Journal of Integer Sequences, Vol. 19 (2016),
Article 16.3.8

Geometric Representations of the n-anacci Constants and Generalizations Thereof

Igor Szczyrba
School of Mathematical Sciences
University of Northern Colorado
Greeley, CO 80639
USA
igor.szczyrba@unco.edu

Rafał Szczyrba
Funiosoft, LLC
Silverthorne, CO 80498
USA
rafals@funiosoft.com

Martin Burtscher
Department of Computer Science
Texas State University
San Marcos, TX 78666
USA
burtscher@txstate.edu

Abstract

We introduce geometric representations of the sequence of the n-anacci constants and generalizations thereof that consist of the ratio limits generated by linear recurrences of an arbitrary order n with equal real weights $p > 0$. We represent the n-anacci constants and their generalizations geometrically by means of the dilation factors of

dilations transforming collections of compact convex sets with increasing dimensions n.

1 Introduction

The study of n-step Fibonacci and Lucas sequences was initiated in the 1960s by Miles [19], Flores [12], and Fiedler [11]. In the last half century, these studies and the exploration of the Horadam sequences [14] evolved into a diverse investigation of integer sequences generated by linear recurrences with signatures $(m, \ldots, m)$, $1 \le m \in \mathbb{N}$.

Sloane [25] and Khovanova [17] catalog over 340 such integer sequences, e.g., Fibonacci and Lucas type sequences with the signatures $(1, \ldots, 1)$, $2 \le n \le 16$, and numerous sequences with $1 < m$, in particular Horadam sequences with the signatures (m, m), $2 \le m \le 10$.

Applications of the n-step Fibonacci sequences with signatures $(1, \ldots, 1)$ include, e.g., electrical networks [6, 7] and the modeling spreading of infectious diseases when each sick person infects periodically one[1] healthy person at a time [8].

Recent research results regarding sequences with signatures $(m, \ldots, m)$, $1 \le m$, not only concern number theory [1, 5, 10, 20] but overlap also with linear algebra [4, 16], abstract algebra [13, 15], group theory [3, 18], combinatorics [2], complex numbers [4], probability [23], and statistics [24].

In a previous paper [26], we linked the ratio limits of the successive terms of sequences with signatures $(m, \ldots, m)$ with *analytic functions*. We achieved this by studying a general case of the ratio limits $\Phi^{(n)}(p)$ of the sequences $\big(F_k^{(n)}(p)\big)_{k=0}^{\infty}$ generated by linear recurrences of an arbitrary order n with equal *real* weights $p > 0$ and real initial conditions, i.e.,

$$F_k^{(n)}(p) \equiv p\big(F_{k-1}^{(n)}(p) + \cdots + F_{k-n}^{(n)}(p)\big), \; n \le k, \;\; \text{and} \;\; F_k^{(n)}(p) = a_k \in \mathbb{R}, \; 0 \le k < n, \tag{1}$$

$$\Phi^{(n)}(p) \equiv \lim_{k\to\infty} F_{k+1}^{(n)}(p)/F_k^{(n)}(p), \;\; k > k_0, \tag{2}$$

where $F_k^{(n)}(p) \ne 0$ if $k > k_0$.

We now draw on these analytic results to establish an overlap of the ratio limits with *affine geometry*. If $p = 1$, the sequence $\big(\Phi^{(n)}(1)\big)_{n=1}^{\infty} \equiv \big(\Phi^{(n)}\big)_{n=1}^{\infty}$ is referred to as the n-anacci constants [21], and we refer to the set $\big\{\Phi^{(n)}(m) \,|\, m, n \in \mathbb{N}\big\}$ as the (m, n)-anacci constants. Using prior results [26], we show that the set of (m, n)-anacci constants can be represented geometrically by means of the dilation factors of the dilations transforming infinite collections of compact convex sets with increasing dimensions n about homothetic centers contained in the sets but not being their centers of mass.

Our representations have clear geometric interpretations if the centers of mass of the sets used are determined by a simple formula in terms of some boundary points of the sets.

[1]Modeling spreading of infectious diseases when a sick individual infects periodically more than one person at a time involves sequences with more elaborate signatures that are not considered here, cf. [8].

For instance, in the n-balls, n-parallelepipeds, n-ellipsoids, solid n-cones, n-pyramids, and generally in the compact convex n-polytopes, the centers of mass divide the interval linking some boundary points by the ratio $1:1$ or $n:1$.

We describe in detail representations of the (m,n)-anacci constants obtained by using the collections of n-balls and n-cones. The analogous representations of the $\Phi^{(n)}(m)$s can be introduced using, e.g., n-ellipsoids, n-parallelepipeds[2], and n-pyramids.

We proved elsewhere [26] that the set $\{\Phi^{(n)}(m) \mid m,n \in \mathbb{N}\}$ of (m,n)-anacci constants is totally ordered as follows:

if $m_2 > m_1$, then $\Phi^{(n_1)}(m_2) > \Phi^{(n_2)}(m_1)$ for any n_1 and n_2, whereas

if $m_2 = m_1 = m$ and $n_2 > n_1$, then $\Phi^{(n_2)}(m) > \Phi^{(n_1)}(m)$.

The representations of the (m,n)-anacci constants that we construct using the collections of n-balls, n-ellipsoids, and n-parallelepipeds fully replicate geometrically the order in the set $\{\Phi^{(n)}(m) \mid m,n \in \mathbb{N}\}$ — the representations by means of the n-cones and n-pyramids are strictly correlated with the order, too. The geometric representations of the $\Phi^{(n)}(m)$s that we present can be extended to the limits $\Phi^{(n)}(p)$ with $0 < p \in \mathbb{R}$ by substituting p for $m \in \mathbb{N}$.

2 Dilations, centers of mass, and linear recurrences

The existence and the value of the ratio limit of the successive terms of the sequence generated by a linear recurrence is determined by two theorems. In 1966, Ostrowski [22, Theorem 12.2] showed that, if the weights b_i in the signature $(b_1, \ldots, b_n)$ of the linear recurrence are nonnegative and the gcd of the indices i of the weights $b_i > 0$ is 1, then the linear recurrence's characteristic polynomial

$$P^{(n)}(\lambda) \equiv \lambda^n - b_1\lambda^{n-1} - \cdots - b_n \tag{3}$$

has the unique simple positive dominant zero $\lambda^{(n)}$. In 1997, Dubeau et al. [9] proved that, if the weights b_i satisfy these conditions, the ratio limit corresponding to polynomial (3) exists at least for the initial conditions $a_k = 0,\ 0 \le k < n-1,\ a_{n-1} = 1$.

