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On Arithmetic Partial Differential Equations

Pentti Haukkanen and Jorma K. Merikoski
School of Information Sciences
FI-33014 University of Tampere
Finland
pentti.haukkanen@uta.fi
jorma.merikoski@uta.fi

Timo Tossavainen
School of Applied Educational Science and Teacher Education
University of Eastern Finland
P. O. Box 86
FI-57101 Savonlinna
Finland
timo.tossavainen@uef.fi

Abstract

Kovič, and implicitly Ufnarovski and Åhlander, defined a notion of arithmetic partial derivative. We generalize the definition for rational numbers and study several arithmetic partial differential equations of the first and second order. For some equations, we give a complete solution, and for others, we extend previously known results. For example, we determine under which conditions two consecutive partial derivations are commutative.

1 Introduction

Let the symbols $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ have their ordinary meaning. We also write $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$, and $\mathbb{P}$ for the set of primes.

Let $a \in \mathbb{Q} \setminus \{0\}$. There exists a unique sequence of integers (with only finitely many nonzero terms)

$$(\nu_p(a))_{p \in \mathbb{P}}$$

such that

$$a=(\operatorname{sgn} a)\prod_{p\in\mathbb{P}} p^{\nu_p(a)}, \tag{1}$$

where sgn denotes the sign function. If $\nu_p(a)\neq 0$, we say that a *is divisible by* p. We use the notation $p\mid a$ (respectively, $p\nmid a$) when a is divisible by p (respectively, a is not divisible by p). The formula (1) is also valid for $a=0$ as we define $\nu_p(0)=0$ for all $p\in\mathbb{P}$.

We define the *arithmetic derivative* of $a\in\mathbb{Q}\setminus\{0\}$ by

$$a'=a\sum_{p\in\mathbb{P}}\frac{\nu_p(a)}{p}=\sum_{p\in\mathbb{P}}a'_p,$$

where

$$a'_p=\frac{\nu_p(a)}{p}a$$

is the *arithmetic partial derivative* of a with respect to p. In particular, $(\pm1)'=(\pm1)'_p=0$ for all $p\in\mathbb{P}$. We also set $0'=0$ and $0'_p=0$ for all $p\in\mathbb{P}$. Further, we define the *second arithmetic partial derivative* of $a\in\mathbb{Q}$ with respect to $p\in\mathbb{P}$ and $q\in\mathbb{P}$ to be

$$a''_{pq}=(a'_p)'_q.$$

If $a\in\mathbb{Q}\setminus\{0\}$, there are unique $\alpha\in\mathbb{Z}$ and $\tilde{a}\in\mathbb{Q}\setminus\{0\}$ such that

$$a=\tilde{a}p^{\alpha} \tag{2}$$

and $p\nmid\tilde{a}$; in fact, $\alpha=\nu_p(a)$. Then

$$a'_p=\tilde{a}\alpha p^{\alpha-1}.$$

For the second partial derivative,

$$a''_{pp}=\tilde{a}\alpha(\alpha-1)p^{\alpha-2}$$

if and only if $p\nmid\alpha$.

The starting point for the present study was set by Barbeau [1], who defined the arithmetic derivative for $a\in\mathbb{N}$. Ufnarovski and Åhlander [6] generalized this concept for $a\in\mathbb{Q}$. Among other things, they solved certain arithmetic differential equations. The present authors and Mattila [3] studied whether the arithmetic derivative can be defined on a non-unique factorization domain. Mistri and Pandey [5] defined the derivative of an ideal ("ideal derivative") on a number ring and studied its connections with the arithmetic derivative. Among other things, they solved certain ideal differential equations.

According to our knowledge, the idea of the arithmetic partial derivative is due to Kovič [4] for $a\in\mathbb{Z}_+$; yet it was already implicitly used by Ufnarovski and Åhlander. Our definition extends this notion to the set of rational numbers.

