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On the Product Representation of Number Sequences, with Applications to the Family of Generalized Fibonacci Numbers

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Abstract

We investigate general properties of number sequences which allow explicit representation in terms of products. We find that such sequences form whole families of number sequences sharing similar recursive identities. Applying the proposed identities to power sequences and the sequence of Pochhammer numbers, we recover and generalize known recursive relations. Restricting to the cosine of fractional angles, we then study the special case of the family of k-generalized Fibonacci numbers, and present general recursions and identities which link these sequences.

1 Introduction

It has long been known that Fibonacci (A000045) and Pell (A000129) numbers, defined by

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \tag{1}$$

and

$$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2} \tag{2}$$

with $n \geq 2$, respectively, can be represented in product form (see, e.g., [1, 2, 3, 4, 5]), specifically

$$F_n = \prod_{l=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(3 + 2\cos\left(\frac{2l\pi}{n}\right)\right) = \prod_{l=1}^{n-1} \left(1 - 2i\cos\left(\frac{l\pi}{n}\right)\right) \tag{3}$$

and

$$P_n = 2^{\lfloor \frac{n}{2} \rfloor} \prod_{l=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(3 + \cos\left(\frac{2l\pi}{n}\right)\right) = \prod_{l=1}^{n-1} \left(2 - 2i\cos\left[\frac{l\pi}{n}\right]\right), \tag{4}$$

where $n \in \mathbb{N}, n \geq 2$. The above sequences F_n and P_n are specific examples of general Lucas sequences, the latter being defined by the recursive relation

$$L_0^{(m,p)} = 0, L_1^{(m,p)} = 1, L_n^{(m,p)} = mL_{n-1}^{(m,p)} - pL_{n-2}^{(m,p)} \tag{5}$$

for $n \geq 2$ [4]. Already Zeitlin [3] showed that

$$L_n^{(m,p)} = p^{\frac{n-1}{2}} \prod_{l=1}^{n-1} \left(\frac{m}{\sqrt{p}} - 2\cos\left(\frac{l\pi}{n}\right)\right), \tag{6}$$

$m, p \in \mathbb{R}$, provides a valid product representation of all members in general Lucas sequences. Later, expressions of the form (6) were used to obtain other explicit representations of the corresponding number sequences in terms of finite power series in the sequence parameters (see, e.g., [6, 7, 8, 9]), thus highlighting the importance and usefulness of such product representations for the investigation of number sequences.

In this contribution, we will show that product representations of number sequences can also be utilized to establish direct links between different sequences. To that end, we formulate

Definition 1. (family of number sequences) Let $\{x_{n,l}\}$ with $x_{n,l} \in \mathbb{C}$ and $n, l \in \mathbb{N}$ be an arbitrary two-parameter set of numbers. The corresponding family of number sequences $\{X_{n,m}\}$ is defined by the set of all $X_{n,m}$ with

$$X_{n,m} = \prod_{l=1}^{n} (m + x_{n,l}), \tag{7}$$

where $m \in \mathbb{C}$ labels the individual number sequences within the family, and n the members of each sequence.

In what follows, we will restrict to the subset of families which are constructed by $m \in \mathbb{Z}$. A concrete example of such a family is given if we set $x_{n,l} = -2i\cos\left(\frac{l\pi}{n+1}\right)$. In this case, using (6) with $p = -1$ and $m \in \mathbb{Z}$, we have

$$X_{n,m} = \prod_{l=1}^{n} \left(m - 2i\cos\left(\frac{l\pi}{n+1}\right)\right) = L_{n+1}^{(m,-1)}, \tag{8}$$

thus $X_{n,m}$ defines the family of generalized Fibonacci sequences $F_n^{(m)} := L_n^{(m,-1)}, n \geq 2$ obeying the recursive relation

$$F_0^{(m)} = 0, F_1^{(m)} = 1, F_n^{(m)} = mF_{n-1}^{(m)} + F_{n-2}^{(m)}. \tag{9}$$

We will call this family the Fibonacci family, and explore some of its properties in Section 4.

The paper is organized as follows. In Section 2, we will prove various general properties of the number sequences $X_{n,m}$ within a given family, with focus on linear recursive relations between the individual sequences. Two simple examples will be investigated in Section 3, and Section 4 will focus on the Fibonacci family defined in (8). Some generalizations will be discussed at the end.

2 Product representation of certain number sequences

For any given set of numbers $\{x_{n,l}\}$, we define

$$\mathscr{X}_n := \sum_{l=1}^{n} x_{n,l}. \tag{10}$$

We will first express $\mathscr{X}_n$ in terms of the associated number sequences $X_{n,m}$ defined in (7).

Lemma 2. *For any given set of numbers $\{x_{n,l}\}$ with $x_{n,l} \in \mathbb{C}, n, l \in \mathbb{N}$, the sum over $x_{n,l}$ is given by*

$$\mathscr{X}_n = \frac{(-1)^n}{n!} \sum_{l=1}^{n} (-1)^l \binom{n}{l} \, l \, X_{n,l} - \frac{1}{2} n(n+1), \tag{11}$$

where $X_{n,m}, m \in \mathbb{Z}$ denotes the number sequences associated with $\{x_{n,l}\}$.

Proof. We first construct a system of equations by explicitly factorizing (7) for successive $m \in [1, n]$. To that end, we define for $p \in [1, n]$

$$\mathcal{X}_n^{(p)} := \sum_{l_1=1}^{n-p+1} \sum_{l_2=1}^{n-p+2} \cdots \sum_{l_p=1}^{n} x_{n,l_1} x_{n,l_2} \cdots x_{n,l_p},$$

where the summation is subject to the constraints $l_{i+1} > l_i$ for all $l_i, i \in [1, p]$. Specifically, for $p = 1$, we have

$$\mathcal{X}_n^{(1)} = \sum_{l_1=1}^{n} x_{n,l_1} = x_{n,1} + x_{n,2} + \cdots + x_{n,n},$$

for $p = 2$

$$\begin{aligned}
\mathcal{X}_n^{(2)} &= \sum_{l_1=1}^{n-1} \sum_{\substack{l_2=1 \\ l_2>l_1}}^{n} x_{n,l_1} x_{n,l_2} \\
&= x_{n,1}x_{n,2} + x_{n,1}x_{n,3} + \cdots + x_{n,n-1}x_{n,n},
\end{aligned}$$

and for $p = 3$

$$\begin{aligned}
\mathcal{X}_n^{(3)} &= \sum_{l_1=1}^{n-2} \sum_{\substack{l_2=1 \\ l_2>l_1}}^{n-1} \sum_{\substack{l_3=1 \\ l_3>l_2}}^{n} x_{n,l_1} x_{n,l_2} x_{n,l_3} \\
&= x_{n,1}x_{n,2}x_{n,3} + x_{n,1}x_{n,2}x_{n,4} + \cdots + x_{n,n-2}x_{n,n-1}x_{n,n}.
\end{aligned}$$

