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An Extended Version of Faulhaber's Formula

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Abstract

This paper presents an extended version of the well-known Faulhaber formula, which is used to compute the sum of the m-th powers of the first n natural numbers, where m and n are two natural numbers. Our expression is analogous to Faulhaber's formula, but sums the m-th powers of the natural numbers $\leq x$ for any non-negative real number x.

1 Introduction

For two natural numbers $m, n \in \mathbb{N}_0$, the Faulhaber formula [1], which was found by Jacob Bernoulli around 1700, provides a very efficient way to compute the sum of the m-th powers of the first n natural numbers. It is given by

$$\sum_{k=0}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} (-1)^k \binom{m+1}{k} B_k n^{m-k+1},$$

where the B_k's are the Bernoulli numbers.

In this paper, we will prove the analogous expression for the sum $\sum_{k=0}^{\lfloor x \rfloor} k^m$, where $x \in \mathbb{R}_0^+$ and $m \in \mathbb{N}_0$, in terms of Bernoulli polynomials $B_k(x)$ instead of Bernoulli numbers B_k. This expression is given by

Theorem 1. *(extended Faulhaber formula)*
For any $x \in \mathbb{R}_0^+$ we have that

$$\sum_{k=0}^{\lfloor x \rfloor} k^m = \frac{1}{m+1} x^{m+1} + (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} B_k(\{x\}) \, x^{m-k+1}.$$

We have searched this version of Faulhaber's formula in the literature, but we have not found it and therefore we believe that this result is new.

2 Definitions

As usual, we denote the floor of x by $\lfloor x \rfloor$ and the fractional part of x by $\{x\}$.

Definition 2. For $k \in \mathbb{N}_0$ we define the *k-th Bernoulli polynomial* $B_k(x)$ via the following exponential generating function [2]:

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k \quad \forall t \in \mathbb{C} \text{ with } |t| < 2\pi.$$

Definition 3. The *k-th Bernoulli number* B_k is defined as the value of the k-th Bernoulli polynomial $B_k(x)$ at $x = 0$ [2], that is

$$B_k := B_k(0).$$

Moreover, we get from the definition of the Bernoulli polynomials [1] that

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \quad \forall t \in \mathbb{C} \text{ with } |t| < 2\pi.$$

3 Proof of the extended Faulhaber formula

In this section we will prove our extended version of Faulhaber's formula.

Proof. Let $m, n \in \mathbb{N}_0$ be two natural numbers. Starting from [1] the usual Faulhaber formula

$$\sum_{k=0}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} (-1)^k B_k \binom{m+1}{k} n^{m-k+1},$$

we obtain

$$\sum_{k=0}^{n} k^m = (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=0}^{m+1} (-1)^k B_k \binom{m+1}{k} n^{m-k+1}.$$

Setting here $n := \lfloor x \rfloor = x - \{x\}$ for some $x \in \mathbb{R}_0^+$, we get

$$\begin{aligned}
\sum_{k=0}^{\lfloor x \rfloor} k^m &= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=0}^{m+1} (-1)^k B_k \binom{m+1}{k} (x - \{x\})^{m-k+1} \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=0}^{m+1} (-1)^k B_k \binom{m+1}{k} \sum_{l=0}^{m-k+1} (-1)^{m-k-l+1} \binom{m-k+1}{l} x^l \{x\}^{m-k-l+1} \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=0}^{m+1} B_k \binom{m+1}{k} \sum_{l=0}^{m-k+1} (-1)^{m-l+1} \binom{m-k+1}{l} x^l \{x\}^{m-k-l+1},
\end{aligned}$$

where we have used the binomial theorem

$$(a+b)^n = \sum_{l=0}^{n} \binom{n}{l} a^l b^{n-l}$$

for $a := x$, $b := \{x\}$ and $n := m - k + 1$.

