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Defining Sums of Products of Power Sums

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Abstract

We study the sums of products of power sums of positive integers and their generalizations, using the multiple products of their exponential generating functions. The generalizations include a closed form expression for the sums of products of infinite series of the form $\sum_{n=0}^{\infty} \alpha^n n^k$, $0 < |\alpha| < 1$, $k \in \mathbb{N}_0$ and the related Abel sum, which define, in a unified way, the sums of products of the power sums for all integers k and connect them with the zeta function.

1 Background

The discrete power sum of a positive integer n is defined as $S_k(n) = 1^k + 2^k + \cdots + n^k$, $k \in \mathbb{N}_0$. The power sums are not new and have been studied since Pascal (1654) who obtained the following expression for $S_k(n)$ in terms of $S_{k-1}(n), \ldots, S_0(n)$

$$(n+1)^{k+1} - 1 = \sum_{m=0}^{k} \binom{k+1}{m} S_m(n), \tag{1}$$

which shows that $S_k(n)$ is a polynomial of degree $k+1$ in n. Properties of these polynomials were observed by Faulhaber (1631). Since then, the power sums have remained fascinating even today in one or the other way. So, the literature is vast in context of different approaches and generalizations of the power sums. However, for a quick review of power sums, the reader may refer to a relatively new work in [1, 2, 3, 4, 5, 6].

As an extension of the discrete power-sum, the polynomial $S_k(x)$ for $k \in \mathbb{N}_0$ and complex number x is defined via its exponential generating function (hereafter referred to as egf) [8]

$$\frac{e^{(1+x)t} - e^t}{e^t - 1} = \sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!}, \ |t| < 2\pi. \tag{2}$$

For fixed k, $S_k(x)$ is given by the classical Faulhaber formula

$$S_k(x) = \frac{1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} (-1)^m B_m x^{k+1-m}, \tag{3}$$

where the rational sequence $\{B_k\}_{k \in \mathbb{N}_0}$ is generated by the egf

$$\frac{t}{e^t - 1}, \ |t| < 2\pi, \tag{4}$$

such that B_k is the k-th Bernoulli number.

For $N \in \mathbb{N}$, the sum of products of Bernoulli numbers denoted B_k^N is generated by the multiple product of the egf (4) in the ring of egfs over $\mathbb{Q}$. The study has attracted the attention of researchers for several number theoretical aspects such as in evaluation of the sums of products of Riemann zeta functions. Srivastava and Todorov [9] have obtained a closed form expression for B_k^N.

Dilcher [10] has also evaluated B_k^N in closed form as the following expression:

$$B_k^N = N \binom{k}{N} \sum_{i=0}^{N} (-1)^{N-1-i} s(N, N-j) \frac{B_{k-i}}{k-i}, \ k > N, \tag{5}$$

where $s(N, N-j)$ is the Stirling number of first kind.

In a short paper, Petojević and Srivastava [11] have computed the Dilcher sums of products of Bernoulli numbers in an elegant way. More generally, Kamano [12] has investigated sums of products of hypergeometric Bernoulli numbers and used them to study multiple hypergeometric zeta function.

On the other hand, Kim [13] has obtained sums of products of Bernoulli numbers, using an analytic continuation of the multiple hypergeometric zeta function. The work has been used by Kim and Hu [14] to describe the sums of products of Apostol-Bernoulli polynomials.

In the review of the sums of products, the author did not find any work on the sums of products of the power sums and their connection with the Riemann zeta function except for some work on the sum of products of the power sums of $\varphi(n)$ integers in Singh [7], where φ denotes the Euler's phi function. One may argue that the power sum is defined via its Bernoulli polynomial up to a constant, so all the properties of $S_k(x)$ and its sums of products are contained in the study of the Bernoulli polynomial $B_k(x)$. However, the power sums are fundamental, and the study of the multiple products of their egfs is important in its own right. The present work is an attempt to consider multiple products of egfs of the power sums and their generalizations in a very natural way via a considerably simple theory.

The paper is outlined as follows. The sums of products of power sums via the multiple product of their egfs are discussed in Section 2. Section 3 and 4 are devoted to explore the properties of the higher order α-Euler numbers [8] (a variant of the Apostol-Bernoulli numbers), their connection with the higher order Bernoulli numbers, and the sums of products of the alternating power sums. The related sums of products of the Abel sum of alternating infinite series are obtained in Section 4. In Section 5, the sums of products of the power sums are generalized to negative integer values, and their connection with the sum of products of zeta function is established.

2 Sums of products

Definition 1. In the ring of formal power series over a field $\mathbb{F}$, where $\mathbb{F}$ is one of the fields, $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$; an element f is said to be an exponential generating function (egf) if f is equal to the formal power series $\sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$, where each $a_k \in \mathbb{F}$. The set of all egfs is also a ring with the component-wise addition $(+)$ and multiplication $(\cdot)$, such that for $f = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ and $g = \sum_{k=0}^{\infty} b_k \frac{t^k}{k!},\ a_k, b_k \in \mathbb{F}$,

$$f + g = \sum_{k=0}^{\infty} (a_k + b_k) \frac{t^k}{k!};\ f \cdot g = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} a_m b_{k-m} \frac{t^k}{k!}, \tag{6}$$

where the sequence defined by $c_k = \binom{k}{m} a_m b_{k-m},\ k \in \mathbb{N}_0$ in $\mathbb{F}$ is the Cauchy type product [15] of the sequences $(a_k)_{k\in\mathbb{N}_0}$ and $(b_k)_{k\in\mathbb{N}_0}$.