With this, the product in (7) yields

$$\begin{aligned}
X_{n,m} &= (m + x_{n,1})(m + x_{n,2})(m + x_{n,3}) \cdots (m + x_{n,n}) \\
&= m(m + x_{n,2})(m + x_{n,3}) \cdots (m + x_{n,n}) + \; x_{n,1}(m + x_{n,2})(m + x_{n,3}) \cdots (m + x_{n,n}) \\
&= m^2(m + x_{n,3}) \cdots (m + x_{n,n}) + \; m(x_{n,1} + x_{n,2})(m + x_{n,3}) \cdots (m + x_{n,n}) \\
&\quad + x_{n,1}x_{n,2}(m + x_{n,3}) \cdots (m + x_{n,n}) \\
&= m^n + m^{n-1}(x_{n,1} + x_{n,2} + x_{n,3} + \cdots + x_{n,n}) + \cdots + x_{n,1}x_{n,2}x_{n,3} \cdots x_{n,n} \\
&= m^n + m^{n-1}\mathcal{X}_n^{(1)} + \cdots + \mathcal{X}_n^{(n)},
\end{aligned}$$

from which we obtain for $m \in [1, n]$ the following system of linear equations in $\mathcal{X}_n^{(p)}, p \in [1, n]$:

$$\begin{aligned}
X_{n,1} &= 1^n + 1^{(n-1)}\mathcal{X}_n^{(1)} + 1^{(n-1)}\mathcal{X}_n^{(2)} + \cdots + \mathcal{X}_n^{(n)} \\
X_{n,2} &= 2^n + 2^{(n-1)}\mathcal{X}_n^{(1)} + 2^{(n-2)}\mathcal{X}_n^{(2)} + \cdots + \mathcal{X}_n^{(n)} \\
X_{n,3} &= 3^n + 3^{(n-1)}\mathcal{X}_n^{(1)} + 3^{(n-2)}\mathcal{X}_n^{(2)} + \cdots + \mathcal{X}_n^{(n)} \\
&\vdots \\
X_{n,n} &= n^n + n^{(n-1)}\mathcal{X}_n^{(1)} + n^{(n-2)}\mathcal{X}_n^{(2)} + \cdots + \mathcal{X}_n^{(n)}.
\end{aligned}$$

This system can be written in more compact form as

$$X_{n,i} - i^n = \sum_{j=1}^{n} a_{ij}\mathcal{X}_n^{(j)}, \tag{12}$$

where $a_{ij} = i^{n-j}$, $i, j \in [1, n]$. What remains is to solve (12) for $\mathcal{X}_n^{(1)} = \mathscr{X}_n$. To that end, we note that $a_{in} = 1, \forall i \in [1, n]$, which allows us to construct a new system of $n - 1$ equations by subtracting successive equations in (12). We obtain

$$X_{n,i+1} - X_{n,i} - ((i + 1)^n - i^n) = \sum_{j=1}^{n-1} a_{ij}^{(1)}\mathcal{X}_n^{(j)}, \tag{13}$$

where

$$a_{ij}^{(1)} = a_{i+1,j} - a_{ij} = \left((i + 1)^{n-j} - i^{n-j}\right) = \sum_{l=0}^{1} (-1)^{l+1} \binom{1}{l} (i + l)^{n-j}$$

for $i,j \in [1,n-1]$. Again, $a^{(1)}_{i,n-1} = 1, \forall i \in [1,n-1]$, and we can further reduce the system (13) by subtracting successive equations. After m repetitions, we have

$$(-1)^m \sum_{l=0}^{m} (-1)^l \binom{m}{l} X_{n,i+l} - (-1)^m \sum_{l=0}^{m} (-1)^l \binom{m}{l} (i+l)^n = \sum_{j=1}^{n-m} a^{(m)}_{ij} \mathcal{X}^{(j)}_n \tag{14}$$

for $i,j \in [1,n-m]$, where

$$a^{(m)}_{ij} = (-1)^m \sum_{l=0}^{m} (-1)^l \binom{m}{l} (i+l)^{n-j}$$

with $a^{(m)}_{i,n-m} = m!$ for all $i \in [1,n-m]$. For $m = n-1$, we finally obtain

$$(-1)^{n-1} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} X_{n,1+l} - (-1)^{n-1} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} (1+l)^n = (n-1)!\, \mathcal{X}^{(1)}_n.$$

Changing the summation variable $l \to l+1$ in both sums, and observing that $\binom{n-1}{l-1} = \frac{l}{n}\binom{n}{l}$ and

$$\frac{(-1)^n}{n!} \sum_{l=1}^{n} (-1)^l \binom{n}{l} l^{n+1} = \frac{1}{2} n(n+1)$$

[10, Equation (1.14)], we finally arrive at (11). $\square$

Lemma 2 provides, for any given $n \geq 1$, an explicit representation of the sum over $x_{n,l}$, Equation (11), in terms of a finite linear combination of the number sequences $X_{n,m}$. With this, we can immediately formulate

Lemma 3. *For any given set of numbers $\{x_{n,l}\}$ with $x_{n,l} \in \mathbb{C}, l \in \mathbb{N}$ and associated family of number sequences $\{X_{n,m}\}, m \in \mathbb{Z}$, the sum over $x_{n,l}$ obeys for any given $n \in \mathbb{N}$ the identities*

$$\mathscr{X}_n = \frac{(-1)^n}{n!} \sum_{l=1}^{n} (-1)^l \binom{n}{l} l\, X_{n,l+m} - \frac{1}{2} n(n+1) - nm \tag{15}$$

and, for $m \neq 0$,

$$\mathscr{X}_n = \frac{(-1)^n}{n!\, m^{n-1}} \sum_{l=1}^{n} (-1)^l \binom{n}{l} l\, X_{n,lm} - \frac{1}{2} n(n+1)m. \tag{16}$$

Proof. We first prove (15). Let us define $X_{n,m}$ and $\mathscr{X}_n$ for the set of numbers $(m' + x_{n,l})$ as follows:

$$X^{(m')}_{n,m} := \prod_{l=1}^{n} (m + (m' + x_{n,l}))$$

$$\mathscr{X}^{(m')}_n := \sum_{l=1}^{n} (m' + x_{n,l})$$

for arbitrary $m' \in \mathbb{Z}$. From the first equation and definition (7), it follows immediately that $X^{(m')}_{n,m} = X_{n,m+m'}$ and $\mathscr{X}^{(m')}_n = nm' + \mathscr{X}_n$, which together with (11) yield (15).