Also, if the ratio limit exists for any set of the initial conditions, it must coincide with the dominant zero $\lambda^{(n)}$.

We use the dominant zeros $\lambda^{(n)}(p) = \Phi^{(n)}(p)$ of the characteristic polynomials

$$P_p^{(n)}(\lambda) \equiv \lambda^n - p\,(\lambda^{n-1} + \cdots + 1) \tag{4}$$

of recurrences (1) to link the limits (2) with affine transformations of compact convex sets.

So, let Λ be a dilation, with the dilation factor $\mu > 0$, acting in the Euclidean space $\mathbb{R}^n$, and let O be the homothetic center of the dilation Λ. Let $\mathfrak{A} \subset \mathbb{R}^n$ be a compact convex n-dimensional set with the center of mass A. If $O \in \mathfrak{A}$, then $\mathfrak{A}$ is contained in the image

[2] A different geometric representation of the n-anacci constants $\Phi^{(n)}$ using n-parallelepipeds has been introduced in 1989 by Dubeau [6].

$\Lambda(\mathfrak{A})$ as a proper subset iff $\mu > 1$, whereas $\Lambda(\mathfrak{A})$ is a proper subset of $\mathfrak{A}$ iff $0 < \mu < 1$. Thus, if $\mu > 1$, we define the set $\mathfrak{B} \equiv \Lambda(\mathfrak{A}) \setminus \mathfrak{A}$, and if $0 < \mu < 1$, we define it as $\mathfrak{B} \equiv \mathfrak{A} \backslash \Lambda(\mathfrak{A})$.

We derive a geometric representation of the ratio limits $\Phi^{(n)}(p) = \lambda^n(p)$ using dilations of an infinite collection of compact convex sets $\mathfrak{A}(n) \subset \mathbb{R}^n$, $n \in \mathbb{N}$, and analyzing for any n relations among the distances $d(\cdot, \cdot)$ between the following four points in $\mathbb{R}^n$:

1. the center of mass A of the set $\mathfrak{A} \equiv \mathfrak{A}(n)$;
2. the homothetic center $O \in \mathfrak{A}$, $O \neq A$, of the dilation Λ;
3. the center of mass $\Lambda(A)$ of the image $\Lambda(\mathfrak{A})$; and
4. the center of mass $B(\mu)$ of the set $\mathfrak{B}$.

For any $\mathfrak{A}$ and O, the mapping $\{\mathbb{R}_+ \setminus \{1\}\} \ni \mu \to B(\mu) \in \mathbb{R}^n$ is continuous and has its limit at 1 since the mapping $\mathbb{R}_+ \ni \mu \to \Lambda(\mu)$ is continuous. Thus, we can define the center of mass $B(1) \equiv \lim_{1 \neq \mu \to 1} B(\mu)$ despite that, for $\mu = 1$, our definition would create an empty set $\mathfrak{B}$. For a given homothetic center O and a given dilation factor $\mu \neq 1$, the center of mass $B(1)$ lies on the line $\mathcal{L}(O, A)$, determined by the homothetic center O and the center of mass A, within the set $\mathfrak{A}$ between the centers of mass A and $B(\mu)$; see Figures 1 and 4. We determine the position of the center of mass $B(1)$ relative to the center of mass $\Lambda(A)$ in terms of varying dilation factors $\mu > 0$ in the Appendix.

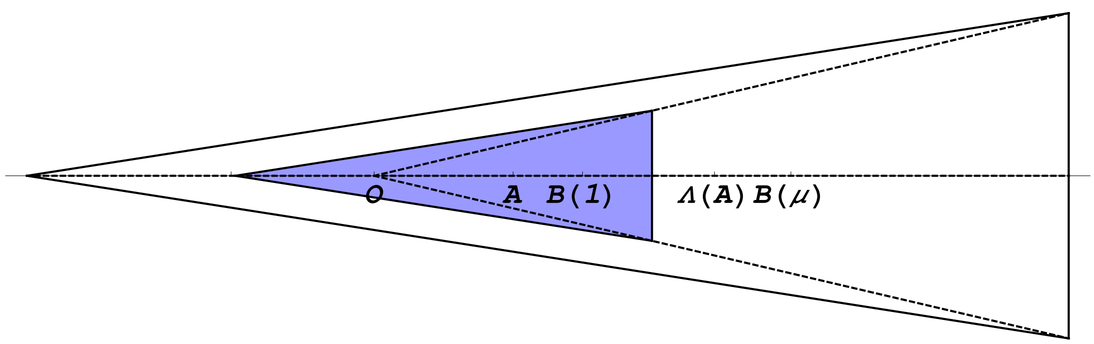

Figure 1: The dilation with $\mu = 2.5$ of the blue triangle $\mathfrak{A}$ about O positioned inside of $\mathfrak{A}$ creates the white hollow set $\mathfrak{B}$. Here, if $\mu \to 1$, $B(\mu) \to B(1)$, where $B(1)$ lies inside of $\mathfrak{A}$.

Our main result links the ratio limits $\Phi^{(n)}(p) = \lambda^n(p)$ with the dilation factors μ.