For a detailed introduction to the ring of formal power series and the ring of egfs, the reader may refer to the review by Wilf [16].

Definition 2. Let $N \in \mathbb{N}$, and let f_i is the egf of the sequence $(a_{ik})_{k\in\mathbb{N}_0}$, for each $i = 1, 2, \ldots, N$, where $a_{ik} \in \mathbb{F}$. In the multiple product of egfs, i.e., $f_1 \cdots f_N = \sum_{k=0}^{\infty} c_k \frac{t^k}{k!}$, the sequence defined by $c_k = \sum_{k_1+\cdots+k_N=k} \binom{k}{k_1,\ldots,k_N} a_{1k_1} \cdots a_{Nk_N}$ in $\mathbb{F}$ is called the sum of products of the sequences $(a_{1k})_{k\in\mathbb{N}_0}, \ldots, (a_{Nk})_{k\in\mathbb{N}_0}$.

Definition 3. For $x \in \mathbb{C}$, let $T_k^N(x)$ denotes the sum of products of the power sums, which is generated by the egf

$$H(x,t,N) = \Big(\frac{e^{(1+x)t} - e^t}{e^t - 1}\Big)^N = \sum_{k=0}^{\infty} T_k^N(x) \frac{t^k}{k!},\ |t| < 2\pi,\ N \in \mathbb{N}_0. \tag{7}$$

Remark 4. Observe from (7) that $T_0^0(x) = 1;\ T_k^0(x) = 0$ for all $k \in \mathbb{N};\ \ T_k^N(1) = N^k$ and $T_k^1(x) = S_k(x)$.

Remark 5. Since for each $k \in \mathbb{N}_0$, $S_k(x)$ is a polynomial in x of degree $k+1$ over $\mathbb{Q}$ with coefficient of the leading term $(k+1)^{-1}$ (see (3)), it follows that for each fixed $N \in \mathbb{N}$ and partition $\{k_1, \ldots, k_N\}$ of k, the product $S_{k_1}(x) \cdots S_{k_N}(x)$ is a polynomial in x over $\mathbb{Q}$ of degree $\sum_{i=1}^{N}(k_i+1)=k+N$ with leading coefficient $\prod_{i=1}^{N}(k_i+1)^{-1}>0$. Also, as $S_{k_i}(0)=0$ for all $i=1,\dots,k_N$, x^N is always a factor of the polynomial $S_{k_1}(x)\cdots S_{k_N}(x)$. Thus, $T_k^N(x)$ is a polynomial in x of degree $k+N$ over $\mathbb{Q}$ since coefficient of x^{k+N} in $T_k^N(x)$ is equal to the N-fold product of the egf of $(k+1)^{-1}$. Therefore, $T_k^N(x)$ is a linear combination of $\{x^N, x^{N+1},\dots,x^{N+k}\}$ over $\mathbb{Q}$. In particular, the coefficient of $x^i,\ i=0,1,\dots,N-1,\ N>0$, in the expansion of $T_k^N(x)$ in powers of x, is always zero. Also, note that $T_k^N(-1)=0$ for all $k\in\mathbb{N}$. Thus, $x^N(x+1)$ is always a factor of $T_k^N(x)$.

Proposition 6. *For $N\in\mathbb{N}_0$ and $x\in\mathbb{R}$,*

(1) $T_0^N(x)=x^N$; $T_k^N(x)=Nk\int_0^x T_{k-1}^N(t)\ dt+N\sum_{m=0}^{k}\binom{k}{m}(-1)^m B_m\int_0^x T_{k-m}^{N-1}(x),\ k\in\mathbb{N}.$

(2) $T_k^N(x)=(-1)^{k+N}\sum_{n=0}^{N}\binom{N}{n}T_k^n(-1-x)$ *for all* $k\in\mathbb{N}_0$.

Proof. (1) Observe that $T_0^N(x)=\lim_{t\to 0}H(x,t,N)=x^N$. The recurrence follows from the following

$$\frac{\partial H}{\partial x}=NtH(x,t,N)+NH(x,t,N-1)\frac{te^t}{e^t-1}. \tag{8}$$

(2) Follows from the following symmetry of H

$$(-1)^N H(x,t,N)=\Big(\frac{e^{(1-1-x)(-t)}-e^{-t}}{e^{-t}-1}+1\Big)^N=\sum_{n=0}^{N}\binom{N}{n}H(-1-x,-t,n) \tag{9}$$

$\square$

Definition 7. Higher order Bernoulli polynomial $B_k^N(x),\ x\in\mathbb{C}$ is defined by the following egf:

$$\frac{t^N e^{xt}}{(e^t-1)^N}=\sum_{k=0}^{\infty}B_k^N(x)\frac{t^k}{k!},\ N\in\mathbb{N}_0, |t|<2\pi. \tag{10}$$

Definition 8. Higher order Bernoulli number is defined by $B_k^N=B_k^N(0)$.