The second relation (16) can be shown in a similar fashion. We define $X_{n,m}$ and $\mathscr{X}_n$ for the set of numbers $x_{n,l}/m', m' \in \mathbb{Z}, m \neq 0$ as follows:

$$\tilde{X}^{(m')}_{n,m} := \prod_{l=1}^{n} \left(m + \frac{x_{n,l}}{m'} \right)$$

$$\tilde{\mathscr{X}}^{(m')}_n := \sum_{l=1}^{n} \frac{x_{n,l}}{m'},$$

from which follows that

$$\tilde{X}^{(m')}_{n,m} = \frac{1}{m'^n} \prod_{l=1}^{n} (mm' + x_{n,l}) = \frac{1}{m'^n} X_{n,mm'} \tag{17}$$

and $\tilde{\mathscr{X}}^{(m')}_n = \mathscr{X}_n/m'$. Using again (11), we obtain (16). □

Lemma 3 is interesting in various respects. It not just generalizes (11), but also shows that, for an given $n \in \mathbb{N}$, various combinations of $X_{n,m}$ within a given family of number sequences must yield the same result $\mathscr{X}_n$. This, in turn, allows us to construct general relations between $X_{n,m}$, which will hold for all families of number sequences $\{X_{n,m}\}$ representable in product form (7), and constitutes the main result of this contribution. We can formulate

Proposition 4. *The members $X_{n,m}$ of a given family of number sequences $\{X_{n,m}\}, m \in \mathbb{Z}$ and $n \in \mathbb{N}$, obey the general recursive relation*

$$X_{n,m+1} = (-1)^n \sum_{l=1}^{n} (-1)^l \binom{n}{l-1} X_{n,l+m-n} + n! \tag{18}$$

and are subject to the identity

$$\frac{1}{m^{n-1}} \sum_{l=1}^{n} (-1)^l \binom{n}{l} l\, X_{n,lm} = \sum_{l=1}^{n} (-1)^l \binom{n}{l} l\, X_{n,l} + \frac{(-1)^{n-1}}{2} (1-m) n (n+1)! \tag{19}$$

for $m \neq 0$.

Proof. The proof of (18) utilizes (15) for $m \to m-n+1$ and $m \to m-n$, yielding

$$\begin{aligned} \mathscr{X}_n &= \frac{(-1)^n}{n!} \sum_{l=1}^{n} (-1)^l \binom{n}{l} l\, X_{n,l+m-n+1} - \frac{1}{2} n(n+1) - n(m-n+1) \\ &= \frac{(-1)^n}{n!} \sum_{l=1}^{n-1} (-1)^l \binom{n}{l} l\, X_{n,l+m-n+1} + \frac{1}{(n-1)!} X_{n,m+1} - \frac{1}{2} n(n+1) - n(m-n+1) \end{aligned}$$

and

$$
\begin{aligned}
\mathscr{X}_n &= \frac{(-1)^n}{n!}\sum_{l=1}^{n}(-1)^l\binom{n}{l}\,l\,X_{n,l+m-n} - \frac{1}{2}n(n+1) - n(m-n)\\
&= \frac{(-1)^n}{n!}\sum_{l=1}^{n-1}(-1)^{l+1}\binom{n}{l+1}\,(l+1)\,X_{n,l+m-n+1} + \frac{(-1)^{n+1}}{(n-1)!}X_{n,m-n+1}\\
&\quad -\frac{1}{2}n(n+1) - n(m-n),
\end{aligned}
$$

respectively, where in the last step $l \to l-1$ was used. Subtracting both identities and observing that $(l+1)\binom{n}{l+1} + l\binom{n}{l} = n\binom{n}{l}$, we obtain

$$
\frac{(-1)^{n+1}}{(n-1)!}\sum_{l=1}^{n-1}(-1)^l\binom{n}{l}X_{n,l+m-n+1} + \frac{(-1)^{n+1}}{(n-1)!}X_{n,m-n+1} - \frac{1}{(n-1)!}X_{n,m+1} + n = 0,
$$

from which

$$
\frac{(-1)^{n+1}}{(n-1)!}\sum_{l=0}^{n-1}(-1)^l\binom{n}{l}X_{n,l+m-n+1} - \frac{1}{(n-1)!}X_{n,m+1} + n = 0
$$

follows. After a change of the summation variable $l \to l+1$, the last relation yields (18).

In a similar fashion, (16) yields together with (11)

$$
\frac{(-1)^n}{n!}\sum_{l=1}^{n}(-1)^l\binom{n}{l}\,l\,X_{n,l} - \frac{1}{2}n(n+1) = \frac{(-1)^n}{n!\,m^{n-1}}\sum_{l=1}^{n}(-1)^l\binom{n}{l}\,l\,X_{n,lm} - \frac{1}{2}n(n+1)m,
$$

from which (19) follows. □

Equation (18) in Proposition 4 provides general linear recursions in $m \in \mathbb{Z}$ for $X_{n,m}$. The form of these recursions depends on n, and contains an increasing number of terms for increasing n. Specifically, for any given n, (18) expresses $X_{n,m}$ in terms of $X_{n,m'}$ with $m' \in [m-n, m-1]$. Based on these recursions, using the generating function approach, we can deduce explicit identities which express $X_{n,m}$ and $X_{n,-m}$ for $m \geq n$, in terms of $X_{n,m'}$ with $m' \in [0, n-1]$ and $m' \in [-n+1, 0]$, respectively.

Corollary 5. *For any given family of number sequences $\{X_{n,m}\}$, the following identities hold*

$$
X_{n,m} = \sum_{l=0}^{n-1}(-1)^{n+l}\frac{n-l}{l-m}\binom{m}{n}\binom{n}{l}X_{n,l} + \frac{m!}{(m-n)!} \tag{20}
$$

$$
X_{n,-m} = \sum_{l=0}^{n-1}(-1)^{n+l}\frac{n-l}{l-m}\binom{m}{n}\binom{n}{l}X_{n,-l} + (-1)^n\frac{m!}{(m-n)!} \tag{21}
$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $m \geq n$.

Proof. We start with (18) for $m+1 \to m$, i.e.,

$$X_{n,m} = \sum_{l=1}^{n} (-1)^{n+l} \binom{n}{l-1} X_{n,l+m-n-1} + n!,$$

and define the general generating function

$$A(z) := \sum_{m \geq 0} X_{n,m} z^m \tag{22}$$

for arbitrary $z \in \mathbb{R}, |z| < 1, z \neq 0$. Multiplication of $X_{n,m}$ with z^m and summation over $m \geq n$ yields

$$\sum_{m \geq n} X_{n,m} z^m = \sum_{m \geq n} \sum_{l=1}^{n} (-1)^{n+l} \binom{n}{l-1} X_{n,l+m-n-1} z^m + n! \sum_{m \geq n} z^m. \tag{23}$$

For any given $n \in \mathbb{N}$, the term on the left-hand side equals

$$\sum_{m \geq n} X_{n,m} z^m = X_{n,n} z^n + X_{n,n+1} z^{n+1} + \cdots = A(z) - \sum_{l=0}^{n-1} X_{n,l} z^l,$$

and the second term on the right-hand side simplifies to

$$n! \sum_{m \geq n} z^m = n! \left(z^n + z^{n+1} + \cdots \right) = n! \, \frac{z^n}{1-z}.$$

To treat the first term on the right-hand side, we first exchange the order of both sums, and observe that

$$\sum_{m \geq n} X_{n,l+m-n-1} z^m = X_{n,l-1} z^n + X_{n,l} z^{n+1} + \cdots = \left(A(z) - \sum_{m=0}^{l-2} X_{n,m} z^m \right) z^{n-l+1},$$

which can easily be shown by generalizing the result for consecutive $l \geq 2$. For instance, for $l = 1$, we have

$$\sum_{m \geq n} X_{n,m-n} z^m = X_{n,0} z^n + X_{n,1} z^{n+1} + \cdots = A(z) z^n,$$

and for $l = 2$

$$\sum_{m \geq n} X_{n,m-n+1} z^m = X_{n,1} z^n + X_{n,2} z^{n+1} + \cdots = \left(A(z) - X_{n,0} \right) z^{n-1}.$$