The question when

$$a''_{pq}=a''_{qp} \tag{3}$$

arises. Kovič gave a sufficient condition for $a \in \mathbb{Z}_+$ to satisfy this. In Section 4, we will solve the arithmetic partial differential equation $x''_{pq} = x''_{qp}$, obtaining a necessary and sufficient condition for $a \in \mathbb{Q}$ to satisfy (3). We will also solve several other partial differential equations: $x'_p = ax^n$ ($n \in \mathbb{Z}$) in Sections 2 and 3, and, extending Kovič's results, the equations $x'_p = x'_q$, $x''_{pq} = x''_{qp}$, $x''_{pp} = x$, and $x''_{pp} + x''_{qq} = x$ in Section 4. In Section 5, we will discuss some examples. We will complete our paper by investigating the cardinality and density of solution sets in Section 6 and by making a few concluding remarks in Section 7.

2 The equation $x'_p = a$

Given $p \in \mathbb{P}$ and $a \in \mathbb{Q}$, we consider the equation

$$x'_p = a, \tag{4}$$

where $x \in \mathbb{Q}$ is unknown. For $a = 0$, this holds if and only if $\nu_p(x) = 0$, i.e., $p \nmid x$.

Assuming that $a \neq 0$, we factorize $a = \tilde{a}p^{\alpha}$ as in (2). Because $\nu_p(\tilde{a}) = 0$, we have

$$(\tilde{a}x)'_p = \tilde{a}x'_p.$$

So, the equation

$$x'_p = p^{\alpha} \tag{5}$$

is equivalent to

$$(\tilde{a}x)'_p = a.$$

We can therefore solve (4) by solving (5) and multiplying the solution by $\tilde{a}$.

Let

$$x = \tilde{x}p^{\xi}, \tag{6}$$

where

$$\xi \in \mathbb{Z}, \quad p \nmid \tilde{x} \in \mathbb{Q} \setminus \{0\}.$$

Then (5) is equivalent to

$$\tilde{x}\xi p^{\xi-1} = p^{\alpha}.$$

Here $\xi \neq 0$, since otherwise $0 = p^{\alpha}$. Therefore,

$$\tilde{x} = \frac{p^{\alpha-\xi+1}}{\xi} \tag{7}$$

under the condition

$$p \nmid \tilde{x}. \tag{8}$$

This condition is satisfied if and only if

$$\xi = \beta p^{\alpha-\xi+1}, \tag{9}$$

where

$$p \nmid \beta \in \mathbb{Z} \setminus \{0\}. \tag{10}$$

We determine the applicable values of ξ. Since $\xi \in \mathbb{Z}$, (10) implies that $\alpha - \xi + 1 \geq 0$, i.e.,

$$\xi \leq \alpha + 1.$$

Case 1. $\alpha \geq 0$. We study (7) under (8).

Subcase 1a. $1 \leq \xi \leq \alpha$. We go through all possible values of ξ by writing

$$\xi = \alpha - i, \quad i = 0, 1, \ldots, \alpha - 1;$$

then (9) reads

$$\alpha - i = \beta p^{i+1}. \tag{11}$$

(Actually $\beta \in \mathbb{Z}_+$ there, but we also want to cover the case $\alpha < 0$.) We therefore obtain

$$x = \tilde{x} p^{\xi} = \frac{p^{\alpha-\xi+1}}{\xi} p^{\xi} = \frac{p^{\alpha-(\alpha-i)+1}}{\alpha - i} p^{\alpha-i} = \frac{p^{\alpha+1}}{\alpha - i}. \tag{12}$$

Given $i \in \{0, 1, \ldots, \alpha - 1\}$, this is a solution of (5) if and only if i satisfies (11). Denoting by $d \parallel m$ that $d \mid m$ and $\gcd(m, m/d) = 1$, we rewrite (11) as

$$p^{i+1} \parallel (\alpha - i).$$

Subcase 1b. $\xi = \alpha + 1$. Then $\tilde{x} = 1/\xi$ satisfies (8) if and only if $p \nmid \xi$. Thus,

$$x = \tilde{x} p^{\alpha+1} = \frac{p^{\alpha+1}}{\alpha + 1}$$

is a solution of (5) if and only if $p \nmid (\alpha + 1)$.

