_{\substack{n\le x\\ (n,P_y)=1}} \Psi(n,y)\\
&= \sum_{n\le x}\Psi(n,y) - \sum_{\substack{n\le x\\ 2\mid n}}\Psi(n,y) - \sum_{\substack{n\le x\\ 3\mid n}}\Psi(n,y) - \cdots\\
&\quad + \sum_{\substack{n\le x\\ 2\cdot 3\mid n}}\Psi(n,y) + \sum_{\substack{n\le x\\ 2\cdot 5\mid n}}\Psi(n,y) + \sum_{\substack{n\le x\\ 2\cdot 7\mid n}}\Psi(n,y) + \cdots\\
&\quad + (-1)^{\omega(P_y)} \sum_{\substack{n\le x\\ P_y\mid n}}\Psi(n,y)\\
&= \left(1 - \frac{1}{2} - \cdots + \frac{1}{2\cdot 3} + \cdots + \frac{(-1)^{\omega(P_y)}}{P_y}\right)\frac{x(\log x)^{\pi(y)}}{\pi(y)!\prod_{p\le y}\log p}\\
&\quad + O(x(\log x)^{\pi(y)-1})\\
&= \left(\sum_{d\mid P_y}\frac{\mu(d)}{d}\right)\frac{x(\log x)^{\pi(y)}}{\pi(y)!\prod_{p\le y}\log p} + O(x(\log x)^{\pi(y)-1})\\
&= \frac{1}{\pi(y)!}\prod_{p\le y}\left(\frac{p-1}{p\log p}\right)x(\log x)^{\pi(y)} + O(x(\log x)^{\pi(y)-1}),
\end{aligned}$$

where we used (2), the fact that for every integer $k \geq 1$

$$\sum_{n \leq x} (\log n)^k = x(\log x)^k + O(x(\log x)^{k-1}),$$

and the fact that

$$\sum_{d | P_y} \frac{\mu(d)}{d} = \frac{\phi(P_y)}{P_y} = \prod_{p \leq y} \frac{p-1}{p}. \qquad \square$$

Lemma 8. *For each integer $n \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{n+k} = \frac{1}{n\binom{2n}{n}}.$$

Proof. We will show that, for every $n \geq 1$ and for all x that do not cause a division by 0,

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{x+k}. \tag{6}$$

It is easy to see that (6) is true if $n = 1$. Assume that (6) is true for some $n \geq 1$. Then,

$$\begin{aligned} \frac{(n+1)!}{x(x+1)\cdots(x+n)(x+n+1)} &= \left(\frac{n+1}{x+n+1}\right) \frac{n!}{x(x+1)(x+2)\cdots(x+n)} \\ &= (n+1) \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(x+n+1)(x+k)}. \end{aligned}$$

Using partial fractions, we find that

$$\frac{1}{(x+n+1)(x+k)} = \frac{1}{(n+1-k)(x+k)} - \frac{1}{(n+1-k)(x+n+1)}.$$

Since

$$\frac{n+1}{n+1-k}\binom{n}{k} = \binom{n+1}{k},$$

we have

$$\begin{aligned}
\frac{(n+1)!}{x(x+1)\cdots(x+n)(x+n+1)} &= (n+1)\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(x+n+1)(x+k)} \\
&= (n+1)\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{1}{(n+1-k)(x+k)} - \frac{1}{(n+1-k)(x+n+1)}\right) \\
&= \sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k}{x+k} - \frac{1}{x+n+1}\sum_{k=0}^{n}\binom{n+1}{k}(-1)^k \\
&= \sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k}{x+k} - \frac{1}{x+n+1}\left(\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k - (-1)^{n+1}\right) \\
&= \sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k}{x+k} + \frac{(-1)^{n+1}}{x+n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(-1)^k}{x+k},
\end{aligned}$$

the simplification on the penultimate line because, by the binomial theorem,

$$\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k = (1-1)^{n+1} = 0.$$

Hence, by induction, (6) is true for all $n \geq 1$. Setting $x = n$, we obtain the desired result. $\square$

Lemma 9. *For all* $2 \leq y \leq x$, *we have*

$$G(x,y) = \sum_{\substack{i \leq x \\ P(i) \leq y}} \left(x\frac{g(i)}{i} - g(i)\right) + O(\Psi(x,y)).$$