With this, (23) takes the form

$$\begin{aligned} & A(z) - \sum_{l=0}^{n-1} X_{n,l} z^l \\ &= (-1)^n A(z) z^n + \sum_{l=2}^{n} (-1)^{n+l} \binom{n}{l-1} \left(A(z) - \sum_{m=0}^{l-2} X_{n,m} z^m \right) z^{n-l+1} + n! \, \frac{z^n}{1-z}, \end{aligned}$$

from which we obtain

$$
\begin{aligned}
& A(z)\left(1-(-1)^n z^n-\sum_{l=2}^{n}(-1)^{n+l}\binom{n}{l-1} z^{n-l+1}\right) \\
&= \sum_{l=0}^{n-1} X_{n,l} z^l-\sum_{l=2}^{n} \sum_{m=0}^{l-2}(-1)^{n+l}\binom{n}{l-1} X_{n,m} z^{m+n-l+1}+n!\,\frac{z^n}{1-z}.
\end{aligned} \tag{24}
$$

The term on the left-hand side in the last equation can be further simplified by observing that

$$
\begin{aligned}
& 1-(-1)^n z^n-\sum_{l=2}^{n}(-1)^{n+l}\binom{n}{l-1} z^{n-l+1} \\
&= 1-\sum_{l=1}^{n}(-1)^{n+l}\binom{n}{l-1} z^{n-l+1} \\
&= 1-\left((-1)^{n+1}\binom{n}{0} z^n+(-1)^{n+2}\binom{n}{1} z^{n-1}+\cdots+(-1)^{2n}\binom{n}{n-1} z\right) \\
&= 1-(-1)^{2n}\left(\binom{n}{n-1} z+(-1)^{-1}\binom{n}{n-2} z^2+\cdots+(-1)^{-(n-1)}\binom{n}{0} z^n\right) \\
&= 1-\sum_{l=0}^{n-1}(-1)^{-l}\binom{n}{n-1-l} z^{l+1} \\
&= \sum_{l=0}^{n}(-1)^l\binom{n}{l} z^l \\
&= (1-z)^n,
\end{aligned}
$$

where in the third step we reversed the order of the terms in the sum. Inserting the last relation back into (24) yields

$$
A(z) = \sum_{l=0}^{n-1} X_{n,l} \frac{z^l}{(1-z)^n}-\sum_{l=2}^{n} \sum_{m=0}^{l-2}(-1)^{n+l}\binom{n}{l-1} X_{n,m} \frac{z^{m+n-l+1}}{(1-z)^n}+n!\,\frac{z^n}{(1-z)^{n+1}}. \tag{25}
$$

We now develop the z-terms in a power series around $z=0$. Specifically, for the first term in (25), we have

$$
\begin{aligned}
\frac{z^l}{(1-z)^n} &= \frac{1}{(n-1)!}\, z^l\, \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \frac{1}{1-z} \\
&= \frac{1}{(n-1)!}\, z^l\, \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \sum_{m\geq 0} z^m \\
&= \sum_{m\geq n-1} \frac{m!}{(m-n+1)!} \frac{1}{(n-1)!} z^{m-n+l+1} \\
&= \sum_{q\geq l} \binom{q-l+n-1}{n-1} z^q .
\end{aligned}
$$

In a similar fashion, one obtains

$$
\frac{z^{m+n-l+1}}{(1-z)^n} = \sum_{q\geq m+n-l+1} \binom{q-m+l-2}{n-1} z^q
$$

and

$$
\frac{z^n}{(1-z)^{n+1}} = \sum_{q\geq n} \binom{q}{n} z^q .
$$

With this, (25) takes the form

$$
\begin{aligned}
A(z) &= \sum_{l=0}^{n-1} \sum_{q\geq l} \binom{q-l+n-1}{n-1} X_{n,l}\, z^q \\
&\quad - \sum_{l=2}^{n} \sum_{m=0}^{l-2} \sum_{q\geq m+n-l+1} (-1)^{n+l} \binom{n}{l-1} \binom{q-m+l-2}{n-1} X_{n,m}\, z^q + n! \sum_{q\geq n} \binom{q}{n} z^q ,
\end{aligned}
\tag{26}
$$

which, we recall, holds for any $n \in \mathbb{N}, n > 0$. What remains is to reorder the terms in the above sums, and collect all terms of equal power in z. The first term in (26) yields

$$
\begin{aligned}
&\sum_{l=0}^{n-1} \sum_{q\geq l} \binom{q-l+n-1}{n-1} X_{n,l}\, z^q \\
&\quad = \sum_{q=0}^{n-1} \left(\sum_{l=0}^{q} \binom{q-l+n-1}{n-1} X_{n,l} \right) z^q + \sum_{q\geq n} \left(\sum_{l=0}^{n-1} \binom{q-l+n-1}{n-1} X_{n,l} \right) z^q .
\end{aligned}
$$

Similarly, observing that the second term in (26) vanishes for $n = 1$ and yields a power series with minimum degree of 1 for $n \geq 2$, we obtain

$$
\begin{aligned}
&\sum_{l=2}^{n}\sum_{m=0}^{l-2}\sum_{q\geq m+n-l+1}(-1)^{n+l}\binom{n}{l-1}\binom{q-m+l-2}{n-1}X_{n,m}\,z^q \\
&\quad= \sum_{q=1}^{n-1}\left(\sum_{m=n-q+1}^{n}\sum_{l=0}^{q-n+m-1}(-1)^{n+m}\binom{n}{m-1}\binom{q-l+m-2}{n-1}X_{n,l}\right)z^q \\
&\qquad+\sum_{q\geq n}\left(\sum_{m=2}^{n}\sum_{l=0}^{m-2}(-1)^{n+m}\binom{n}{m-1}\binom{q-l+m-2}{n-1}X_{n,l}\right)z^q
\end{aligned}
$$

The first term in the last relation can further be simplified by changing the summation variable $m \to m-n+q-1$ and collecting all terms $X_{n,l}$ for a given l:

$$
\begin{aligned}
&\sum_{m=n-q+1}^{n}\sum_{l=0}^{q-n+m-1}(-1)^{n+m}\binom{n}{m-1}\binom{q-l+m-2}{n-1}X_{n,l} \\
&\quad= -\sum_{m=0}^{q-1}\sum_{l=0}^{m}(-1)^{m-q}\binom{n}{m+n-q}\binom{m+n-l-1}{n-1}X_{n,l} \\
&\quad= -\sum_{l=0}^{q-1}X_{n,l}\sum_{m=l}^{q-1}(-1)^{m-q}\binom{n}{m+n-q}\binom{m+n-l-1}{n-1} \\
&\quad= \sum_{l=0}^{q-1}\binom{q-l+n-1}{n-1}X_{n,l}.
\end{aligned}
$$

Here, we utilized in the last step the binomial identity

$$
\sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\binom{k+x-1}{x-1}=0, \tag{27}
$$

which can easily be shown by induction in the upper summation limit and using Brill's sum formula [10, Equation (3.181)].