Subcase 1c. $\xi \leq -1$. Then $\gamma = -\xi \geq 1$, and

$$\tilde{x} = \frac{p^{\alpha-\xi+1}}{\xi} = -\frac{p^{\alpha+\gamma+1}}{\gamma}.$$

Clearly, $p^{\alpha+\gamma+1} > \gamma$, and thus $p^{\alpha+\gamma+1} \nmid \gamma$. Therefore, $p \mid \tilde{x}$, and no new solution is obtained.

Case 2. $\alpha < 0$. For $\xi \leq \alpha$, we proceed as in Subcase 1a. Again, (8) holds if and only if (9) is satisfied. In order to go through all possible values of ξ, we write

$$\xi = \alpha - i, \quad i = 0, 1, 2, \ldots;$$

then (11) is equivalent to (9). Given $i \in \mathbb{N}$, (12) is a solution if and only if i satisfies (11). There are only finitely many such i's, because $p^{i+1} \nparallel (\alpha - i)$ when i is large. For $\xi = \alpha + 1$, proceed as in Subcase 1b.

We summarize the above reasoning in the following theorem.

Theorem 1. *Let $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}$. We write $I_0 = \{0, 1, \dots, \alpha - 1\}$ for $\alpha > 0$, $I_0 = \emptyset$ for $\alpha = 0$, and $I_0 = \mathbb{N}$ for $\alpha < 0$. Let also*

$$I = \left\{ i \in I_0 \,:\, p^{i+1} \parallel (\alpha - i) \right\}.$$

Then

$$x = \frac{p^{\alpha+1}}{\alpha - i} \tag{13}$$

is a solution of

$$x'_p = p^{\alpha}$$

for each $i \in I$. If $p \nmid (\alpha + 1)$, then also

$$x = \frac{p^{\alpha+1}}{\alpha + 1} \tag{14}$$

is a solution. All solutions are obtained in this way. In particular ($\alpha = 0$), the only solution of $x'_p = 1$ is $x = p$. The equation

$$x'_p = 0$$

holds if and only if $p \nmid x$.

Next, let us take a closer look at I. In the following, we assume that $\alpha > 0$. A simple modification applies to $\alpha < 0$. The case $\alpha = 0$ does not require any further work.

Let

$$\alpha = np + r, \quad 0 \le r < p. \tag{15}$$

Then the solution candidates (13) are

$$x = \frac{p^{np+r+1}}{np + r - i}, \quad i = 0, 1, \dots, np + r - 1.$$

To study the validity of (11), we write

$$i = mp + k,$$

where $m = 0, 1, \dots, n$. If $m < n$, then $k = 0, 1, \dots, p - 1$, and if $m = n$, then $k = 0, 1, \dots, r - 1$. (So, here m and k are variables while p, n, and r are constants.) Now

$$\alpha - i = np + r - (mp + k) = (n - m)p + r - k.$$

By (11), p must divide $(n - m)p + r - k$. Hence, $r = k$, and

$$\alpha - i = (n - m)p.$$

It is convenient to replace β with h in (11); then

$$(n - m)p = hp^{mp+r+1}. \tag{16}$$

Since $h \in \mathbb{Z}_+$, $m = n$ cannot hold; Thus, $m < n$, and the candidates are of the form

$$x = \frac{p^{np+r+1}}{(n-m)p} = \frac{p^{np+r+1}}{hp^{mp+r+1}} = \frac{p^{(n-m)p}}{h}.$$

They are solutions if and only if m and h satisfy (16), i.e.,

$$n - m = hp^{mp+r}. \tag{17}$$

It turns out that the cases $m = 0$ and $m = n - 1$ are easy to solve, but the case $1 \le m \le n-2$ is remarkably more difficult. Namely, if $m = 0$, then (17) implies that $p^r \mid n$ and $p^{r+1} \nmid n$. Then $h = n/p^r$, and

$$x = \frac{1}{h}p^{np} = \frac{p^r}{n}p^{np} = \frac{p^\alpha}{n}$$

is a solution. In the case of $r = 0$, we can eliminate n by writing

$$x = \frac{p^{np}}{n} = \frac{p^{np+1}}{np} = \frac{p^{\alpha+1}}{\alpha}.$$

For $m = n-1$, (17) implies that $hp^{(n-1)p+r} = 1$, which is impossible unless $n = 1$, $r = 0$, and $h = 1$. Then $\alpha = p$, see Example 17.