Proof. Using the fact that, for every k, $\gcd(i,n) = \gcd(i, n+ki)$, we easily find that

$$\sum_{n \leq x}(i,n) = x\frac{g(i)}{i} + O(1) \qquad (x \to \infty).$$

Thus, changing the order of summation gives

$$\begin{aligned}
G(x,y) = \sum_{n \leq x} g(n,y) &= \sum_{n \leq x}\sum_{\substack{i \leq n \\ P(i) \leq y}}(i,n) = \sum_{\substack{i \leq x \\ P(i) \leq y}}\sum_{i \leq n \leq x}(i,n) \\
&= \sum_{\substack{i \leq x \\ P(i) \leq y}}\left(x\frac{g(i)}{i} - g(i)\right) + O(\Psi(x,y)).
\end{aligned}$$

$\square$

In order to evaluate sums of multiplicative functions over friable numbers, we will require a result due to Tenenbaum and Wu [9, Cor. 2.3], which is a generalization of a theorem of Song [8, Main Theorem].

Theorem 10. *Let $\epsilon > 0$ and let $f(n)$ be an arithmetic function which satisfies the two conditions*

$$\sum_{p\le x} f(p)\log p = \kappa x + O\left(x\exp(-(\log x)^{\frac{3}{5}-\epsilon})\right) \tag{7}$$

$$\sum_{p}\sum_{k\ge 2}\frac{f(p^k)}{p^{(1-\eta)k}} \le A, \tag{8}$$

for some constants $\kappa > 0, 0 < \eta < \frac{1}{2}, A > 0$. Then

$$\sum_{\substack{n\le x\\ P(n)\le y}} f(n) = C_\kappa(f)x\rho_\kappa(u)(\log y)^{\kappa-1}\left(1+O\left(\frac{\log(u+1)}{\log y}+\frac{1}{(\log y)^\kappa}\right)\right),$$

uniformly for

$$\exp((\log\log x)^{\frac{5}{3}+\epsilon}) \le y \le x,$$

where

$$C_\kappa(f) := \prod_p\left(1-\frac{1}{p}\right)^\kappa\sum_{j=0}^\infty\frac{f(p^j)}{p^j},$$

and $\rho_\kappa(u)$ is the continuous solution to the delayed differential equation

$$\begin{cases}\rho_\kappa(u) = u^{\kappa-1}/\Gamma(\kappa), & \text{if } 0\le u\le 1;\\ u\rho'_\kappa(u)+(1-\kappa)\rho_\kappa(u)+\kappa\rho_\kappa(u-1)=0, & \text{if } u>1.\end{cases} \tag{9}$$

In our case, it is the function $\rho_2(u)$ that will be of interest. We will need the following lemma that summarizes some basic properties of $\rho_\kappa(u)$.

Lemma 11. *Let $\rho_\kappa(u)$ be defined as in (9), for some $\kappa > 1$. Then*

(i) As $u\to\infty$,

$$\rho'_\kappa(u) \asymp \rho_\kappa(u)\log(u+1).$$

(ii) As $x\to\infty$,

$$\int_1^x \rho_\kappa\left(\frac{\log t}{\log y}\right)dt \ll x\rho_\kappa(u).$$

Proof. For (i), see Tenenbaum–Wu [9]; for (ii), see Song [8, Lem. 8]. ☐

Finally, we will need the following sharp form of the prime number theorem; see for example Ellison [2, Thm. 11.3, p. 425].

Theorem 12 (Prime Number Theorem)**.** *Let $\vartheta(x) := \sum_{p\le x}\log p$, and $\epsilon > 0$. Then*

$$\vartheta(x) = x + O(x\exp(-(\log x)^{\frac{3}{5}-\epsilon})) \qquad (x\to\infty).$$

4 Proof of Theorem 3

Let

$$C = C(y) := \frac{1}{\pi(y)!} \prod_{p \le y} \left(\frac{p-1}{p \log p} \right).$$

We have, by Lemma 7,

$$\begin{aligned} G(x,y) &= \sum_{\substack{n \le x \\ P(n) \le y}} n \Phi\left(\frac{x}{n}, y\right) \\ &= \sum_{\substack{n \le x \\ P(n) \le y}} n \left(C \frac{x}{n} \left(\log\left(\frac{x}{n}\right)\right)^{\pi(y)} + O\left(\frac{x}{n} \left(\log \left(\frac{x}{n}\right)\right)^{\pi(y)-1}\right)\right) \\ &= Cx \sum_{\substack{n \le x \\ P(n) \le y}} (\log x - \log n)^{\pi(y)} + O(x(\log x)^{\pi(y)-1}\Psi(x,y)). \end{aligned}$$