Now, collecting all terms for $q < n$ and $q \geq n$, we can then rewrite (26) as

$$\begin{aligned}
&A(z)\\
&= X_{n,0} + \sum_{q=1}^{n-1}\left(\sum_{l=0}^{q}\binom{q-l+n-1}{n-1} - \sum_{l=0}^{q-1}\binom{q-l+n-1}{n-1}\right) X_{n,l}\, z^q + \sum_{q\geq n}\binom{q}{n} n!\, z^q\\
&\quad + \sum_{q\geq n}\left(\sum_{l=0}^{n-1}\binom{q-l+n-1}{n-1} - \sum_{m=2}^{n}\sum_{l=0}^{m-2}(-1)^{n+m}\binom{n}{m-1}\binom{q-l+m-2}{n-1}\right) X_{n,l}\, z^q\\
&= \sum_{q=0}^{n-1} X_{n,q}\, z^q + \sum_{q\geq n}\binom{q}{n} n!\, z^q\\
&\quad + \sum_{q\geq n}\left(\sum_{l=0}^{n-1}\binom{q-l+n-1}{n-1} X_{n,l} - \sum_{l=0}^{n-2} X_{n,l}\sum_{m=l}^{n-2}(-1)^{n+m}\binom{n}{m+1}\binom{q+m-l}{n-1}\right) z^q\\
&= \sum_{q=0}^{n-1} X_{n,q}\, z^q + \sum_{q\geq n}\binom{q}{n} n!\, z^q\\
&\quad - \sum_{q\geq n}\left(\binom{q}{n-1} X_{n,n-1} - \sum_{l=0}^{n-2}(-1)^{n+l}\,\frac{n-l}{l-q}\binom{q}{n}\binom{n}{l} X_{n,l}\right) z^q, \qquad (28)
\end{aligned}$$

where in the penultimate step we, again, collected terms $X_{n,l}$ for any given l by reordering the sum, and in the last step we utilized the binomial relation

$$\sum_{m=l}^{n-2}(-1)^{n+m}\binom{n}{m+1}\binom{q+m-l}{n-1} = \binom{q-l+n-1}{n-1} - (-1)^{n+l}\frac{n-l}{l-q}\binom{q}{n}\binom{n}{l}.$$

Noting that $q = n + a$ with $a \in \mathbb{N}, a \geq 0$, this relation is a direct consequence of the binomial identity

$$\sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\binom{k+x-1+a}{x-1} = -\frac{a(n-x)}{x(n+a)}\binom{x}{n}\binom{x-1+a}{x-1}, \qquad (29)$$

which itself is a generalization of (27) and can be shown by induction in the upper summation limit using again Brill's sum formula [10, Equation (3.181)] and the binomial identities

$$\begin{aligned}
\binom{n}{k} &= \frac{n+1-k}{k}\binom{n}{k-1}\\
\binom{n}{k} &= \frac{n+1-k}{n+1}\binom{n+1}{k}.
\end{aligned}$$

From (28), we finally obtain

$$A(z) = \sum_{q=0}^{n-1} X_{n,q}\, z^q + \sum_{q\geq n}\left(\sum_{l=0}^{n-1}(-1)^{n+l}\,\frac{n-l}{l-q}\binom{q}{n}\binom{n}{l} X_{n,l} + \binom{q}{n} n!\right) z^q,$$

which, after comparison with (22), yields (20). Equation (21) can be shown in a similar fashion. □

Equation (18) also allows to deduce a number of general identities the number sequences $X_{n,m}$ of any given family must obey. Specifically, we have

Corollary 6. *For $n, p, q \in \mathbb{N}, n \geq p+1$ with $p \geq 1$, $0 \leq q < p$ and $m \in \mathbb{Z}$, the sequences $X_{n,m}$ of any given family of number sequences $\{X_{n,m}\}$ are subject to the following identities:*

$$\sum_{l=0}^{n}(-1)^l \binom{n}{l} l^q X_{n-p,m-n+l} = 0 \tag{30}$$

$$\sum_{l=0}^{n}(-1)^l \binom{n}{l} l^p X_{n-p,m-n+l} - (-1)^n n! = 0. \tag{31}$$

Proof. We first show that (30) is valid for the special case $q = 0$ by induction in p. From (18), after change of the summation variable $l \to l+1$, we have for $n \to n-1 \geq 1$ and arbitrary m

$$(-1)^n \sum_{l=0}^{n-1}(-1)^l \binom{n-1}{l} X_{n-1,m-n+l+1} + (n-1)! = 0,$$

and for $m \to m-1$

$$(-1)^n \sum_{l=0}^{n-1}(-1)^l \binom{n-1}{l} X_{n-1,m-n+l} + (n-1)! = 0.$$

Subtracting the last two identities yields

$$(-1)^n \sum_{l=0}^{n-2}(-1)^l \tbinom{n-1}{l} X_{n-1,m-n+l+1} - X_{n-1,m}$$
$$-(-1)^n X_{n-1,m-n} - (-1)^n \sum_{l=1}^{n-1}(-1)^l \tbinom{n-1}{l} X_{n-1,m-n+l} = 0.$$

Using $\binom{n-1}{l-1} + \binom{n-1}{l} = \binom{n}{l}$, we obtain

$$(-1)^{n+1} \sum_{l=0}^{n}(-1)^l \binom{n}{l} X_{n-1,m-n+l} = 0,$$

which proves (30) for the special case $q = 0$ for $p = 1$. In a similar fashion, assuming (30) is true for $q = 0$ and a given $p \geq 1$, we subtract the resulting relations for $n \to n-1$, m arbitrary, and $m \to m-1$, and obtain

$$-(-1)^n X_{n-(p+1),m-n} + (-1)^{n+1} \sum_{l=1}^{n-1}(-1)^l \binom{n}{l} X_{n-(p+1),m-n+l} - X_{n-(p+1),m} = 0,$$

which yields

$$(-1)^{n+1}\sum_{l=0}^{n}(-1)^l\binom{n}{l}X_{n-(p+1),m-n+l}=0$$

and, thus, proves (30) for $q=0$ and $p+1$.

Identity (30) for $0<q<p$ can be shown by induction in q. Assuming that (30) is true for a given $q\geq 0$ and all $p\geq q+1$, we obtain for $n\to n-1$ and $m\to m-1$

$$\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}\,l^q\,X_{n-p,m-n+l}=0.$$

Using the binomial identity $\binom{n-1}{l}=\frac{n-l}{n}\binom{n}{l}$, the last relation can be rewritten as

$$\sum_{l=0}^{n-1}(-1)^l\binom{n}{l}\,l^q\,X_{n-p,m-n+l}-\frac{1}{n}\sum_{l=0}^{n-1}(-1)^l\binom{n}{l}\,l^{q+1}\,X_{n-p,m-n+l}=0$$

and further simplified to

$$-(-1)^n n^q X_{n-p,m}-\frac{1}{n}\sum_{l=0}^{n-1}(-1)^l\binom{n}{l}\,l^{q+1}\,X_{n-p,m-n+l}=0,$$

from which finally

$$\sum_{l=0}^{n}(-1)^l\binom{n}{l}\,l^{q+1}\,X_{n-p,m-n+l}=0$$

follows, thus proving (30) for $q<p$.