If $1 \le m \le n-2$, it seems difficult to find explicit solutions in general. In this paper, we only note that, if $n < p^p$ and $m > 0$, then

$$hp^{mp} = h(p^p)^m \ge p^p > n.$$

Hence (17) does not hold, and no solution is obtained.

We also record that the candidate (14),

$$x = \frac{p^{np+r+1}}{np+r+1},$$

is a solution if and only if $p \nmid (np+r+1)$, i.e., $r \ne p-1$.

Including the case $\alpha < 0$, we now obtain the following summary.

Theorem 2. *Let $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z} \setminus \{0\}$. Express α as in (15). If $h \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}$ satisfy (17) and*

$$\begin{gathered} h > 0 \text{ and } 0 \le m \le n-1 \text{ for } \alpha > 0, \\ h < 0 \text{ and } m \ge 0 \text{ for } \alpha < 0, \end{gathered} \tag{18}$$

then

$$x = \frac{p^{(n-m)p}}{h} \tag{19}$$

is a solution of

$$x'_p = p^\alpha.$$

If $r \ne p-1$, then also

$$x = \frac{p^{\alpha+1}}{\alpha+1} \tag{20}$$

is a solution. All solutions are obtained in this way.

3 The equation $x'_p = ax^n$

Given $p \in \mathbb{P}$, $a \in \mathbb{Q}$, and $n \in \mathbb{Z}$, we now study the equation

$$x'_p = ax^n.$$

If $n \geq 1$, then it has a trivial solution $x = 0$. We have already settled the cases $a = 0$ and $n = 0$ in Theorem 1. Next, we consider the case $n = 1$.

Theorem 3. *Let $p \in \mathbb{P}$ and $a \in \mathbb{Q}$. The equation*

$$x'_p = ax$$

has a nontrivial solution if and only if $ap \in \mathbb{Z}$. Then all nontrivial solutions are of the form

$$x = cp^{ap},$$

where $p \nmid c \in \mathbb{Q} \setminus \{0\}$. Conversely, all numbers of this form are nontrivial solutions.

Proof. Let $x = \tilde{x}p^{\xi}$ be as in (6). Then

$$x'_p = ax \iff \tilde{x}\xi p^{\xi-1} = a\tilde{x}p^{\xi} \iff \xi = ap,$$

and the theorem follows (as we write $c = \tilde{x}$). □

Corollary 4. *Let $p_1, p_2, \ldots, p_m \in \mathbb{P}$ be different, and let $a_1, a_2, \ldots, a_m \in \mathbb{Q}$. The system of equations*

$$x'_{p_1} = a_1 x, \quad x'_{p_2} = a_2 x, \quad \ldots, \quad x'_{p_m} = a_m x$$

has a nontrivial solution if and only if $a_1p_1, a_2p_2, \ldots, a_mp_m \in \mathbb{Z}$. All nontrivial solutions are then of the form

$$x = cp_1^{a_1p_1} p_2^{a_2p_2} \cdots p_m^{a_mp_m},$$

where $p_1, p_2, \ldots, p_m \nmid c \in \mathbb{Q} \setminus \{0\}$. Conversely, all numbers of this form are nontrivial solutions.

Proof. The claim follows by induction. □

We tackle the remaining values of n in the following theorem. Therein "nontrivial" is relevant only for $n \geq 2$ because $x = 0$ is not a solution whenever $n \leq -1$.