Theorem 1. *Let $\mathfrak{A} \subset \mathbb{R}^n$ be a compact convex set with the center of mass A. Let Λ be a dilation with the dilation factor μ about a homothetic center $O \in \mathfrak{A}$, $O \neq A$, let $B(\mu)$ be the center of mass of the set $\mathfrak{B} \equiv \Lambda(\mathfrak{A}) \setminus \mathfrak{A}$ if $\mu > 1$, and of the set $\mathfrak{B} \equiv \mathfrak{A} \backslash \Lambda(\mathfrak{A})$ if $0 < \mu < 1$. Then*

(i) if $\mu > 1$, $\mu = \Phi^{(n)}(p)$ iff $d\big(A, B(\mu)\big) = p \cdot d(O, A)$ where $p > 1/n$, i.e., $\Phi^{(n)}(p) = d\big(O, \Lambda(A)\big)/d(O, A)$;

(ii) if $0 < \mu < 1$, $\mu = 1/\Phi^{(n)}(p)$ iff $d\big(\Lambda(A), B(\mu)\big) = p \cdot d\big(O, \Lambda(A)\big)$ where $p > 1/n$,

i.e., $\Phi^{(n)}(p) = d(O,A)/d\big(O,\Lambda(A)\big)$;

(iii) if $\mu = 1$, $\mu = \Phi^{(n)}(p)$ *iff* $d\big(A, B(1)\big) = p \cdot d(O,A)$ *where* $p = 1/n$.

Proof. (i) If $\mu > 1$, the set $\mathfrak{B}$ is constructed by removing some mass from the set $\Lambda(\mathfrak{A})$. Consequently, the distances between the homothetic center O and the centers of mass satisfy $d(O,A) < d\big(O,\Lambda(A)\big) < d\big(O,B(\mu)\big)$, cf. Figures 1 and 4. Moreover, the location of the center of mass $B(\mu)$ coincides with the fulcrum of the lever that is in equilibrium when two forces, $\mathbf{F}_A$ and $\mathbf{F}_{\Lambda(A)}$, with magnitudes proportional to the n-volumes of the sets $\mathfrak{A}$ and $\Lambda(\mathfrak{A})$, act in opposite directions at the centers of mass A and $\Lambda(A)$, respectively.

Dilation factors μ change the distances proportionally to μ and the n-volumes as μ^n, i.e.,

$$d\big(O,\Lambda(A)\big) = \mu\, d(O,A) \quad \text{and} \tag{5}$$

$$\mathbf{F}_{\Lambda(A)} = -\mu^n\, \mathbf{F}_A. \tag{6}$$

Taking into account the order of points O, A, $\Lambda(A)$ and $B(\mu)$, and using (5), we obtain

$$d\big(\Lambda(A), B(\mu)\big) = d(O,A) + d\big(A, B(\mu)\big) - \mu\, d(O,A), \tag{7}$$

whereas the following equilibrium relation for the lever is implied by formula (6):

$$d\big(A, B(\mu)\big) = \mu^n d\big(\Lambda(A), B(\mu)\big). \tag{8}$$

It follows from relations (7) and (8) that

$$d\big(A, B(\mu)\big) = \mu^n \big[d\big(A, B(\mu)\big) + (1-\mu) d(O,A)\big]. \tag{9}$$

In turn, equation (9) implies that the dilation factor μ is a zero of the polynomial

$$\mu^{n+1} - (p+1)\mu^n + p = 0 \tag{10}$$

iff $d\big(A, B(\mu)\big) = p \cdot d(O,A)$.

Now, since it is true that

$$\mu^{n+1} - (p+1)\mu^n + p = (\mu - 1)\big(\mu^n - p\,(\mu^{n-1} + \cdots + 1)\big), \tag{11}$$

a dilation factor $\mu > 1$ is the dominant zero of polynomial (4) iff $d\big(A, B(\mu)\big) = p \cdot d(O,A)$.

In another paper [26], we showed that, for any n, the function $\mathbb{R}_+ \ni p \to \Phi^{(n)}(p) \in \mathbb{R}_+$ is continuous, and

$$1 < \Phi^{(n)}(p) \ \text{ iff } \ p \cdot n > 1 \ \text{ and} \tag{12}$$

$$\Phi^{(n)}(p) = 1 \ \text{ iff } \ p \cdot n = 1. \tag{13}$$

Thus, according to formulas (5), (10) and (12), $\mu = \lambda^{(n)}(p) = \Phi^{(n)}(p) = d\big(O,\Lambda(A)\big)/d(O,A)$ iff $d\big(A, B(\mu)\big) = p \cdot d(O,A)$, where $p > 1/n$.

(ii) If $0 < \mu < 1$, the distances $d\big(O, \Lambda(A)\big) < d(O, A) < d\big(O, B(\mu)\big)$. Renaming the center of mass $\Lambda(A)$ as A' leads to $A = \Lambda^{-1}(A')$, where Λ^{-1} is the dilation about O with the dilation factor $1/\mu > 1$.

Part (i) implies that $1/\mu = \Phi^{(n)}(p) = d(O, A)/d\big(O, \Lambda(A)\big)$ iff $d\big(\Lambda(A), B(\mu)\big) = p \cdot d\big(O, \Lambda(A)\big)$, where $p > 1/n$.

(iii) To prove the *only if* part of the assertion, let us consider $\lim_{p \to 1/n} B\big(\Phi^{(n)}(p)\big)$ where $p > 1/n$. The continuity of the function $\mathbb{R}_+ \ni p \to \Phi^{(n)}(p) \in \mathbb{R}_+$, formulas (12) and (13), and part (i) of the theorem imply that $\lim_{p \to 1/n} B\big(\Phi^{(n)}(p)\big) = B(1)$ and $d\big(A, B(1)\big) = \frac{1}{n} d(O, A)$. Moreover, formula (13) implies that $\Phi^{(n)}(p) = 1$ only if $p = \frac{1}{n}$. The *if* part of the assertion follows from formula (13).

□

3 Geometric representation of (m,n)-anacci constants by means of n-balls

We first describe the geometric representation of the

n-anacci sequence $\big(\Phi^{(n)}\big)_{n=1}^{\infty}$ by means of the collection of nested unit n-balls with the centers at $A(n) = A \equiv (1, 0, \ldots, 0) \in \mathbb{R}^n$, i.e., A is the common center for all unit n-balls, and we treat $[0, 2]$ as the unit 1-ball.

Specifically,

(i) we set the homothetic center O of the dilation $\Lambda(n)$ at $(0, \ldots, 0) \in \mathbb{R}^n$, i.e., O is common for all $\Lambda(n)$s and is included in all unit $(n-1)$-spheres bordering the unit n-balls;

(ii) for $n > 1$, we choose the dilation factor $\mu(n)$ of $\Lambda(n)$ such that the unit $(n-1)$-sphere and the dilated $(n-1)$-sphere, bordering the dilated n-ball, enclose the set $\mathfrak{B}(n)$ with its center of mass at the point $B\big(\mu(n)\big) = B \equiv (2, 0, \ldots, 0) \in \mathbb{R}^n$— common for all sets $\mathfrak{B}(n)$.