Identity (31) can be shown in an equivalent fashion through induction in p, utilizing (18) and (30). □

We finally note that the recursive relation (18) and identities listed in Corollaries 5 and 6 are general and hold for each family of number sequences $\{X_{n,m}\}$, thus suggesting that all families constructed from number sequences of the form (11) are governed by identical relationships between their members. In the next section, we will elaborate on this property, and briefly consider two simple examples, before illustrating the application to Fibonacci numbers in Section 4.

Table 1: The first members of the family of power sequences $X_{n,m} = m^n, n \in \mathbb{N}$ and $m \in \mathbb{Z}$, Equation (32) with $c = 0$.

$n \backslash m$	0	1	2	3	4	5	6	7
1	0	1	2	3	4	5	6	7
2	0	1	4	9	16	25	36	49
3	0	1	8	27	64	125	216	343
4	0	1	16	81	256	625	1296	2401
5	0	1	32	243	1024	3125	7776	16807
6	0	1	64	729	4096	15625	46656	117649
7	0	1	128	2187	16384	78125	279936	823543
$\vdots$								

3 Two simple examples of families of number sequences

3.1 The family of power sequences

Let $x_{n,l} = c =\in \mathbb{C}$. In this case, we have

$$X_{n,m} = \prod_{l=1}^{n}(m+c) = (m+c)^n \tag{32}$$

$$\mathscr{X}_n = \sum_{l=1}^{n} c = nc. \tag{33}$$

The first few members of this family, for $c = 0$, are listed in Table 1. With the results presented in the previous section, we can immediately formulate

Corollary 7. *The family of power sequences $X_{n,m} = (m+c)^n$ obeys for all $c \in \mathbb{C}$*

$$\begin{aligned}
\sum_{l=1}^{n}(-1)^l\binom{n}{l}l(l+m+c)^n &= \frac{1}{2}(-1)^n(2m+2c+n+1)nn! \\
\sum_{l=1}^{n}(-1)^l\binom{n}{l}l(lm+c)^n &= \frac{1}{2}(-1)^n m^{n-1}\left(2c+m(n+1)\right)nn! \\
\sum_{l=0}^{n}(-1)^l\binom{n}{l}(m+c+1-l)^n &= n! \\
\sum_{l=1}^{n}(-1)^l\binom{n}{l}l\left((lm+c)^n - m^{n-1}(l+c)^n\right) &= \frac{1}{2}(-1)^{n-1}m^{n-1}(1-m)n(n+1)!
\end{aligned}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$,

$$\begin{aligned}\sum_{l=0}^{n-1}(-1)^l\binom{n}{l}\frac{n-l}{l-m}(c+l)^n &= (-1)^n\left(\frac{(m-n)!}{m!}(c+m)^n-1\right)n! \\ \sum_{l=0}^{n-1}(-1)^l\binom{n}{l}\frac{n-l}{l-m}(c-l)^n &= (-1)^n\left(\frac{(m-n)!}{m!}(c-m)^n-(-1)^n\right)n!\end{aligned}$$

for all $n, m \in \mathbb{N}$ with $m \geq n$, and

$$\begin{aligned}\sum_{l=0}^{n}(-1)^l\binom{n}{l}(n-l)^q(m+c-l)^{n-p} &= 0 \qquad \text{for } 0 \leq q < p \\ \sum_{l=0}^{n}(-1)^l\binom{n}{l}(n-l)^p(m+c-l)^{n-p} &= n!\end{aligned}$$

for all $n, p \in \mathbb{N}$ with $n \geq p+1, p \geq 1$ and $m \in \mathbb{Z}$.

Proof. All identities in Corollary 7 are a direct consequence of Lemmata 2 and 3, Proposition 4 and Corollaries 5 and 6, using (32) and (33). □

We note that Corollary 7 yields a number of interesting combinatorial, in particular binomial, identities and their generalizations. Specifically, for $c = 0$, Gould (1.13), (1.14) and (1.47) are recovered [10]. Furthermore, for any fixed n, $X_{n,m}$ yields the sequence of nth powers of subsequent integers m. The third relation in Corollary 7, for $c = 0$, provides then the general form of the nth-order linear homogeneous recursions in m with constant coefficients for such sequences:

$$(m+1)^n = \sum_{l=0}^{n-1}(-1)^l\binom{n}{l+1}(m-l)^n + n! \tag{34}$$

$\forall m \in \mathbb{Z}$. For example, restricting to $m \geq 0$, we obtain for $n = 2$ the sequence of square numbers $a_m = m^2$, obeying the known linear recursion

$$a_0 = 0, a_1 = 1, a_{m+1} = 2a_m - a_{m-1} + 2, m \geq 1$$

(A000290, M. Kristof, 2005), for $n = 3$ the sequence of cubes $a_m = m^3$, obeying

$$a_0 = 0, a_1 = 1, a_2 = 2^3, a_{m+1} = 3a_m - 3a_{m-1} + a_{m-2} + 6, m \geq 2$$

(A000578, A. King, 2013), and for $n = 4$ the sequence $a_m = m^4$, subject to the 4th-order linear recursion

$$a_0 = 0, a_1 = 1, a_2 = 2^4, a_3 = 3^4, a_{m+1} = 4a_m - 6a_{m-1} + 4a_{m-2} - a_{m-3} + 24, m \geq 3$$

(A000583, A. King, 2013).

3.2 The family of Pochhammer numbers

Let $x_{n,l} = l \in \mathbb{N}$. In this case, we have

$$\begin{aligned} X_{n,m} &= \prod_{l=1}^{n}(m+l) = (m+1)_n && (35) \\ \mathscr{X}_n &= \sum_{l=1}^{n} l = \frac{1}{2}n(n+1), && (36) \end{aligned}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol. The first members of this family of sequences are visualized in Table 2. With the results presented in the last section, we can immediately formulate

Corollary 8. *The family of Pochhammer sequences $X_{n,m} = (m+1)_n$ obeys*

$$\begin{aligned} \sum_{l=1}^{n}(-1)^l\binom{n}{l}l(l+m)_n &= (-1)^n(m+n)nn! \\ \sum_{l=1}^{n}(-1)^l\binom{n}{l}(lm)_{n+1} &= \frac{1}{2}(-1)^n m^n(1+m)n(n+1)! \\ \sum_{l=0}^{n}(-1)^l\binom{n}{l}(l+m-n+1)_n &= (-1)^n n! \\ \sum_{l=1}^{n}(-1)^l\binom{n}{l}\left((lm)_{n+1} - m^n(l)_{n+1}\right) &= \frac{1}{2}(-1)^{n+1}m^n(1-m)n(n+1)! \end{aligned}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$,

$$\begin{aligned} \sum_{l=0}^{n-1}(-1)^l\binom{n}{l}\frac{n-l}{l-m}(1+l)_n &= (-1)^n\left(\frac{(m-n)!}{m!}(1+m)_n - 1\right)n! \\ \sum_{l=0}^{n-1}(-1)^l\binom{n}{l}\frac{n-l}{l-m}(1-l)_n &= (-1)^n\left(\frac{(m-n)!}{m!}(1-m)_n - (-1)^n\right)n! \end{aligned}$$

for all $n, m \in \mathbb{N}$ with $m \geq n$, and

$$\begin{aligned} \sum_{l=0}^{n}(-1)^l\binom{n}{l}l^q(m-n+l+1)_{n-p} &= 0 \qquad \text{for } 0 \leq q < p \\ \sum_{l=0}^{n}(-1)^l\binom{n}{l}l^p(m-n+l+1)_{n-p} &= n! \end{aligned}$$

for all $n, p \in \mathbb{N}$ with $n \geq p+1, p \geq 1$ and $m \in \mathbb{Z}$.