Theorem 5. *Let $p \in \mathbb{P}$, $a \in \mathbb{Q} \setminus \{0\}$, and $n \in \mathbb{Z} \setminus \{0, 1\}$. The equation*

$$x'_p = ax^n \tag{21}$$

has a nontrivial solution if and only if the following conditions are satisfied:

(i) *There is $\xi \in \mathbb{Z} \setminus \{0\}$ such that*

$$\nu_p(\xi) = \nu_p(a) + (n-1)\xi + 1 \tag{22}$$

and

$$\frac{\nu_q(\xi) - \nu_q(a)}{n-1} \in \mathbb{Z} \tag{23}$$

for all $q \in \mathbb{P} \setminus \{p\}$.

(ii) *For odd* n, $\operatorname{sgn} \xi = \operatorname{sgn} a$.

Then all nontrivial solutions are of the form

$$x = \Big(\frac{\xi}{ap}\Big)^{\frac{1}{n-1}}, \tag{24}$$

where ξ *satisfies* (i) *and* (ii). *Conversely, all numbers of this form are nontrivial solutions.*

Proof. To solve (21), we write again $x = \tilde{x}p^\xi$ as in (6). Then $\xi \neq 0$ because $x = \tilde{x}$ is not a solution. Since

$$x'_p = ax^n \iff \tilde{x}\xi p^{\xi-1} = a\tilde{x}^n p^{n\xi} \iff \xi = a\tilde{x}^{n-1}p^{(n-1)\xi+1},$$

the equation

$$\xi = a\tilde{x}^{n-1}p^{(n-1)\xi+1} \tag{25}$$

is equivalent to (21).

If ξ satisfies (i) and (ii), our task is to find $\tilde{x}$ satisfying (25), i.e.,

$$\tilde{x}^{n-1} = \frac{\xi}{ap^{(n-1)\xi+1}} =: y.$$

By (i), $(n-1) \,|\, \nu_q(y)$ for all $q \in \mathbb{P}$. By (ii), $y > 0$ if n is odd. Therefore,

$$\tilde{x} = y^{\frac{1}{n-1}} \in \mathbb{Q} \setminus \{0\},$$

and

$$x = \tilde{x}p^\xi = \Big(\frac{\xi}{ap^{(n-1)\xi+1}}\Big)^{\frac{1}{n-1}} p^\xi = \Big(\frac{\xi}{ap}\Big)^{\frac{1}{n-1}}$$

is a nontrivial solution of (21).

Conversely, if x satisfies (21), then (25) holds. Hence,

$$\nu_p(\xi) = \nu_p(a) + (n-1)\xi + 1,$$

i.e., (22) follows. For $q \in \mathbb{P} \setminus \{p\}$,

$$\nu_q(\xi) = \nu_q(a) + (n-1)\nu_q(\tilde{x}),$$

and

$$\nu_q(\tilde{x}) = \frac{\nu_q(\xi) - \nu_q(a)}{n-1}.$$

Since $\nu_q(\tilde{x}) \in \mathbb{Z}$, (23) follows. If n is odd, then $\tilde{x}^{n-1} > 0$. Hence, by (25), ξ and a have the same signs, which verifies (ii). $\square$

Corollary 6. *Let $p \in \mathbb{P}$, $p \nmid a \in \mathbb{Q} \setminus \{0\}$, and $n \in \mathbb{Z} \setminus \{0, 1\}$. The equation*

$$x'_p = ax^n$$

has a nontrivial solution if and only if $n = 2$. The solution is

$$x = -\frac{1}{ap}. \tag{26}$$

Proof. By Theorem 5,

$$x = \left(\frac{\xi}{ap}\right)^{\frac{1}{n-1}}, \tag{27}$$

where $\xi \in \mathbb{Z} \setminus \{0\}$ satisfies (i) and (ii). Let

$$\xi = \tilde{\xi} p^\gamma, \tag{28}$$

where

$$\gamma \in \mathbb{N}, \quad p \nmid \tilde{\xi} \in \mathbb{Z} \setminus \{0\}. \tag{29}$$

Then (22) reduces to

$$\gamma = (n-1)\tilde{\xi}p^\gamma + 1$$

or, equivalently,

$$\tilde{\xi} = \frac{\gamma - 1}{(n-1)p^\gamma}.$$

To complete the proof, we study which values of γ apply.