Both of these conditions can be satisfied only if $\mu > 1$. Since, we have $d(A, B) = d(O, A) = 1$, Theorem 1(i) implies that the radius of the dilated $(n-1)$-sphere equals $\mu(n) = \Phi^{(n)}$ and its center $\Lambda(n)\big(A\big)$ is at $(\Phi^{(n)}, 0, \ldots, 0) \in \mathbb{R}^n$.

If $n = 1$, it follows from Theorem 1(iii) that the dilation of the unit 1-ball about the origin with the dilation factor $\mu(1) = 1 = \Phi^{(1)}$ leads to $B(1) = 2$. Thus, we can extend our construction for $n = 1$ by defining the set $\mathfrak{B}(1) \equiv \{2\}$. All sets $\mathfrak{B}(n)$, $n \geq 1$, are nested one inside the other as proper subsets following the order of the sequence $\big(\Phi^{(n)}\big)_{n=1}^{\infty}$.

In Figure 2, we depict the cyan unit disc and the blue set $\mathfrak{B}(2)$ with its center of mass at $B = (2, 0)$. It is shaped as an eclipsed moon enclosed by the unit circle and by the dilated solid circle with the radius equal to $\Phi^{(2)} = \Phi$ and the center at $(\Phi, 0)$. The dashed circles with radii $\Phi^{(n)}$ and centers at $(\Phi^{(n)}, 0)$, n=3, 4, 5, are the projections on the plane spanned by the basic vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathbb{R}^n$ of the dilated $(n-1)$-spheres enclosing the corresponding sets $\mathfrak{B}(n)$. The dashed circles intersect the first coordinate line $\mathcal{L}(\mathbf{e_1})$ at the points $2\,\Phi^{(n)}$ approaching 4.

It has been known since the 1960s that $\lim_{n\to\infty} \Phi^{(n)} = 2$, cf. [12]. In 1989, Dubeau [6] and then in 1998, Wolfram [27] proved that

$$2(1-\frac{1}{2^n}) < \Phi^{(n)}, \quad n > 1, \tag{14}$$

which quantizes the way the sets $\mathfrak{B}(n)$ are nested one inside the other following the order in the set of (m,n)-anacci constants.

We extend our representation to the (m,n)-anacci constants $\Phi^{(n)}(m)$ with $m > 1$ by

(i) keeping the common homothetic center O at the origin, and

(ii) choosing the dilation factors $\mu(m,n)$ so that, for a fixed m and any n, the centers of mass $B\big(\mu(m,n)\big)$ of the sets $\mathfrak{B}(m,n)$ are at the common point $B(m) \equiv (m+1, 0, \ldots, 0) \in \mathbb{R}^n$.

Thus, we have $d\big(A, B(m)\big) = m \cdot d(O,A)$ and, according to Theorem 1(i), each compact non-convex (if $n > 1$) set $\mathfrak{B}(m,n)$ is enclosed by the unit $(n-1)$-sphere and the dilated $(n-1)$-sphere with the radius $\Phi^{(n)}(m)$ and the center at $\big(\Phi^{(n)}(m), 0\ldots, 0\big) \in \mathbb{R}^n$.

The (m,n)-anacci constants are now uniformly represented along the line $\mathcal{L}(\mathbf{e_1})$ by the radii $\Phi^{(n)}(m)$ $\big($respectively the diameters $2\,\Phi^{(n)}(m)\big)$ spanning between the homothetic center O and the centers of the dilated n-balls $\big($respectively the points at which the line $\mathcal{L}(\mathbf{e_1})$ is intersected by the dilated $(n-1)$-spheres$\big)$.

The sets $\mathfrak{B}(m,n)$ as well as the intervals representing the $\Phi^{(n)}(m)$s are nested inside each other as proper subsets in a way that replicates the order in the set of the (m,n)-anacci constants.

Figure 2 shows the sets $\mathfrak{B}(2,2)$ and $\mathfrak{B}(3,2)$ with the centers of mass at $(3,0)$ and $(4,0)$, respectively. They are shaped as eclipsed moons, enclosed by the unit circle and the solid circles with radii $\Phi^{(2)}(2)$ and $\Phi^{(2)}(3)$, and centers at $\big(\Phi^{(2)}(2),0\big)$ and $\big(\Phi^{(2)}(3),0\big)$, respectively. The dashed circles with the radii $\Phi^{(n)}(m)$ and the centers at $\big(\Phi^{(n)}(m),0\big)$ with m=$2,3$ and n=$3,4,5$ are the projections of the dilated $(n-1)$-spheres enclosing the sets $\mathfrak{B}(m,n)$ with the centers of mass at $(3,\ldots,0)$ and $(4,\ldots,0)$, respectively.