Remark 9. Note that $B_k^1(x)=B_k(x)$ is the k-th Bernoulli polynomial and $B_k^1(0)=B_k$. Now the following recurrences are immediate: $B_0^0(x)=1,\ B_k^0(x)=x^k$ for $k\in\mathbb{N}$, and $B_k^N(x)=\sum_{m=0}^{k}\binom{k}{m}B_m(x)B_{k-m}^{N-1}$, which, for $N=1$, reduces to the usual identity of the Bernoulli polynomials. Also,

$$B_0^{N+1}=1;\ B_k^{N+1}=\Big(1-\frac{k}{N}\Big)B_k^N-kB_{k-1}^N,\ k\in\mathbb{N}, \tag{11}$$

which can be used to evaluate B_k^N for $N=2,3,\dots$.

Remark 10. It is not hard to see that $T_k^N(x)$ can be expressed in terms of $B_k^N(x)$ by

$$T_k^N(x) = \frac{k!}{(N+k)!}\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}B_{N+k}^N(N+(1+x)n). \tag{12}$$

Proposition 11. *Let $N,\ k \in \mathbb{N}_0$, and let $x \in \mathbb{C}$. Then*

$$T_k^N(x) = \frac{k!}{(k+N)!}\sum_{m=0}^{k}\binom{k+N}{m}B_m^N\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}(N+nx)^{k+N-m}. \tag{13}$$

The case $N=1$ recovers the classical Faulhaber formula.

Proof. Consider the following computational steps:

$$\begin{aligned}
\sum_{k=0}^{\infty}T_k^N(x)\frac{t^k}{k!} &= \Big(\frac{e^{(1+x)t}-e^t}{e^t-1}\Big)^N \\
&= \frac{1}{t^N}\Big(\frac{t}{e^t-1}\Big)^N(e^{(1+x)t}-e^t)^N \\
&= \sum_{m=0}^{\infty}B_m^N\frac{t^{m-N}}{m!}\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}e^{(N+nx)t} \\
&= \sum_{m=0}^{\infty}B_m^N\frac{t^{m-N}}{m!}\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}\sum_{s=0}^{\infty}(N+nx)^s\frac{t^s}{s!} \\
&= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}B_m^N\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}(N+nx)^s\frac{t^{s+m-N}}{s!m!} \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}B_m^N\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}(N+nx)^{k-m}\frac{t^{k-N}}{k!},
\end{aligned} \tag{14}$$

which on comparing the like powers of t gives

$$\sum_{m=0}^{k}\binom{k}{m}B_m^N\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}(N+nx)^{k-m}=0, \tag{15}$$

for each $k=0,1,\dots,N-1$, and

$$T_k^N(x) = \frac{k!}{(k+N)!}\sum_{m=0}^{k+N}\binom{k+N}{m}B_m^N\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}(N+nx)^{k+N-m}. \tag{16}$$

As remarked earlier, $T_k^N(x)$ is a linear combination of $x^N,\dots,x^{N+k}$, and for $N\in\mathbb{N}$, the coefficient of x^i for $i=0,1,\dots,N-1,k+N+1,\dots$ vanishes in (16). So, we have

$$\sum_{m=k+1}^{k+N}\binom{k+N}{m}B_m^N\sum_{n=0}^{N}\binom{N}{n}(-1)^{N-n}(N+nx)^{k+N-m}=0. \tag{17}$$

Therefore, (16) reduces through (17) to (13). ∎

Corollary 12. *For k and $N \in \mathbb{N}_0$,*

$$T_k^N(x) = \sum_{m=0}^{k} \binom{k}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{k-m} \binom{k-m}{s} \frac{s!(nx)^{N+s} N^{k-m-s}}{(N+s)!} \tag{18}$$

Proof. For $N = 0$, (13) gives $T_k^0(x) = \sum_{m=0}^{k} \binom{k}{m} B_m^0 (1+x)^{k-m}$, which takes value 1 for $k = 0$ and vanishes otherwise. Thus the case $N = 0$ follows. Now let $N \in \mathbb{N}$, and observe that for each $i = 0, 1, \ldots, N-1$,

$$\sum_{m=0}^{k} \binom{k+N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \binom{k+N-m}{i} (nx)^i N^{k+N-m-i} = 0. \tag{19}$$

Using (19) in (13), we get

$$\begin{aligned}
T_k^N(x) &= \sum_{m=0}^{k} \binom{k+N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{i=N}^{k+N-m} \binom{k+N-m}{i} \frac{k!(nx)^i N^{k+N-m-i}}{(k+N)!} \\
&= \sum_{m=0}^{k} \binom{k+N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{k-m} \binom{k+N-m}{N+s} \frac{k!(nx)^{N+s} N^{k-m-s}}{(k+N)!} \\
&= \sum_{m=0}^{k} \binom{k}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{k-m} \binom{k+N-m}{N+s} \frac{(k-m)!(nx)^{N+s} N^{k-m-s}}{(k+N-m)!} \\
&= \sum_{m=0}^{k} \binom{k}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{k-m} \binom{k-m}{s} \frac{s!(nx)^{N+s} N^{k-m-s}}{(N+s)!}
\end{aligned}$$

as required. □

3 Apostol-Bernoulli numbers

Definition 13. For $\alpha \in \mathbb{C}$, the Apostol-Bernoulli number $\beta_k(\alpha)$ is defined by the following egf:

$$\frac{t}{\alpha e^t - 1} = \sum_{k=0}^{\infty} \beta_k(\alpha) \frac{t^k}{k!}, \ k \in \mathbb{N}_0, \ \alpha \neq 1, \ |t| < |\log \alpha|. \tag{20}$$