Case 1. $\gamma = 0$. Then $\tilde{\xi} = -(n-1)^{-1} \in \mathbb{Z}$ if and only if $n - 1 = \pm 1$. Since $n \neq 0, 1$, necessarily $n = 2$. Hence, $\tilde{\xi} = -1$ implying also $\xi = -1$. To check (23), now

$$\frac{\nu_q(\xi) - \nu_q(a)}{n-1} = \frac{0 - \nu_q(a)}{2-1} = -\nu_q(a) \in \mathbb{Z}$$

for all $q \in \mathbb{P}$.

Case 2. $\gamma = 1$. Then $\tilde{\xi} = 0$, contradicting (29).

Case 3. $\gamma \geq 2$. Then

$$|n-1|p^\gamma \geq p^\gamma > \gamma - 1 > 0,$$

so $0 < |\tilde{\xi}| < 1$, again contradicting (29).

Substituting $n = 2$ and $\xi = -1$ in (27) yields (26). □

Corollary 7. *Let $p \in \mathbb{P} \setminus \{2\}$ and $n \in \mathbb{Z} \setminus \{0, 1\}$. The equation*

$$x'_p = px^n$$

has a nontrivial solution if and only if $n = 2$ *or* $n = -1$*. The solution is*

$$x = -\frac{2}{p^2}$$

in the first-mentioned case and

$$x = p \tag{30}$$

in the second one. For $p = 2$*, only* (30) *holds.*

Proof. With the above notation, (22) reads $\gamma = 1 + (n-1)\tilde{\xi}p^\gamma + 1$, i.e.,

$$\tilde{\xi} = \frac{\gamma - 2}{(n-1)p^\gamma}.$$

Case 1. $\gamma = 0$. Then $\tilde{\xi} = -2(n-1)^{-1} \in \mathbb{Z}$ if and only if $n - 1 = \pm 1$ or $n - 1 = \pm 2$. Since $n \neq 0, 1$, necessarily $n = 2$, $n = 3$, or $n = -1$. We have $\tilde{\xi} = -2$ (for $p \neq 2$), $\tilde{\xi} = -1$, and $\tilde{\xi} = 1$, respectively. Further, $\xi = \tilde{\xi}$. The validity of (23) is easily verified. However, (ii) in Theorem 5 is violated for $n = 3$.

Case 2. $\gamma = 1$. Then

$$\tilde{\xi} = -\frac{1}{(n-1)p} \notin \mathbb{Z}.$$

Case 3. $\gamma = 2$. As Case 2 in the previous proof.

Case 4. $\gamma \geq 3$. As Case 3 in the previous proof.

Thus, $n = 2$, $\xi = -2$ (for $p \neq 2$), and $n = -1$, $\xi = 1$ remain. Substituting them in (24) yields the claims of the corollary. □

4 Equations involving x'_p and x'_q or their derivatives

In this section, we extend Kovič's results from $x \in \mathbb{Z}_+$ to $x \in \mathbb{Q}$ (and revise some of them). We begin by [4, Proposition 34].

Theorem 8. *Let* $p, q \in \mathbb{P}$*,* $p \neq q$*. All nontrivial solutions of*

$$x'_p = x'_q \tag{31}$$

are of the form

$$x = cp^{kp}q^{kq},$$

where

$$k \in \mathbb{Z}, \quad p, q \nmid c \in \mathbb{Q} \setminus \{0\}.$$

Conversely, all numbers of this form are nontrivial solutions.

Proof. Let

$$x = \tilde{x} p^{\xi} q^{\eta}, \tag{32}$$

where

$$\xi, \eta \in \mathbb{Z}, \quad p, q \nmid \tilde{x} \in \mathbb{Q} \setminus \{0\}.$$

Then (31) is equivalent to

$$\tilde{x} \xi p^{\xi-1} q^{\eta} = \tilde{x} \eta p^{\xi} q^{\eta-1},$$

i.e.,

$$\xi q = \eta p. \tag{33}$$

Hence, $p \mid \xi$, implying that $\xi = kp$ for some $k \in \mathbb{Z} \setminus \{0\}$. Similarly, $\eta = hq$ for some $h \in \mathbb{Z} \setminus \{0\}$. Because $kpq = hqp$ by (33), we have $h = k$, and the theorem follows (as we write $c = \tilde{x}$). □