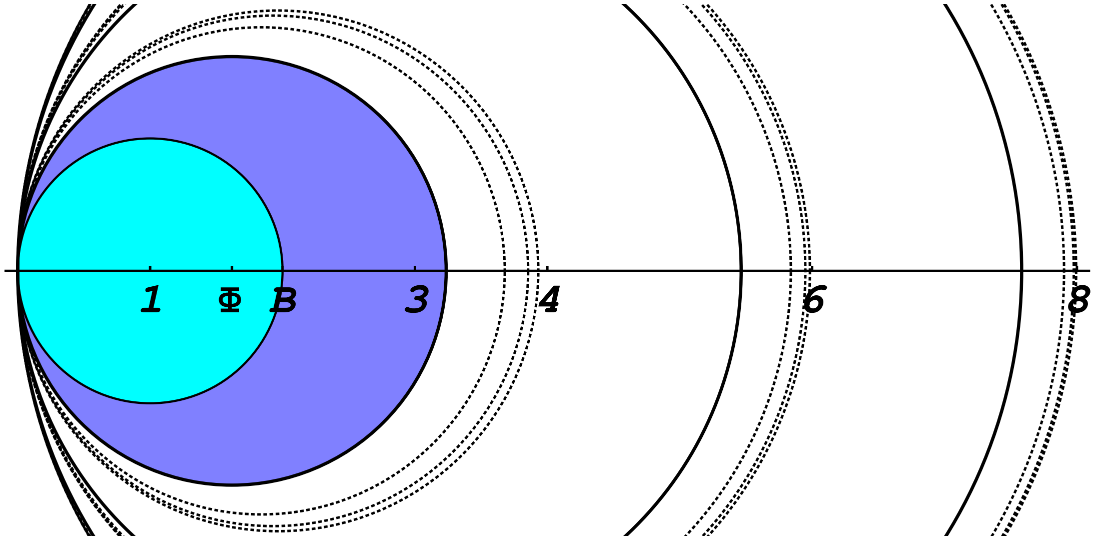

Figure 2: The representation of the (m, n)-anacci constants $\Phi^{(n)}(m)$ with $m = 1, 2, 3$ and $n = 1, 2, 3, 4, 5$ by means of the dilations of unit n-balls with centers at $(1, 0, \ldots, 0)$ about the fixed homothetic center positioned at the origin in $\mathbb{R}^n$. If $m = 1$, we have $B = (2, \ldots, 0)$.

We showed previously [26] that, for $p > 0$, the sequence $\left(\Phi^{(n)}(p)\right)_{n=1}^{\infty}$ strictly increases to

$$\lim_{n\to\infty} \Phi^{(n)}(p) = p + 1. \tag{15}$$

Thus, when n goes to ∞, the projections of the dilated $(n-1)$-spheres with m=2, 3 intersect the line $\mathcal{L}(\mathbf{e_1})$ at the points $2\,\Phi^{(n)}(m)$ converging to 6 and 8, respectively.

We also proved [26] that, for any $p > 1/\Phi$ and $n > 1$,

$$(p + 1) - 1/(p + 1) < \Phi^{(n)}(p), \tag{16}$$

and that, for each n, the function $\mathbb{R}_+ \ni p \to \Phi^{(n)}(p) \in \mathbb{R}_+$ is strictly increasing and has the same asymptotic line $\mathbb{R}_+ \ni p \to p+1 \in \mathbb{R}_+$. Thus, for any given n, the value $p+1-\Phi^{(n)}(p)$ decreases when p increases. For $m = 1$ and $n > 1$, the Dubeau-Wolfram formula (14) implies that this value is less than $1/2^{n-1}$. Hence, we obtain that, for any m and $n > 1$, it also holds that

$$m + 1 - \frac{1}{2^{n-1}} < \Phi^{(n)}(m). \tag{17}$$

For a given m, formula (16) sets a larger lower bound for the (m, n)-anacci constants than formula (17) if $n < \frac{\ln(m+1)}{\ln 2} + 1$, i.e., for sufficiently small n, formula (16) provides a better estimate for the $\Phi^{(n)}(m)$s than formula (17), whereas for larger n formula (17) is better.

The analogous uniform representations of the (m, n)-anacci constants $\Phi^{(n)}(m)$ can be obtained by means of n-ellipsoids or n-parallelepipeds if the following holds:

(i) the sets are nested within each other so that they have a common center of mass,

(ii) the fixed homothetic center O belongs to all $(n-1)$-enclosures of the sets, and

(iii) the common center of mass is selected for all sets $\mathfrak{B}(m,n)$ with a given m.

4 Geometric representation of (m,n)-anacci constants by means of n-cones

Our second geometric representation of the (m,n)-anacci constants utilizes the fact that the center of mass of a *solid* n-cone divides the interval between the cone's apex and the center of mass of the cone's base by the ratio $n:1$.

For simplicity of notation, we consider the collection of regular, spherical, unit (i.e., with a height of 1) solid n-cones, each with the apex at the origin in $\mathbb{R}^n$, and the center of mass $A(n)$ at $(\frac{n}{n+1}, 0, \ldots, 0) \in \mathbb{R}^n$. For $n > 1$, we assume the radii of the $(n-1)$-balls that form the n-cones' bases to all be equal. We treat interval $[0,1]$ as the 1-cone, i.e., all unit n-cones are nested inside each other as proper subsets, and their bases are nested inside each other as well.

Now, for any given m and n,

(i) we position the homothetic center $O(m,n)$ of the dilation $\Lambda(m,n)$ at the point $\big(\frac{m\cdot n-1}{m\cdot(n+1)}, 0, \ldots, 0\big) \in \mathbb{R}^n$, and

(ii) we select the dilation factor $\mu(m,n)$ such that the center of mass $B\big(\mu(m,n)\big)$ of the set $\mathfrak{B}(m,n)$ is at $B \equiv (1,0,\ldots,0) \in \mathbb{R}^n$— the common center of mass for all the bases of the unit n-cones.

For $m \cdot n \neq 1$, our choice leads to $\mu(m,n) > 1$ and $d\big(A(n), B\big) = m \cdot d\big(O(m,n), A(n)\big)$. Thus, it follows from Theorem 1(i) that the dilation factor $\mu(m,n) = \Phi^{(n)}(m)$.

If $m = n = 1$, Theorem 1(iii) implies that the dilation of the unit 1-cone about the origin with the dilation factor $\mu(1,1) = 1 = \Phi^{(1)}(1)$ leads to $B(1) = 1$, and we define $\mathfrak{B}(1,1) \equiv \{1\}$.

The (m,n)-anacci constants $\Phi^{(n)}(m)$ are now represented uniformly as the heights of the dilated n-cones, i.e., as the following intervals on the coordinate line $\mathcal{L}(\mathbf{e_1})$:

$$\big[L(m,n), R(m,n)\big] \equiv \Big[\big(1-\Phi^{(n)}(m)\big)\frac{mn-1}{m(n+1)}, \big(1-\Phi^{(n)}(m)\big)\frac{mn-1}{m(n+1)} + \Phi^{(n)}(m)\Big]. \quad (18)$$

The center of mass $\Lambda(m,n)\big(A(n)\big)$ of the dilated n-cone is at $\big(\frac{\Phi^{(n)}(m)+mn-1}{m(n+1)}, 0, \ldots, 0\big)$, i.e., it lies inside the unit n-cone in the interval $[A(n), B]$, cf. proof of Theorem 1.