Table 2: The first members of the family of Pochhammer sequences $X_{n,m} = (m+1)_n, n \in \mathbb{N}, m \in \mathbb{Z}$, Equation (35).

n \ m	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	8
2	2	6	12	20	30	42	56	72
3	6	24	60	120	210	336	504	720
4	24	120	360	840	1680	3024	5040	7920
5	120	720	2520	6720	15120	30240	55440	95040
6	720	5040	20160	60480	151200	332640	665280	1235520
7	5040	40320	181440	604800	1663200	3991680	8648640	17297280
$\vdots$								

Proof. All identities in Corollary 8 are a direct consequence of Lemmata 2 and 3, Proposition 4 and Corollaries 5 and 6, using (32) and (33) and the Pochhammer identity $l(lm+1)_n = (lm)_{n+1}/m$, valid $\forall m \in \mathbb{Z}$. $\square$

As in the case of the family of power sequences, for any given n, the relations listed in Corollary 8 provide links between Pochhammer numbers $(m)_n$ for different m. Specifically, the third identity yields the general linear recursive rule for sequences defined by $(m+1)_n$ for any fixed n, namely

$$(m+1)_n = \sum_{l=0}^{n-1} (-1)^l \binom{n}{l+1} (m-l)_n + n!, \tag{37}$$

which is valid $\forall m \in \mathbb{Z}$. Restricting again to $m \geq 0$, $n = 2$ yields the sequence of Oblong numbers $a_m = m(m+1)$ (A002378), subject to the recursion

$$a_0 = 0, a_1 = 2, a_{m+1} = 2a_m - a_{m-1} + 2, m \geq 1,$$

for $n = 3$ we obtain the sequence $a_m = m(m+1)(m+2)$, obeying

$$a_0 = 0, a_1 = 3!, a_2 = 4!, a_{m+1} = 3a_m - 3a_{m-1} + a_{m-2} + 6, m \geq 2$$

(A007531, Z. Seidov, 2006), and for $n = 4$ the sequence of products of four consecutive integers $a_m = m(m+1)(m+2)(m+3)$ (A052762) with

$$a_0 = 0, a_1 = 4!, a_2 = 5!, a_3 = \frac{1}{2}6!, a_{m+1} = 4a_m - 6a_{m-1} + 4a_{m-2} - a_{m-3} + 24, m \geq 3.$$

We note that already for $n = 2$ and $n = 4$, the recursions obtained here differ in form from those provided in OEIS [11] for the corresponding sequences.

4 The family of k-generalized Fibonacci numbers

In the remainder of this contribution, we will apply the general results presented in Section 2 to a less trivial case, namely the generalized Fibonacci sequences defined in (9). To that end, we set

$$x_{n,l} = -2i\cos\left(\frac{l\pi}{n+1}\right), \tag{38}$$

from which, using (8), immediately follows that

$$X_{n,m} = \prod_{l=1}^{n}\left(m - 2i\cos\left(\frac{l\pi}{n+1}\right)\right) = F_{n+1}^{(m)}. \tag{39}$$

Furthermore, noting that $\cos\left(\frac{l\pi}{n+1}\right), l \in [0, n]$ are the zeros of Chebyshev polynomials of the second kind

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left(\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right)$$

(see, e.g., [13, Chapter 22], [14, §8.94]), we have

$$\mathscr{X}_n = -2i\sum_{l=1}^{n}\cos\left(\frac{l\pi}{n+1}\right) = 0, \tag{40}$$

where the orthogonality relations for Chebyshev polynomials were used. The first members of the family of sequences formed by (39) are visualized in Table 3. Specifically, the second column $m = 1$ contains the original Fibonacci sequence F_n (A000045), Equation (1), and the third column $m = 2$ the sequence of Pell numbers P_n (A000129), Equation (2), for $n \geq 1$.

While individual columns yield subsequent sequences of generalized Fibonacci numbers, each row, for fixed n, generates new integer sequences whose elements are all generalized Fibonacci numbers. Specifically, for $n = 2$, we obtain, for $m \geq 0$, the sequence $a_m = m^2 + 1$ (A002522), for $n = 3$ the sequence $a_m = m^3 + 2m$ (A054602) and for $n = 4$ the sequence $a_m = m^4 + 3m^2 + 1$ (A057721). In general, for any given $n \in \mathbb{N}$, integer sequences are obtained whose explicit form is given in terms of Fibonacci polynomials (see, e.g., [6]), polynomials of nth order in the sequence index m of the form

$$a_m = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-l}{l} m^{n-2l} = F_{n+1}^{(m)}. \tag{41}$$

From Lemmata 2 and 3, we can immediately formulate

Table 3: The first members of the family of generalized Fibonacci numbers $X_{n,m} = F^{(m)}_{n+1}, n \in \mathbb{N}, m \in \mathbb{Z}$, defined explicitly in (39) and subject to the recursive relation (9).

n \ m	0	1	2	3	4	5	6	7
1	0	1	2	3	4	5	6	7
2	1	2	5	10	17	26	37	50
3	0	3	12	33	72	135	228	357
4	1	5	29	109	305	701	1405	2549
5	0	8	70	360	1292	3640	8658	18200
6	1	13	169	1189	5473	18901	53353	129949
7	0	21	408	3927	23184	98145	328776	927843
$\vdots$								

Proposition 9. *The family of generalized Fibonacci sequences $\{F^{(m)}_n\}$, defined in (39), obeys the following identities:*

$$\sum_{l=1}^{n}(-1)^l\binom{n}{l}lF^{(l+m)}_{n+1} = \frac{1}{2}(-1)^n(2m+n+1)nn! \tag{42}$$

$$\sum_{l=1}^{n}(-1)^l\binom{n}{l}lF^{(lm)}_{n+1} = \frac{1}{2}(-1)^n m^n n(n+1)! \tag{43}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

Proof. Both identities are a direct consequence of (15) and (16), using (39) and (40). □