Therefore, by the binomial theorem,

$$\begin{aligned} G(x,y) &= Cx \sum_{\substack{n \le x \\ P(n) \le y}} \left(\sum_{j=0}^{\pi(y)} \binom{\pi(y)}{j} (\log x)^{\pi(y)-j} (-1)^j (\log n)^j \right) \\ &\quad + O(x(\log x)^{2\pi(y)-1}) \\ &= Cx(\log x)^{\pi(y)} \sum_{j=0}^{\pi(y)} \binom{\pi(y)}{j} (\log x)^{-j} (-1)^j \left(\sum_{\substack{n \le x \\ P(n) \le y}} (\log n)^j \right) \\ &\quad + O(x(\log x)^{2\pi(y)-1}). \end{aligned} \tag{10}$$

However, for each $j \geq 0$,

$$\begin{aligned}
\sum_{\substack{n \leq x \\ P(n) \leq y}} (\log n)^j &= \Psi(x, y)(\log x)^j - \int_1^x \frac{j\Psi(t, y)(\log t)^{j-1}}{t} dt \\
&= \frac{(\log x)^{\pi(y)+j}}{\pi(y)! \prod_{p \leq y} \log p} - \int_1^x \frac{j(\log t)^{\pi(y)+j-1}}{t\pi(y)! \prod_{p \leq y} \log p} dt \\
&\quad + O((\log x)^{\pi(y)+j-1}) + \int_1^x O\left(\frac{(\log t)^{\pi(y)+j-2}}{t}\right) dt \\
&= \frac{(\log x)^{\pi(y)+j}}{\pi(y)! \prod_{p \leq y} \log p} - \frac{j(\log x)^{\pi(y)+j}}{(\pi(y)+j)\pi(y)! \prod_{p \leq y} \log p} \\
&\quad + O((\log x)^{\pi(y)+j-1}) \\
&= \frac{(\log x)^{\pi(y)+j}}{(\pi(y)+j)(\pi(y)-1)! \prod_{p \leq y} \log p} + O((\log x)^{\pi(y)+j-1}).
\end{aligned}$$

Substituting this in (10) yields

$$\begin{aligned}
&G(x, y) \\
&= Cx(\log x)^{\pi(y)} \sum_{j=0}^{\pi(y)} \binom{\pi(y)}{j} (-\log x)^{-j} \left(\frac{(\log x)^{\pi(y)+j}}{(\pi(y)+j)(\pi(y)-1)! \prod_{p \leq y} \log p}\right) \\
&\quad + O(x(\log x)^{2\pi(y)-1}) \\
&= Dx(\log x)^{2\pi(y)} + O(x(\log x)^{2\pi(y)-1}),
\end{aligned}$$

where

$$D = D(y) := \frac{C}{(\pi(y)-1)! \prod_{p \leq y} \log p} \sum_{j=0}^{\pi(y)} \binom{\pi(y)}{j} \frac{(-1)^j}{\pi(y)+j}.$$

But by Lemma 8, we have

$$\begin{aligned}
D &= \frac{1}{\pi(y)!(\pi(y)-1)!} \left(\prod_{p \leq y} \frac{p-1}{p \log^2 p}\right) \frac{1}{\pi(y)\binom{2\pi(y)}{\pi(y)}} \\
&= \left(\prod_{p \leq y} \frac{p-1}{p \log^2 p}\right) \frac{1}{(\pi(y)!)^2 \binom{2\pi(y)}{\pi(y)}} = \left(\prod_{p \leq y} \frac{p-1}{p \log^2 p}\right) \frac{1}{(2\pi(y))!}.
\end{aligned}$$

Substituting C and D by their appropriate expressions gives the required estimate for $G(x, y)$. $\square$

5 Proof of Theorem 4

Using Lemma 5, we have that

$$\sum_{p\le x}\frac{g(p)}{p}\log p=\sum_{p\le x}\left(2-\frac{1}{p}\right)\log p=2\sum_{p\le x}\log p-\sum_{p\le x}\frac{\log p}{p}$$
$$=2x+O(x\exp(-(\log x)^{\frac{3}{5}-\epsilon})),$$

where we used Theorem 12 and the fact that

$$\sum_{p\le x}\frac{\log p}{p}\ll\log x\qquad(x\to\infty).$$

Hence, setting $f(n):=g(n)/n$, condition (7) of Theorem 10 is satisfied with $\kappa=2$. Also, we have

$$\sum_p\sum_{k\ge2}\frac{f(p^k)}{p^{(1-\eta)k}}=\sum_p\sum_{k\ge2}\frac{(k+1)p^k-kp^{k-1}}{p^{(2-\eta)k}}\le\sum_p\sum_{k\ge2}\frac{2kp^k}{p^{(2-\eta)k}}$$
$$=\sum_p\frac{4-2p^{\eta-1}}{(1-p^{1-\eta})^2}\le\sum_p\frac{4}{(p^{1-\eta}-1)^2}<\infty,$$