Formula (16) implies that, for any $p > 1/\Phi$ and $n > 1$, we see that[3]

$$p < \Phi^{(n)}(p). \quad (19)$$

[3]Formula (19) is, in fact, valid for any $p > 0$ and $n > 1$, cf. [26].

Using formulas (15) and (19), we find that, for any m and $n > 1$,

$$1 - \frac{1}{m(n+1)} < \frac{\Phi^{(n)}(m) + mn - 1}{m(n+1)} < 1, \tag{20}$$

i.e., for any n, the centers of mass $\Lambda(m,n)\big(A(n)\big)$ converge to B when m goes to ∞, and for any m, the non-zero component of $\Lambda(m,n)\big(A(n)\big)$ converges to 1 when n goes to ∞.

Figure 3 illustrates our construction in 2D by showing the dilation of the unit 2-cone resulting in the dilated 2-cone with the height equal to the golden ratio Φ.

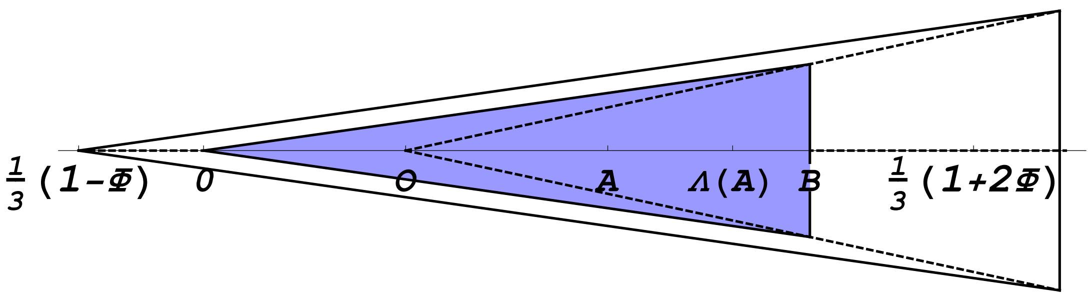

Figure 3: The representation of the golden ration Φ as the height of the white 2-cone obtained by the dilation of the blue unit 2-cone about O positioned 1/3 from the blue cone's apex 0.

Contrary to the radii of the dilated n-balls representing the (m,n)-anacci constants in our first construction, the n-cones' heights (18) representing the $\Phi^{(n)}(m)$s in the second construction are not nested according to the order in the set $\big\{\Phi^{(n)}(m) \mid m,n \in \mathbb{N}\big\}$. Nevertheless, a strict correlation between the order and the locations of heights (18) along the line $\mathcal{L}(\mathbf{e_1})$ is still maintained.

Lemma 2. *(i) For any n, the sequence $\big(L(m,n)\big)_{m=1}^{\infty}$ strictly decreases to $-\infty$, and the sequence $\big(R(m,n)\big)_{m=1}^{\infty}$ strictly increases to ∞. Thus, if n is fixed and $m \to \infty$, heights (18) are nested inside each other according to the order in the set of the (m,n)-anacci constants.*

(ii) For any m, the sequence $\big(L(m,n)\big)_{n=1}^{\infty}$ strictly decreases to $-m$ and the sequences $\big(R(m,n)\big)_{n=1}^{\infty}$ with $m > 2$, $\big(R(2,n)\big)_{n=2}^{\infty}$, and $\big(R(1,n)\big)_{n=3}^{\infty}$ strictly decrease to 1. Consequently, if m is fixed and $n \to \infty$, heights (18) shift in the negative direction along the line $\mathcal{L}(\mathbf{e_1})$ according to the order in the set of the (m,n)-anacci constants with the exception of the heights representing $\Phi^{(2)}(2)$, $\Phi^{(1)}(1)$, and $\Phi^{(3)}(1)$.

Proof. (i) For any n, the sequence $\big(\Phi^{(n)}(m)\big)_{m=1}^{\infty}$ strictly increases to ∞, cf. formulas (15) and (19), while the sequence $\big(\frac{mn-1}{m(n+1)}\big)_{m=1}^{\infty}$ strictly increases to 1. So, the sequence $\big(L(m,n)\big)_{m=1}^{\infty}$ strictly decreases to $-\infty$.

The expression for the right heights' ends $R(m,n)$ in (18) can be transformed into

$$R(m,n) = \frac{mn-1}{m(n+1)} + \frac{m+1}{m(n+1)}\Phi^{(n)}(m). \tag{21}$$

According to our prior results [26], the sequence

$$\Big(\frac{m+1}{m}\Phi^{(n)}(m)\Big)_{m=1}^{\infty} \tag{22}$$

strictly increases to ∞. Thus, the sequence $\big(R(m,n)\big)_{m=1}^{\infty}$ strictly increases to ∞, too.

(ii) For any m, the sequence $\big(\Phi^{(n)}(m)\big)_{n=1}^{\infty}$ strictly increases to $m+1$, cf. (15). Consequently, the sequence $\big(L(m,n)\big)_{n=1}^{\infty}$ strictly decreases to $-m$, while $\lim_{n\to\infty} R(m,n) = 1$. To show that the sequences $\big(R(m,n)\big)_{n=1}^{\infty}$ are monotonically decreasing beginning at some n, we use formula (21), the upper bound (15), and the lower bound (17) to derive

$$R(m,n+1) < \frac{m(n+1)-1}{m(n+2)} + \frac{(m+1)^2}{m(n+2)} \leq \frac{mn-1}{m(n+1)} + \frac{m+1}{m(n+1)}\Big(m+1-\frac{1}{2^{n-1}}\Big) < R(m,n). \tag{23}$$

The middle inequality in formula (23) is true if $m \geq (n+2)/2^{n-1}$. Hence, the latter inequality implies that the sequences $\big(R(m,n)\big)_{n=2}^{\infty}$ with $m > 2$, $\big(R(2,n)\big)_{n=3}^{\infty}$, and $\big(R(1,n)\big)_{n=4}^{\infty}$, are strictly decreasing.