Corollary 9. *Let $p_1, p_2, \ldots, p_m \in \mathbb{P}$ be different. Then all nontrivial solutions of*

$$x'_{p_1} = x'_{p_2} = \cdots = x'_{p_m}$$

are of the form

$$x = c p_1^{kp_1} p_2^{kp_2} \cdots p_m^{kp_m},$$

where $k \in \mathbb{Z}$ and $p_1, p_2, \ldots, p_m \nmid c \in \mathbb{Q} \setminus \{0\}$. Conversely, all numbers of this form are nontrivial solutions.

Proof. The claim follows by induction. □

Let $p, q \in \mathbb{P}$. Kovič [4, Proposition 33] proved for $a \in \mathbb{Z}_+$ that if $\gcd(\nu_p(a), q) = \gcd(\nu_q(a), p) = 1$, then

$$a''_{pq} = a''_{qp}. \tag{34}$$

We next show for $a \in \mathbb{Q}$ that (34) holds if and only if

$$\nu_q(\nu_p(a))\nu_p(a) = \nu_p(\nu_q(a))\nu_q(a).$$

Theorem 10. *Let $p, q \in \mathbb{P}$, $p \neq q$. All nontrivial solutions of*

$$x''_{pq} = x''_{qp} \tag{35}$$

are of the form

$$x = c p^h q^k,$$

where

$$h, k \in \mathbb{Z}, \quad p, q \nmid c \in \mathbb{Q} \setminus \{0\}$$

and

$$h \nu_q(h) = k \nu_p(k).$$

Conversely, all numbers of this form are nontrivial solutions.

Proof. Let x be as in (32) and

$$\xi = \tilde{\xi} p^{\alpha} q^{\beta}, \quad \eta = \tilde{\eta} p^{\gamma} q^{\delta},$$

where

$$\alpha, \beta, \gamma, \delta \in \mathbb{N}, \quad p, q \nmid \tilde{\xi}, \tilde{\eta} \in \mathbb{Z} \setminus \{0\}.$$

Then

$$x'_p = \tilde{x} \xi p^{\xi - 1} q^{\eta} = \tilde{x} \tilde{\xi} p^{\alpha} q^{\beta} p^{\xi - 1} q^{\eta} = \tilde{x} \tilde{\xi} p^{\xi + \alpha - 1} q^{\eta + \beta}$$

and

$$x''_{pq} = \tilde{x} \tilde{\xi} (\eta + \beta) p^{\xi + \alpha - 1} q^{\eta + \beta - 1} = \tilde{x} \tilde{\xi} (\tilde{\eta} p^{\gamma} q^{\delta} + \beta) p^{\xi + \alpha - 1} q^{\eta + \beta - 1} =$$
$$\tilde{x} \tilde{\xi} \tilde{\eta} p^{\xi + \alpha + \gamma - 1} q^{\eta + \beta + \delta - 1} + \tilde{x} \tilde{\xi} \beta p^{\xi + \alpha - 1} q^{\eta + \beta - 1},$$

and similarly

$$x''_{qp} = \tilde{x} \tilde{\xi} \tilde{\eta} p^{\xi + \alpha + \gamma - 1} q^{\eta + \beta + \delta - 1} + \tilde{x} \tilde{\eta} \gamma p^{\xi + \gamma - 1} q^{\eta + \delta - 1}.$$

Therefore, (35) holds if and only if

$$\tilde{\xi} \beta p^{\xi + \alpha - 1} q^{\eta + \beta - 1} = \tilde{\eta} \gamma p^{\xi + \gamma - 1} q^{\eta + \delta - 1}.$$

Substituting $\tilde{\xi} = \xi p^{-\alpha} q^{-\beta}$ and $\tilde{\eta} = \eta p^{-\gamma} q^{-\delta}$, we obtain

$$\xi \beta = \eta \gamma,$$

and the theorem follows (as we write $c = \tilde{x}$, $h = \xi$, $k = \eta$, $\nu_q(h) = \beta$, $\nu_p(k) = \gamma$). ☐

Next, we extend [4, Proposition 35(i)]. (In this proposition, $n = p^{pk} c$ should actually read $n = p^p c$.)