Remark 14. Singh [8] introduced the number $E_k(\alpha)$ via the following egf:

$$\frac{\alpha}{\alpha e^t - 1} = \sum_{k=0}^{\infty} E_k(\alpha) \frac{t^k}{k!}, \ \alpha \neq 1, \ |t| < |\log \alpha|; \ E_k(1) = \frac{E_k(-1)}{1 - 2^{k+1}}, \ k \in \mathbb{N}_0, \tag{21}$$

which is related to the Apostol-Bernoulli number by

$$\frac{E_k(\alpha)}{\alpha} = \frac{\beta_{k+1}(\alpha)}{k+1}, \tag{22}$$

and is known to satisfy the following beautiful identities

$$\alpha^{-1}E_{k+1}(\alpha) = \alpha\frac{\partial}{\partial\alpha}(\alpha^{-1}E_k(\alpha)),\ \alpha^{-1}E_k(\alpha) = (-1)^{k+1}\alpha E_k(\alpha^{-1}). \tag{23}$$

for all $\alpha \neq 0, 1$.

Remark 15. The Apostol-Bernoulli numbers and hence $E_k(\alpha)$'s are used to express the power sums of the type

$$S_k(\alpha, x) = \alpha + \alpha^2 2^k + \cdots + \alpha^x x^k,\ x \in \mathbb{N}_0, \tag{24}$$

and their generalizations. In fact, we have the following

$$\lim_{x\to\infty} S_k(\alpha) = -\alpha^{-1}E_k(\alpha) \text{ for } 0 < |\alpha| < 1. \tag{25}$$

Definition 16. Consider for $h \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, the operators defined by $E_h(f)(x) = f(x+h)$, $\Delta_h(f)(x) = (E_h - I)(f)(x) = f(x+h) - f(x)$, where $I = E_0$ is the identity operator. The map E_h is called the shift operator and Δ_h is called the difference operator. For a nonnegative integer n, the n-th difference for f at x is defined as $\Delta_h^n(f(x)) = (E_h - I)^n(f(x)) = \sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}E_h^m(f(x))$.

Note that $E_h^m(f)(x) = f(x+mh)$. In particular, if we take $f(x) = x^k$ then

$$\Delta_h^n(x^k) = \sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}(x+mh)^k. \tag{26}$$

The closed form expressions for $E_k(\alpha)$ and $E_k^N(\alpha)$ are given by the next result.

Proposition 17. *For $\alpha \neq 0, 1$ and $k \in \mathbb{N}$,*

$$E_k(\alpha) = \sum_{n=1}^{k}\Big(\frac{\alpha}{\alpha-1}\Big)^{n+1}\sum_{m=1}^{n}\binom{n}{m}(-1)^m m^k, \tag{27}$$

$$E_k^{N+1}(\alpha) = -\frac{\alpha^2}{N}\frac{\partial}{\partial\alpha}E_k^N(\alpha),\ N \in \mathbb{N}, \tag{28}$$

and

$$E_k^{N+1}(\alpha) = \sum_{n=1}^{k}\binom{N+n}{N}\Big(\frac{\alpha}{\alpha-1}\Big)^{n+N+1}\sum_{m=1}^{n}\binom{n}{m}(-1)^m m^k,\ N \in \mathbb{N}_0. \tag{29}$$

Proof. Consider the geometric power series $G(\alpha, t) = \dfrac{\alpha}{\alpha e^t - 1} = -\sum_{n=0}^{\infty}\alpha^{n+1}e^{nt}$ which is valid for all $0 < |t| < |\log\alpha|$. Then for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
E_k(\alpha) = \lim_{t\to 0^-}\frac{\partial^k}{\partial t^k}(G(\alpha,t)) &= -\alpha \lim_{t\to 0^-}\sum_{n=0}^{\infty}\alpha^n n^k e^{nt} \\
&= -\alpha \lim_{t\to 0^-}\Big(\sum_{n=0}^{\infty}\alpha^n E_1^n\Big)t^k \\
&= \alpha \lim_{t\to 0^-}(\alpha E_1 - I)^{-1}t^k \\
&= \alpha \lim_{t\to 0^-}(\alpha(I+\Delta_1) - I)^{-1}t^k \\
&= \frac{\alpha}{\alpha-1}\lim_{t\to 0^-}\Big(1+\frac{\alpha}{\alpha-1}\Delta_1\Big)^{-1}t^k \\
&= \frac{\alpha}{\alpha-1}\sum_{i=1}^{k}\Big(\frac{\alpha}{1-\alpha}\Big)^i \lim_{t\to 0}\Delta_1^i(t^k) \\
&= \frac{\alpha}{\alpha-1}\sum_{i=1}^{k}\Big(\frac{\alpha}{\alpha-1}\Big)^i \sum_{j=1}^{i}\binom{i}{j}(-1)^j j^k
\end{aligned}$$

where we have used the fact that $\Delta_1^i(t^k) = 0$ for $i > k$ and (26).