Equation (43) can be used to link generalized Fibonacci numbers $F^{(m)}_n$ for positive and negative m. Specifically, subtracting (43) for a given m and $m \to -m$, we obtain

$$\sum_{l=1}^{n}(-1)^l\binom{n}{l}l\left(F^{(lm)}_{n+1} - F^{(-lm)}_{n+1}\right) = \frac{1}{2}m^n\left((-1)^n - 1\right)n(n+1)! \tag{44}$$

which, for $m = 1$, yields

$$\sum_{l=1}^{n}(-1)^l\binom{n}{l}l\left(F^{(-l)}_{n+1} - F^{(l)}_{n+1}\right) = \frac{1}{2}\left(1-(-1)^n\right)n(n+1)! = \begin{cases} 0, & \text{for } n \text{ even;} \\ n(n+1)!, & \text{for } n \text{ odd.} \end{cases} \tag{45}$$

Furthermore, application of Proposition 4 to the family of Fibonacci sequences leads to

Proposition 10. *The family of Fibonacci sequences $F_n^{(m)}$ obeys $\forall n \in \mathbb{N}$ the following relations*

$$\sum_{l=0}^{n}(-1)^l\binom{n}{l}F_{n+1}^{(l+m-n+1)} = (-1)^n n! \tag{46}$$

$$\sum_{l=1}^{n}(-1)^l\binom{n}{l}l\left(F_{n+1}^{(lm)} - m^{n-1}F_{n+1}^{(l)}\right) = \frac{1}{2}(-1)^{n-1}m^{n-1}(1-m)n(n+1)! \tag{47}$$

for all $m \in \mathbb{Z}$.

Proof. Both identities are a consequence of (18) and (19), using (39) and (40). Relation (47) can be directly obtained also from (43). □

We note that equation (47) is a special application of (43), which for $m = -1$ yields

$$\sum_{l=1}^{n}(-1)^l\binom{n}{l}l\left(F_{n+1}^{(-l)} + (-1)^n F_{n+1}^{(l)}\right) = n(n+1)! \tag{48}$$

$\forall n \in \mathbb{N}$, complementing (45) above. Interestingly, (46) allows to construct, for any given n, general recursive relations in m for generalized Fibonacci numbers $F_n^{(m)}$, namely

$$F_n^{(m+1)} = \sum_{l=0}^{n-1}(-1)^l\binom{n}{l+1}F_n^{(m-l)} + n!, \tag{49}$$

which is valid $\forall m \in \mathbb{Z}$. Restricting to $m \geq 0$, the resulting sequence $a_m = F_2^{(m+1)} = m^2 + 1$ obtained for $n = 2$ from (46), obeys

$$a_0 = 1, a_1 = 2, a_{m+1} = 2a_m - a_{m-1} + 2, m \geq 1$$

(A002522, E. Werley, 2011). Similarly, for $n = 3$, the integer sequence given by the third-order polynomial $a_m = F_3^{(m+1)} = m^3 + 2m$ (A054602), obeys the recursive relation

$$a_0 = 0, a_1 = 3, a_2 = 12, a_{m+1} = 3a_m - 3a_{m-1} + a_{m-2} + 6, m \geq 2,$$

and for $n = 4$, the sequence $a_m = F_4^{(m+1)} = m^4 + 3m^2 + 1$ (A057721) is subject to the recursion

$$a_0 = 1, a_1 = 5, a_2 = 29, a_2 = 109, a_{m+1} = 4a_m - 6a_{m-1} + 4a_{m-2} - a_{m-3} + 24, m \geq 3.$$

Finally, Corollaries 5 and 6 provide explicit representations of generalized Fibonacci numbers in terms of other members of the Fibonacci family:

Proposition 11. *Generalized Fibonacci numbers $F_n^{(m)}$ obey*

$$F_{n+1}^{(m)} = (-1)^n \binom{m}{n} \sum_{l=0}^{n-1} (-1)^l \binom{n}{l} \frac{n-l}{l-m} F_{n+1}^{(l)} + \frac{m!}{(m-n)!} \tag{50}$$

$$F_{n+1}^{(-m)} = (-1)^n \binom{m}{n} \sum_{l=0}^{n-1} (-1)^l \binom{n}{l} \frac{n-l}{l-m} F_{n+1}^{(-l)} + (-1)^n \frac{m!}{(m-n)!} \tag{51}$$

$\forall n, m \in \mathbb{N}$ with $m \geq n$, and

$$F_{n-p+1}^{(m)} = (-1)^{n+1} n^{-q} \sum_{l=0}^{n-1} (-1)^l \binom{n}{l} l^q F_{n-p+1}^{(m-n+l)} \tag{52}$$

$$F_{n-p+1}^{(m)} = (-1)^{n+1} n^{-p} \sum_{l=0}^{n-1} (-1)^l \binom{n}{l} l^p F_{n-p+1}^{(m-n+l)} + n^{-p} n! \tag{53}$$

for all $n, p \in \mathbb{N}$ with $n \geq p+1, p \geq 1$, $0 \leq q < p$ and $m \in \mathbb{Z}$.

Proof. The first two identities are a direct consequence of (20) and (20), the last two can be shown with (30) and (31), using (39) and (40). □

We note that Propositions 9 to 11 provide a number of identities which interlink the set of generalized Fibonacci sequences defined in (39). Specifically, the defining recursive relation in n, Equation (9), for generalized Fibonacci numbers allows to express each $F_n^{(m)}$ in terms of $F_{n'}^{(m)}$, $n' < n$ for fixed m, whereas relations (42)–(43), (46)–(47) and (50)–(53) allow to express each $F_n^{(m)}$ in terms of $F_n^{(m')}$, $m' \neq m$ for any given n. Combining both sets of identities, we arrive at linking all members of the family of generalized Fibonacci numbers.

5 Concluding remarks

In this contribution, we investigated general properties of number sequences generated through products of the form (7). We found several identities and recursive relations which interlink such sequences and suggest their classification in terms of families (Definition 1). Although the families studied here as examples describe different integer sequences, such as Pochhammer numbers, powers of integers or generalized Fibonacci numbers, we find that each of these families is subject to the same set of identities which, in some cases, generalize interesting relations between these known sequences.

The examples presented here constitute but a small set of potential applications. For instance, q-Pochhammer sequences and sequences produced by products of Pochhammer numbers are obtained for $x_{n,l} = a^l, a \in \mathbb{R}$ and $x_{n,l} = l^a, a \in \mathbb{Z}$, respectively. The general relations listed in Section 2 apply in these cases, and provide a number of identities obeyed by the corresponding sequences. By setting

$$x_{n,l} = -2\sqrt{q}\cos\left(\frac{l\pi}{n+1}\right),$$

we obtain with (6) general Lucas sequences (A108299), i.e.,

$$X_{n,m} = \frac{1}{\sqrt{q}}\prod_{l=1}^{n}\left(m - 2\sqrt{q}\cos\left(\frac{l\pi}{n+1}\right)\right) = L_{n+1}^{(m,q)}.$$

For appropriate m and q, interesting identities, such as grandma's identity [12], signed bisections of Fibonacci sequences and relations interlinking powers of Fibonacci numbers, are obtained. The study of these relations, their potential generalization and application to other families of number sequences might provide novel and potentially useful insights into properties shared by qualitatively different number sequences.

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