Now, using the lower bound (16) and the upper bound (15), we obtain

$$(m^2+4m-1)/3m < R(m,2) < (m+4)/3. \tag{24}$$

Since $\Phi^{(1)}(m) = m$, we also have

$$R(m,1) = (m^2+2m-1)/2m. \tag{25}$$

It follows from formulas (24) and (25) that, if $m > 2$, we have $R(m,2) < R(m,1)$ but $R(2,2) > R(2,1)$ and $R(1,2) > R(1,1)$.

If we replace in formula (23) the lower bound (17) by the lower bound (16) that provides a better estimate for small n, we obtain that $R(m,n+1) < R(m,n)$ if $n \leq m^2+m-2$. It follows from the latter inequality that $R(2,3) < R(2,2)$. Thus, the sequences $\big(R(m,n)\big)_{n=1}^{\infty}$ with $m > 2$ and $\big(R(2,n)\big)_{n=2}^{\infty}$ are strictly decreasing. However, since formula (15) implies that $\Phi^{(3)}(2) < 3$, we also have $R(2,3) < 1.75 = R(2,1)$, i.e., only the height representing $\Phi^{(2)}(2)$ does not shift according to the order in the set of the (m,n)-anacci constants.

To show how the values $R(1,2)$, $R(1,3)$, and $R(1,4)$ are related requires more subtle estimations than the existing lower and upper bounds provide. The exact solutions of polynomials (4) with $p=1$ and $n=2,3,4$ imply that $1.61 < \Phi^{(2)}(1) = \Phi < 1.62$, $\Phi^{(3)}(1) > 1.83$, and $\Phi^{(4)}(1) < 1.94$.

Thus, it is true that $R(1,2) > R(1,4) < R(1,3) > R(1,2) > R(1,1) = 1$, i.e., the sequence $\big(R(1,n)\big)_{n=3}^{\infty}$ is strictly decreasing, and when m is fixed and n goes to ∞, only the heights representing $\Phi^{(1)}(1)$ and $\Phi^{(3)}(1)$ do not shift according to the order.

□

The analogous uniform representation of the (m,n)-anacci constants can be obtained using the collection of regular n-pyramids with apexes at the origin in $\mathbb{R}^n$, heights equal

1, and bases consisting, e.g., of unit $(n-1)$-cubes with the centers of mass at the same point — which assures the nesting of the pyramids within each other (as well as their bases).

The geometric representation of the $\Phi^{(n)}(m)$s can be constructed also by means of the collection of the n-simplexes. However, this representation does not have a clear uniform geometric interpretation because the proper nesting of the n-simplexes cannot be maintained.

5 Appendix

According to part (iii) of Theorem 1, $d\big(O, B(1)\big) = d(O,A) + \frac{1}{n} d(O,A)$ holds. Thus, the centers of mass A, $\Lambda(A)$, $B(1)$, and $B(\mu)$ are ordered along the line $\mathcal{L}(O,A)$ by their distances from a fixed homothetic center O in terms of varying dilation factors μ in the following way:

$$
\begin{aligned}
&O \prec A \prec B(1) \prec \Lambda(A) \prec B(\mu) \quad \text{if} \quad \mu > 1 + 1/n,\\
&O \prec A \prec B(1) = \Lambda(A) \prec B(\mu) \quad \text{if} \quad \mu = 1 + 1/n,\\
&O \prec A \prec \Lambda(A) \prec B(1) \prec B(\mu) \quad \text{if} \quad 1 < \mu < 1 + 1/n,\\
&O \prec A = \Lambda(A) \prec B(1) = B(\mu) \quad \text{if} \quad \mu = 1,\\
&O \prec \Lambda(A) \prec A \prec B(1) \prec B(\mu) \quad \text{if} \quad 0 < \mu < 1.
\end{aligned}
$$

Interestingly, Theorem 1 implies also that the center of mass in n-cones, n-pyramids, and n-simplexes divides the interval between the apex (vertex) and the center of mass of the set's base according to the ratio $n:1$. Indeed, a dilation with a dilation factor $\mu \neq 1$ of an n-dimensional cone, pyramid, or simplex $\mathfrak{A}$ about the homothetic center O positioned at the set's apex (vertex) results in a *convex* set $\mathfrak{B}$ with the center of mass $B(\mu) \in \mathfrak{B}$. Consequently, $B(1) \equiv \lim_{1 \neq \mu \to 1} B(\mu)$ is the center of mass of the base of $\mathfrak{A}$. Thus, it follows from part (iii) of Theorem 1 that the center of mass A of $\mathfrak{A}$ divides the distance $d\big(O, B(1)\big)$ according to the ratio $n:1$; see Figure 4.

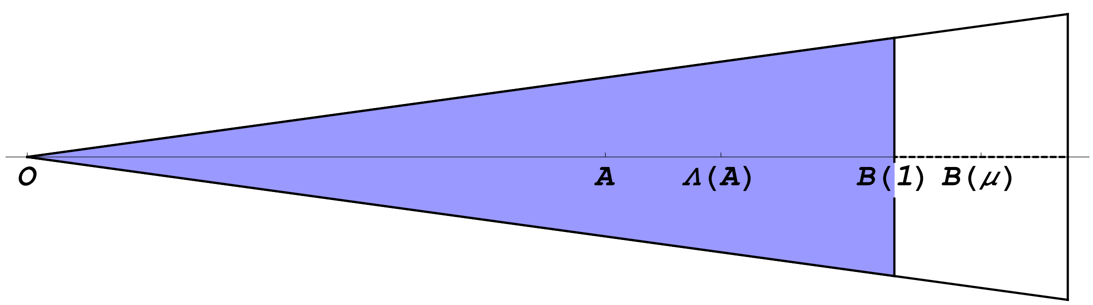

Figure 4: The dilation with $\mu = 1.2$ of the blue 2-cone $\mathfrak{A}$ about the blue cone's apex O creates the white convex set $\mathfrak{B}$. Here, if $\mu \to 1$, $B(\mu) \to B(1)$, where $B(1)$ lies at the base of $\mathfrak{A}$, and $d(O,A) = 2\,d\big(A, B(1)\big)$ according to Theorem 1(iii).