Equation (28) follows from the following for α and N as before

$$\frac{\partial}{\partial\alpha}\Big(G(\alpha,t)\Big)^N = -\frac{N}{\alpha^2}\Big(G(\alpha,t)^{N+1}\Big), \tag{30}$$

and (29) can be proved using induction on N. □

Another way of representing $E_k^N(\alpha)$ is the following.

Theorem 18. *For each $N \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $\alpha \neq 1$,*

$$E_0^N(\alpha) = \frac{\alpha^N}{(\alpha-1)^N};\ \ E_k^N(\alpha) = \frac{(-\alpha)^{N-1}}{(N-1)!}(\Delta_1 + 2I)_{N-3}E_k(\alpha), k \in \mathbb{N}, \tag{31}$$

where $\Delta_1(E_k(\alpha)) = E_{k+1}(\alpha) - E_k(\alpha)$ and $(\Delta_1+2I)_{N-3} = (\Delta_1+2I)(\Delta_1+3I)\cdots(\Delta_1+NI)$.

Proof. Observe that $E_0^N(\alpha) = \lim_{t\to 0}\Big(\frac{\alpha}{\alpha e^t - 1}\Big)^N = \frac{\alpha^N}{(\alpha-1)^N}$. Also,

$$\frac{\partial}{\partial t}\Big(\frac{\alpha}{\alpha e^t-1}\Big)^N = -N\Big(\frac{\alpha}{\alpha e^t-1}\Big)^N - \frac{N}{\alpha}\Big(\frac{\alpha}{\alpha e^t-1}\Big)^{N+1}, \tag{32}$$

which gives

$$
\begin{aligned}
E_k^{N+1}(\alpha) &= -\frac{\alpha}{N}(E_{k+1}^N(\alpha) + NE_k^N(\alpha)) \\
&= -\frac{\alpha}{N}(E_{k+1}^N(\alpha) - E_k^N(\alpha) + (N+1)E_k^N(\alpha)) \\
&= -\frac{\alpha}{N}(\Delta_1 + (N+1)I)E_k^N(\alpha) \\
&= \frac{(-\alpha)^2}{N(N-1)}(\Delta_1 + (N+1)I)(\Delta_1 + NI)E_k^{N-1}(\alpha) \\
&\vdots \\
&= \frac{(-\alpha)^N}{N!}(\Delta_1 + (N+1)I)\cdots(\Delta_1 + 2I)E_k(\alpha).
\end{aligned}
\tag{33}
$$

The last expression in (33) is valid for $N \to 0$ in addition to $N = 1, 2, \ldots$. The result follows on changing N to $N-1$ in (33). □

Lemma 19. *For k, α, N as before,*

$$
E_k^N(\alpha) = \frac{\alpha^{N-1}}{(N-1)!}\sum_{i=0}^{N}(-1)^{i-1}s(N,i)E_{k+i-1}(\alpha) \tag{34}
$$

where $s(n,m)$ is the Stirling number of first kind.

Proof. For $n, m \in \mathbb{N}_0$, the Stirling number of first kind $s(n,m)$ is defined by $x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} s(n,m)x^m$ which on changing x to $-x$ gives $x(x+1)\cdots(x+n-1) = \sum_{m=0}^{n}(-1)^{n-m}s(n,m)x^m$. Replacing the symbol x in the generating function for $s(n,m)$ above by the shift operator $E_1 = \Delta_1 + I$, we have

$$
\begin{aligned}
(\Delta_1 + 2I)\cdots(\Delta_1 + (N-1)I)E_k(\alpha) &= E_1(E_1 + I)\cdots(E_1 + (N-1)I)E_{k-1}(\alpha) \\
&= \sum_{i=0}^{N}(-1)^{N-i}s(N,i)E_1^i(E_{k-1})(\alpha) \\
&= \sum_{i=0}^{N}(-1)^{N-i}s(N,i)E_{k+i-1}(\alpha).
\end{aligned}
\tag{35}
$$

Substituting for $(\Delta_1 + 2I)\cdots(\Delta_1 + (N-1)I)E_k(\alpha)$ from (35) in (31), we get (34). □

Remark 20. The expression (34) is special in a sense that, it connects B_{k+N}^N with $E_k^N(1)$. More precisely, we have the next result.

Theorem 21. *For all k, $N \in \mathbb{N}$,*

$$
B_{k+N}^N = \frac{(k+N)!}{k!}E_k^N(1). \tag{36}
$$

Proof. Using $E_k(1) = \dfrac{B_{k+1}}{k+1}$ for $\alpha = 1$ in (34), we get

$$\begin{aligned}\frac{(k+N)}{k!}E_k^N(1) &= \frac{(k+N)}{k!}\frac{1}{(N-1)!}\sum_{i=0}^{N}(-1)^{i-1}s(N,i)\frac{B_{k+i}}{k+i}\\ &= N\binom{k+N}{N}\sum_{i=0}^{N}(-1)^{N-i-1}s(N,N-i)\frac{B_{k+N-i}}{k+N-i}\\ &= B_{k+N}^N.\end{aligned}$$

where the last step has been obtained using (5). □

4 Alternating power sums

Singh [8] defined for any $\alpha \in \mathbb{C}$, k-th α-power sum $S_k(\alpha, x)$ by egf

$$\frac{\alpha^{x+1}e^{(x+1)t} - \alpha e^t}{\alpha e^t - 1} = \sum_{k=0}^{\infty} S_k(\alpha, x)\frac{t^k}{k!}, \ \alpha \neq 1, \ |t| < |\log \alpha|. \tag{37}$$