We now interchange the order of summation and use the binomial identity

$$\begin{aligned}
\binom{m+1}{k}\binom{m-k+1}{l} &= \frac{(m+1)!(m-k+1)!}{k!l!(m+1-k)!(m-k-l+1)!} \\
&= \frac{(m+1)!}{k!l!(m-k-l+1)!} \\
&= \frac{(m+1)!(m-l+1)!}{k!l!(m-l+1)!(m-k-l+1)!} \\
&= \binom{m+1}{l}\binom{m-l+1}{k}
\end{aligned}$$

to obtain that

$$\begin{aligned}
\sum_{k=0}^{\lfloor x \rfloor} k^m &= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=0}^{m+1} B_k \binom{m+1}{k} \sum_{l=0}^{m-k+1} (-1)^{m-l+1} \binom{m-k+1}{l} x^l \{x\}^{m-k-l+1} \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{l=0}^{m+1} (-1)^{m-l+1} x^l \sum_{k=0}^{m-l+1} B_k \binom{m+1}{k} \binom{m-k+1}{l} \{x\}^{m-k-l+1} \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{l=0}^{m+1} (-1)^{m-l+1} x^l \sum_{k=0}^{m-l+1} B_k \binom{m+1}{l} \binom{m-l+1}{k} \{x\}^{m-k-l+1} \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{l=0}^{m+1} (-1)^{m-l+1} \binom{m+1}{l} x^l \sum_{k=0}^{m-l+1} B_k \binom{m-l+1}{k} \{x\}^{m-k-l+1}.
\end{aligned}$$

If we use now the following explicit formula [3, Proposition 23.2, p. 86] for the Bernoulli polynomials

$$B_n(x) = \sum_{k=0}^{n} B_k \binom{n}{k} x^{n-k}$$

for $n := m - l + 1$ and $x := \{x\}$, we get

$$\begin{aligned}
\sum_{k=0}^{\lfloor x \rfloor} k^m &= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{l=0}^{m+1} (-1)^{m-l+1} \binom{m+1}{l} x^l \sum_{k=0}^{m-l+1} B_k \binom{m-l+1}{k} \{x\}^{m-k-l+1} \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{l=0}^{m+1} (-1)^{m-l+1} \binom{m+1}{l} B_{m-l+1}\left(\{x\}\right) x^l.
\end{aligned}$$

In the above formula we can change variables according to $l := m - k + 1 \Longleftrightarrow k = m - l + 1$ and use the symmetry of the binomial coefficients

$$\binom{m+1}{m-k+1} = \binom{m+1}{k},$$

to conclude that

$$\begin{aligned}
\sum_{k=0}^{\lfloor x \rfloor} k^m &= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{l=0}^{m+1} (-1)^{m-l+1} \binom{m+1}{l} B_{m-l+1}\left(\{x\}\right) x^l \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{m-k+1} B_k\left(\{x\}\right) x^{m-k+1} \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} B_k\left(\{x\}\right) x^{m-k+1}.
\end{aligned}$$

Finally, if we use the fact that $B_0(x) = 1 \ \forall x \in \mathbb{R}$, we get our claimed formula

$$\sum_{k=0}^{\lfloor x \rfloor} k^m = \frac{1}{m+1} x^{m+1} + (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} B_k\left(\{x\}\right) x^{m-k+1}$$

for all $x \in \mathbb{R}_0^+$. □

Remark 4. The ordinary Faulhaber formula follows by setting $x := n \in \mathbb{N}_0$ in our developed extension, because

$$
\begin{aligned}
\sum_{k=0}^{n} k^m &= \sum_{k=0}^{\lfloor n \rfloor} k^m \\
&= \frac{1}{m+1} n^{m+1} + (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} B_k\left(\{n\}\right) n^{m-k+1} \\
&= \frac{1}{m+1} n^{m+1} + (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} B_k n^{m-k+1} \\
&= (-1)^m \frac{B_{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} B_k n^{m-k+1} \\
&= \frac{1}{m+1} \sum_{k=0}^{m} (-1)^k \binom{m+1}{k} B_k n^{m-k+1},
\end{aligned}
$$

where we have used that $B_k\left(\{n\}\right) = B_k(0) = B_k$ for all $k \in \mathbb{N}_0$.

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References

[1] Kevin J. McGown and Harold R. Parks, The generalization of Faulhaber's formula to sums of non-integral powers, *J. Math. Anal. Appl.* **330** (2007), 571–575.

[2] A. Bazsó, Á. Pintér, and H. M. Srivastava, A refinement of Faulhaber's theorem concerning sums of powers of natural numbers, *Appl. Math. Lett.* **25** (2012), 486–489.

[3] Victor Kac and Pokman Cheung, *Quantum Calculus*, Springer, 2002.

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(Concerned with sequences A027641 and A027642.)

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