for any $0<\eta<\frac{1}{2}$. It follows that condition (8) of Theorem 10 is also satisfied. Note that for this particular f,

$$C_2(f)=\prod_p\left(1-\frac{1}{p}\right)^2\sum_{j=0}^{\infty}\frac{g(p^j)}{p^{2j}}=\prod_p\left(\frac{p-1}{p}\right)^2\left(\frac{p^2}{(p-1)^2}-\frac{1}{(p-1)^2}\right)$$
$$=\prod_p\left(1-\frac{1}{p^2}\right)=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}.$$

This gives

$$\sum_{\substack{n\le x\\P(n)\le y}}\frac{g(n)}{n}=\frac{6x\rho_2(u)\log y}{\pi^2}\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right).\tag{11}$$

Using Abel's summation formula, we get

$$\sum_{\substack{n\le x\\P(n)\le y}}g(n)=\sum_{\substack{n\le x\\P(n)\le y}}n\frac{g(n)}{n}=\frac{6x^2\rho_2(u)\log y}{\pi^2}\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right)$$
$$-\int_1^x\frac{6t\rho_2\left(\frac{\log t}{\log y}\right)\log y}{\pi^2}dt\tag{12}$$
$$+\int_1^x O\left(t\rho_2\left(\frac{\log t}{\log y}\right)\log\left(\frac{\log t}{\log y}+1\right)\right)dt.$$

Using integration by parts on the first integral gives

$$\int_1^x \frac{6t\rho_2\left(\frac{\log t}{\log y}\right)\log y}{\pi^2}dt = \frac{3x^2\rho_2(u)\log y}{\pi^2} - \int_1^x \frac{3t\rho_2'\left(\frac{\log t}{\log y}\right)}{\pi^2}dt. \tag{13}$$

From Lemma 11(i), we can see that the remaining integral from (13) is of the same order as the second integral in (12). But using Lemma 11(ii), we have that

$$\begin{aligned}\int_1^x t\rho_2\left(\frac{\log t}{\log y}\right)\log\left(\frac{\log t}{\log y}+1\right)dt &\leq x\log(u+1)\int_1^x \rho_2\left(\frac{\log t}{\log y}\right)dt\\ &\ll x^2\rho_2(u)\log(u+1),\end{aligned}$$

as $x \to \infty$. Gathering these estimates in (12), we obtain

$$\sum_{\substack{n\leq x\\ P(n)\leq y}} g(n) = \frac{3x^2\rho_2(u)\log y}{\pi^2}\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right). \tag{14}$$

Finally, using (14) and (11) in Lemma 9, we obtain the desired estimate. □

6 Acknowledgments

This project was supervised by Jean-Marie De Koninck, to whom I am grateful for his advice and interest.

References

[1] K. A. Broughan, The gcd-sum function, *J. Integer Sequences* **4** (2001), Article 01.2.2.

[2] W. J. Ellison. In collaboration with M. Mendès France, *Les Nombres Premiers*, Hermann, 1975.

[3] R. de la Bretèche and G. Tenenbaum, Propriétés statistiques des entiers friables, *Ramanujan J.* **9** (2005), 139–202.

[4] A. Hildebrand, Integers free of large prime factors and the Riemann hypothesis, *Mathematika* **31** (1984), 258–271.

[5] A. Hildebrand, On the number of positive integers $\leq x$ and free of prime factors $> y$, *J. Number Theory* **22** (1986), 289–307.

[6] A. Hildebrand and G. Tenenbaum, Integers without large prime factors, *J. Théor. Nombres Bordeaux* **5** (1993), 411–484.

[7] J.-M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, American Mathematical Society, 2012.

[8] J. M. Song, Sums of nonnegative multiplicative functions over integers without large prime factors II, *Acta Arith.* **102** (2002), 105–129.

[9] G. Tenenbaum and J. Wu, Moyennes de certaines fonctions multiplicatives sur les entiers friables, *J. Reine Angew. Math.* **564** (2003), 119–166.

2010 *Mathematics Subject Classification*: Primary 11N25.
Keywords: friable number, largest prime factor, summatory function, Dickman function.

(Concerned with sequence A018804.)

Received September 14 2015; revised version received February 8 2016. Published in *Journal of Integer Sequences*, April 5 2016.

Return to Journal of Integer Sequences home page.