Remark 22. The following closed form expression for $S_k(\alpha, x)$ is obtainable:

$$S_k(\alpha, x) = \begin{cases} \sum_{m=0}^{k}\binom{k}{m}E_m(\alpha)\{\alpha^x(1+x)^{k-m} - 1\}, & \text{if } \alpha \neq 1;\\ S_k(x), & \text{if } \alpha = 1.\end{cases} \tag{38}$$

Remark 23. Note that $\lim_{\alpha\to 1} S_k(\alpha, x) \neq S_k(x)$ even when x is a positive integer. However, for $x \in \mathbb{N}$, $S_k(\alpha, x) = \alpha + \alpha^2 2^k + \cdots \alpha^x x^k$. Further, if $0 < |\alpha| < 1$ then $\lim_{x\to\infty} S_k(\alpha, x)$ exists and is equal to $-\alpha^{-1}E_k(\alpha)$ for all $k \in \mathbb{N}$, where for $k = 0$, the limit is $-E_0(\alpha)$. In particular, the Abel sum for the divergent series $\eta(-k) = \sum_{n=0}^{\infty}(-1)^n n^k$ can be obtained in closed form as the one sided limit at $\alpha = -1$, i.e.,

$$\eta(-k) = \lim_{\alpha\to -1^+}(\lim_{x\to\infty} S_k(\alpha, x)) = -E_k(-1) = -\sum_{i=1}^{k}\frac{1}{2^{i+1}}\sum_{j=1}^{i}\binom{i}{j}(-1)^j j^k. \tag{39}$$

We now consider sums of products of α-power sums and the related Abel sums.

Definition 24. For $N \in \mathbb{N}_0$ and complex number x, sum of products of α-power sums $T_k^N(\alpha, x)$ is defined via the following egf:

$$G(\alpha, x, N, t) = \Big(\frac{\alpha^{x+1}e^{(x+1)t} - \alpha e^t}{\alpha e^t - 1}\Big)^N = \sum_{k=0}^{\infty} T_k^N(\alpha, x)\frac{t^k}{k!}, \ \alpha \neq 1, \ |t| < |\log \alpha|. \tag{40}$$

It is immediate to see that $T_k^N(\alpha, 0) = 0$ and $T_k^1(\alpha, x) = S_k(\alpha, x)$.

Remark 25. For $0 < |\alpha| < 1$, $\alpha^x \to 0$ as $x \to \infty$. If this is the case then from (38), we have

$$\lim_{x\to\infty} T_k^N(\alpha, x) = (-1)^N \sum_{m=0}^{k} \binom{k}{m} E_m^N(\alpha) N^{k-m}, \ 0 < |\alpha| < 1, \tag{41}$$

which in particular gives the Abel sum of products

$$\eta^N(-k) = (-1)^N \sum_{m=0}^{k} \binom{k}{m} E_m^N(-1) N^{k-m}. \tag{42}$$

Theorem 26. *For $N \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x \in \mathbb{C}$, and $\alpha \neq 0, 1$,*

$$T_k^N(\alpha, x) = \sum_{n=0}^{k} \binom{k}{m} E_m^N(\alpha) \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \alpha^{nx} (N + nx)^{k-m}. \tag{43}$$

Proof. Follows from following computational steps:

$$\begin{aligned}
\sum_{k=0}^{\infty} T_k^N(\alpha, x) \frac{t^k}{k!} &= \left(\frac{\alpha}{\alpha e^t - 1}\right)^N (\alpha^x e^{xt} - e^t)^N \\
&= \sum_{m=0}^{\infty} E_m(\alpha) \frac{t^m}{m!} \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i} \alpha^{ix} e^{(N+ix)t} \\
&= \sum_{m=0}^{\infty} E_m(\alpha) \frac{t^m}{m!} \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i} \alpha^{ix} \sum_{n=0}^{\infty} (N + ix)^n \frac{t^n}{n!} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} E_m(\alpha) \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i} \alpha^{ix} (N + ix)^n \frac{t^{m+n}}{m!n!} \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} E_m(\alpha) \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i} \alpha^{ix} (N + ix)^{k-m} \frac{t^k}{k!}
\end{aligned}$$

from which the result can be obtained on comparing the like powers of t. $\square$

5 Further extensions

The sequence of functions $\left(T_k^N(\alpha, x)\right)_{k\in\mathbb{N}_0}$, x, $\alpha \in \mathbb{C}$, $\alpha \neq 0$, 1 satisfies simple recurrence relation from which its terms can be obtained successively. Here is the special one, which we state as the following.