References

[1] M. K. Azarian, Identities involving Lucas or Fibonacci and Lucas numbers as binomial sums, *Int. J. Contemp. Math. Sci.* **7** (2012), 2221–2227.

[2] B. Balof, Restricted tilings and bijections, *J. Integer Seq.* **15** (2012), Article 15.4.5.

[3] V. Baltic, On the number of certain types of strongly restricted permutations, *Appl. Anal. Discrete Math.* **4** (2010), 119–135.

[4] N. D. Cahill, J. R. D'Errico, and J. P. Spenc, Complex factorizations of the Fibonacci and Lucas numbers, *Fibonacci Quart.* **41** (2003), 13–19.

[5] B. Chaffin, J. P. Linderman, N. J. A. Sloane, and A. R. Wilks, On curling numbers of integer sequences, *J. Integer Seq.* **16** (2013), Article 13.4.3.

[6] F. Dubeau, On r-generalized Fibonacci numbers, *Fibonacci Quart.* **27** (1989), 221–228.

[7] F. Dubeau, The rabbit problem revisited, *Fibonacci Quart.* **31** (1993), 268–274.

[8] F. Dubeau and A. G. Shannon, A Fibonacci model of infectious disease, *Fibonacci Quart.* **34** (1996), 257–270.

[9] F. Dubeau, W. Motta, M. Rachidi, and O. Saeki, On weighted r-generalized Fibonacci sequences, *Fibonacci Quart.* **35** (1997), 102–110.

[10] S. Falcon, On the Lucas triangle and its relationship with the k-Lucas numbers, *J. Math. Comp. Sci.* **2** (2012), 425–434.

[11] D. C. Fielder, Certain Lucas-like sequences and their generation by partitions of numbers, *Fibonacci Quart.* **5** (1967), 319–324.

[12] I. Flores, Direct calculation of k-generalized Fibonacci numbers, *Fibonacci Quart.* **5** (1967), 259–266.

[13] L. Guo and W. Sit, Enumeration and generating functions for differential Rota-Baxter words, *Math. Comput. Sci.* **4** (2010), 313–337.

[14] A. F. Horadam, A generalized Fibonacci sequence, *Amer. Math. Monthly* **68** (1961), 455–459.

[15] J. Huang, Hecke algebras with independent parameters, published electronically at `http://arxiv.org/abs/1405.1636`.

[16] M. Janjic, Determinants and recurrence sequences, *J. Integer Seq.* **15** (2012), Article 12.3.5.

[17] T. Khovanova, Recursive sequences, available at
http://www.tanyakhovanova.com/RecursiveSequences/RecursiveSequences.html.

[18] S. Linton, J. Propp, T. Roby, and J. West, Equivalence classes of permutations under various relations generated by constrained transpositions, *J. Integer Seq.* **15** (2012), Article 12.9.1.

[19] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* **67** (1960), 745–752.

[20] T. D. Noe and J. V. Post, Primes in Fibonacci n-step and Lucas n-step sequences, *J. Integer Seq.* **8** (2005), Article 05.4.4.

[21] T. D. Noe, T. Piezas, III, and E. W. Weisstein, Fibonacci n-step number, published electronically at http://mathworld.wolfram.com/Fibonaccin-StepNumber.html.

[22] A. M. Ostrowski, *Solutions of Equations and Systems of Equations*, Academic Press, 1966.

[23] A. Rajan, R. V. Rao, A. Rao, and H. S. Jamadagni, Fibonacci sequence, recurrence relations, discrete probability distributions and linear convolution, published electronically at http://arxiv.org/abs/1205.5398.

[24] M. A. Shattuck and C. G. Wagner, Periodicity and parity theorems for a statistic on r-mino arrangements, *J. Integer Seq.* **9** (2006), Article 06.3.6.

[25] N. J. A. Sloane, Online Encyclopedia of Integer Sequences, http://oeis.org.

[26] I. Szczyrba, R. Szczyrba, and M. Burtscher, Analytic representations of the n-anacci constants and generalizations thereof, *J. Integer Seq.* **18** (2015), Article 15.4.5.

[27] D. A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* **36** (1998), 129–145.

2010 *Mathematics Subject Classification*: Primary 11B37; Secondary 11B39.
Keywords: weighted n-step Fibonacci sequence, generalized n-anacci constant, dilation and geometric representation of the (m, n)-anacci constant.

(Concerned with sequences A000073, A000078, A000213, A000288, A000322, A000383, A001590, A001591, A001592, A001630, A001631, A001644, A001648, A002605, A003691, A007486, A010924, A020992, A021006, A023424, A026150, A028831, A028859, A028860, A030195, A052945, A057087, A057088, A057089, A057090, A057091, A057092, A057093, A060455, A066178, A073728, A073817, A074048, A074584, A077835, A077836, A077843, A079262, A080040, A081172, A083337, A084128, A085480, A086192, A086213, A086347, A093175, A094013, A100532, A100683, A101757, A101758, A102026, A103771, A104144, A104621, A105754, A105755, A106435, A106565, A106568, A108051, A108306, A116556, A119826, A121907, A122189, A122265, A122997, A123526, A123620, A123871, A123887, A124312, A124313, A125145, A127193, A127194, A127624, A134927, A135055, A135056, A135491, A135492, A141036, A141523, A145027, A145028, A145029, A145030, A155116, A155127, A155144, A160175, A162562, A163551, A164540, A164545, A164593, A164603, A164607, A168082, A168083, A168084, A170931, A172011, A172012, A180033, A180167, A181140, A186830, A188714, A189737, A189743, A189749, A202206, A207539, A213665, A214727, A214825, A214826, A214827, A214828, A214829, A214830, A214831, A214899, A220469, A220493, A228603, A248959, A249169, A251653, A251654, A251655, A251656, A251672, A251703, A251704, A251705, A251706, A251707, A251708, A251709, A251710, A251711, A251712, A251713, A251714, A251740, A251741, A251742, A251744, A251745, A251746, A251747, A251748, A251749, A251750, A251751, A251752, A251759, A251760, A251761, A251762, A251763, A251764, A251765, and A251766.)

Received December 5 2015; revised version received March 8 2016. Published in *Journal of Integer Sequences*, April 7 2016.

Return to Journal of Integer Sequences home page.