Proposition 27. *For $k, N \in \mathbb{N}_0$, $x \in \mathbb{C}$, and $\alpha \neq 0$, 1,*

$$T_0^N(\alpha, x) = \alpha^N \left(\frac{\alpha^x - 1}{\alpha - 1}\right)^N, \ T_k^N(\alpha, x) = \alpha \frac{\partial}{\partial \alpha} T_{k-1}^N(\alpha, x), \ k \in \mathbb{N}. \tag{44}$$

Proof. Expression for T_0^N is trivially true. It is also easy to see that (44) holds for $N = 0$. For $N \in \mathbb{N}$, the proof follows from $\frac{\partial}{\partial t}G(\alpha, x, N, t) = \alpha\frac{\partial}{\partial \alpha}G(\alpha, x, N, t)$, where $G(\alpha, x, N, t)$ is as in (40). $\square$

The recurrence (44) can be used to extend the definition of the sum of products of the power sums to negative integer values of k.

Definition 28. The sum of products $T_k^N(\alpha, x)$ for a complex number N and real $\alpha \neq 0,\ 1$ is defined successively by

$$T_0^N(\alpha, x) = \alpha^N\Big(\frac{\alpha^x - 1}{\alpha - 1}\Big)^N;\ T_{-k}^N(\alpha, x) = \int_0^\alpha \frac{T_{-k+1}^N(\theta, x)}{\theta}d\theta,\ k \in \mathbb{N},\ \alpha \neq 0,\ 1. \tag{45}$$

Definition 29. For $0 < |\alpha| < 1$, define

$$T_k^N(\alpha, \infty) = \lim_{x\to\infty} T_k^N(\alpha, x). \tag{46}$$

From (45), we have

$$T_{-1}^N(\alpha, \infty) = \int_0^\alpha \theta^{N-1}(1-\theta)^{-N}d\theta. \tag{47}$$

For $N = 1$, Singh [8] proved the following result:

$$T_{-i}^1(\alpha, \infty) = \begin{cases} -\log|1-\alpha|, & \text{if } i = 1; \\ \dfrac{(-1)^{i-1}}{(i-1)!}\displaystyle\int_0^\alpha \frac{(\log(t))^{i-1}}{1-t}dt - \sum_{\beta=1}^{i-1}\frac{(-\log|\alpha|)^\beta}{\beta!}T_{-i+\beta}(\alpha, \infty), & \text{if } i \geq 2. \end{cases} \tag{48}$$

Observe from (48) that

$$\lim_{\alpha\to1^-} T_{-k}^1(\alpha, \infty) = \zeta(k), \quad \lim_{\alpha\to-1^+} T_{-k}^1(\alpha, \infty) = -\eta(k), \tag{49}$$

for all $k \in \mathbb{N}$.

In the next result, we generalize the formula (48) for the underlying sums of products, i.e., for all positive integers $N \neq 1$. So, we have the following main result.

Theorem 30. *For all $k \in \mathbb{N}_0$, $N = 2, 3, \ldots,$ and $0 < |\alpha| < 1$,*

$$\begin{aligned} T_{-k}^N(\alpha, \infty) &= \sum_{i_1=1}^{N-1}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_{k-1}-1}\Big(\frac{\alpha}{1-\alpha}\Big)^{i_k}\frac{(-1)^{N-k-i_k}}{i_1\cdots i_k} \\ &+ \sum_{i=1}^{k-1}\frac{s(N, k+1-i)}{(N-1)!}T_{-i}^1(\alpha, \infty) + (-1)^{N-1}T_{-k}^1(\alpha, \infty). \end{aligned} \tag{50}$$

Proof. From (45) and (47), we have

$$T_{-1}^{N}(\alpha,\infty)=\sum_{i=1}^{N-1}(-1)^{N-1-i}\frac{T_0^i(\alpha,\infty)}{i}+(-1)^{N-1}T_{-1}^1(\alpha,\infty), \tag{51}$$

for $N=2,3,\ldots$. Similarly,

$$T_{-2}^{N}(\alpha,\infty)=\sum_{i=1}^{N-1}\sum_{j=1}^{i-1}(-1)^{N-2-j}\frac{T_0^j(\alpha,\infty)}{ij}+(-1)^{N-1}T_{-1}^1\sum_{i=1}^{N-1}\frac{1}{i}+(-1)^{N-1}T_{-2}^1, \tag{52}$$

which inductively leads to the following:

$$\begin{aligned}T_{-k}^{N}(\alpha,\infty)&=\sum_{i_1=1}^{N-1}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_{k-1}-1}\Big(\frac{\alpha}{1-\alpha}\Big)^{i_k}\frac{(-1)^{N-k-i_k}}{i_1\cdots i_k}\\&+\sum_{i_1=1}^{N-1}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_{k-1}=1}^{i_{k-2}-1}\frac{(-1)^{N-k}}{i_1\cdots i_{k-1}}T_{-1}^1(\alpha,\infty)\\&+\cdots\\&+\sum_{i_1=1}^{N}\frac{(-1)^{N-1}}{i_1}T_{-k+1}^1(\alpha,\infty)+(-1)^{N-1}T_{-k}^1(\alpha,\infty),\end{aligned} \tag{53}$$

where we have used (48) and the fact that $T_0^i(\alpha,\infty)=\big(\frac{\alpha}{1-\alpha}\big)^i$ for all $i\in\mathbb{N}$.

On substituting for the following expression of the Stirling numbers of the first kind [18] in (53)

$$s(n+1,m+1)=(-1)^{n+m}n!\sum_{\ell=1}^{n}\sum_{\ell_2=1}^{\ell_1-1}\cdots\sum_{\ell_m=1}^{\ell_{m-1}-1}\frac{1}{\ell_1\ell_2\cdots\ell_m}, \tag{54}$$

we obtain (50). ◻

Corollary 31. *Let $k,\ N\in\mathbb{N}$, such that $N=2,3,\ldots$. Then*

$$\lim_{\alpha\to1^-}T_{-k}^{N}(\alpha,\infty)=\frac{1}{(N-1)!}\sum_{i=2}^{k-1}s(N,k+1-i)\zeta(i)+(-1)^{N-1}\zeta(k)\ \textit{for}\ N<k, \tag{55}$$

and

$$\begin{aligned}\lim_{\alpha\to-1^+}T_{-k}^{N}(\alpha,\infty)&=\sum_{i_1=1}^{N-1}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_k-1}\Big(\frac{1}{2}\Big)^{i_k}\frac{(-1)^{N-k-i_k}}{i_1\cdots i_k}\\&+\sum_{i=1}^{k-1}\frac{s(N,k+1-i)\eta(i)}{(N-1)!}+(-1)^{N-1}\eta(k).\end{aligned} \tag{56}$$

Remark 32. For a positive integer N and $0 < |\alpha| < 1$, we have $\dfrac{(-\alpha)^N e^{Nt}}{(\alpha e^t - 1)^N} = \displaystyle\sum_{k=0}^{\infty} T_k^N(\alpha, \infty)\frac{t^k}{k!}$, from which, it follows that

$$\begin{aligned}\lim_{\alpha\to 1^-} T_{-k}^N(\alpha,\infty) &= (-1)^N \lim_{t_1\to 0}\Big(\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_k}\frac{e^{Nt_{k+1}}}{(e^{t_{k+1}}-1)^N}dt_{k+1}\cdots dt_2\Big)\\ &= \frac{1}{(N-1)!}\sum_{i=k+1-N}^{k} s(N,k+1-i)\zeta(i) \text{ if } N<k.\end{aligned} \tag{57}$$

Example 33. Table 1 below shows the values of $\lim_{\alpha\to 1^-} T_{-k}^N(\alpha,\infty)$ for few values of k and N. The computations were done in `Mathematica` with reference to (55).

Table 1: $\lim_{\alpha\to 1^-} T_{-k}^N(\alpha,\infty)$ for some values of k and N

$k \setminus N$	2	3	4
3	$\frac{\pi^2}{6} - \zeta(3)$		
4	$-\frac{\pi^4}{90} + \zeta(3)$	$\frac{\pi^2}{12} + \frac{\pi^4}{90} - 3\frac{\zeta(3)}{2}$	
5	$\frac{\pi^4}{90} - \zeta(5)$	$-\frac{\pi^4}{60} + \frac{\zeta(3)}{2} + \zeta(5)$	$\frac{\pi^2}{36} + \frac{11\pi^4}{540} - \zeta(3) - \zeta(5)$
6	$-\frac{\pi^6}{945} + \zeta(5)$	$\frac{\pi^4}{180} + \frac{\pi^6}{945} - \frac{3\zeta(5)}{2}$	$-\frac{\pi^4}{90} - \frac{\pi^6}{945} + \frac{\zeta(3)}{6} + \frac{11\zeta(5)}{6}$

Remark 34. The preceding Example 33, together with (57), establishes that

$$\lim_{\alpha\to 1^-} T_{-k}^N(\alpha,\infty) = \sum_{\sum_{i=1}^N k_i = k,\ 0\le k_i\le k,\ k_i\ne 1}^{k} \binom{k}{k_1,\ldots,k_N}\zeta(k_1)\cdots\zeta(k_N), \tag{58}$$

for $N < k$, where the right hand side is the sum of product of Riemann zeta functions.

The classical Euler's formula is given by

$$\int_0^{\frac{1}{2}} \frac{\log(1-t)}{t}dt = -T_{-2}^1\Big(\frac{1}{2},\infty\Big) = \frac{(\log(2))^2}{2} - \frac{\zeta(2)}{2}. \tag{59}$$

The next result generalize the formula (59) to the sums of products.

Theorem 35. *For $N = 2, 3, \ldots,$ the generalized Euler's formula is given by*

$$(-1)^N T_{-2}^N\Big(\frac{1}{2}, \infty\Big) = \sum_{i=1}^{N-1}\sum_{j=1}^{i-1}\frac{(-1)^j}{ij} + \log(2)\sum_{i=1}^{N-1}\frac{1}{i} + \frac{(\log(2))^2}{2} - \frac{\zeta(2)}{2}. \tag{60}$$

Proof. First observe that for $\alpha = \frac{1}{2}$, $T_{-2}^N(\alpha, \infty) = T_{-2}^N(1-\alpha, \infty)$. Along with this, if we use the following identity due to Singh [8]

$$-\int_0^{1-\alpha}\frac{\log(1-t)}{t}dt = -\log(\alpha)\log(1-\alpha) + \int_0^1\frac{\log(t)}{1-t}dt + \int_0^{\alpha}\frac{\log(1-t)}{t}dt, \tag{61}$$

and the fact that $T_0^N(\frac{1}{2}) = 1$ for all N, we obtain (60) by taking $\alpha = \frac{1}{2}$ in (52), where $\displaystyle\int_0^1 \frac{\log(t)}{1-t}dt = \zeta(2)$. □

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