Theorem 11. *Let $p \in \mathbb{P}$. All nontrivial solutions of*

$$x''_{pp} = x \tag{36}$$

are of the form

$$x = c p^{\pm p}, \tag{37}$$

where $p \nmid c \in \mathbb{Q} \setminus \{0\}$. Conversely, all numbers of this form are nontrivial solutions.

Proof. Let $x = \tilde{x} p^{\xi}$ as in (6). Also, let $\xi = \tilde{\xi} p^{\gamma}$ as in (28). Then

$$x'_p = \tilde{x} \xi p^{\xi - 1} = \tilde{x} \tilde{\xi} p^{\xi + \gamma - 1}$$

and

$$x''_{pp} = \tilde{x} \tilde{\xi} (\xi + \gamma - 1) p^{\xi + \gamma - 2} = x p^{-\xi} \tilde{\xi} (\tilde{\xi} p^{\gamma} + \gamma - 1) p^{\xi + \gamma - 2} = x \tilde{\xi} p^{\gamma - 2} (\tilde{\xi} p^{\gamma} + \gamma - 1). \tag{38}$$

Therefore, (36) is equivalent to

$$\tilde{\xi} p^{\gamma} (\tilde{\xi} p^{\gamma} + \gamma - 1) = p^2 \tag{39}$$

(or $x = 0$). Since $\tilde{\xi} \mid p^2$ but $p \nmid \tilde{\xi}$, necessarily $\tilde{\xi} = \pm 1$ and (39) reduces to

$$p^\gamma[p^\gamma \pm (\gamma - 1)] = p^2.$$

If $\gamma = 0$, then $1 \mp 1 = p^2$, which is impossible. If $\gamma = 2$, then $p = 0$ or $p = \sqrt{2}$, which is again impossible. If $\gamma \geq 3$, then $p^3 \mid p^2$; this is also impossible. But $\gamma = 1$ works, giving us $\xi = \pm p$. Writing $c = \tilde{x}$, we obtain (37). $\square$

Let $p, q \in \mathbb{P}$, $p \neq q$. According to Kovič [4, Proposition 35(iii)], the only integer solutions of $x''_{pp} + x''_{qq} = x$ are $x = cp^p$ and $x = cq^q$, where $\gcd(c, pq) = 1$. We extend this theorem to $\mathbb{Q}$, discovering that other integer solutions also exist. The question about the complete solution, however, remains open.

Theorem 12. *Let $p, q \in \mathbb{P}$, $p \neq q$. If*

$$x \in \{cp^{\pm p}, cq^{\pm q}, cp^{\pm p}q, cpq^{\pm q}\},$$

where $p, q \nmid c \in \mathbb{Q}$, then

$$x''_{pp} + x''_{qq} = x.$$

Proof. A simple calculation. $\square$

Finally, we extend [4, Proposition 35(ii)]. (The proof of this proposition is incorrect in [4]. For example, $(ep^{e-1}c)'_p = e(e-1)p^{e-2}c$ is not generally valid because e may be divisible by p.)

Theorem 13. *Let $p \in \mathbb{P}$, $\alpha \in \mathbb{Z}_+$, and $c \in \mathbb{Q}$. If $\alpha \geq p$ and $p \nmid c$ or $c = 0$, then $x = cp^\alpha$ satisfies*

$$|x''_{pp}| \geq |x|.$$

Proof. Omitting the trivial case, we assume that $c \neq 0$. If $p \nmid \alpha$, then $\alpha \geq p$ implies $\alpha \geq p+1$, and

$$|x''_{pp}| = |c|\alpha(\alpha - 1)p^{\alpha-2} \geq |c|(p+1)pp^{\alpha-2} > |c|p^\alpha = |x|.$$

If $p \mid \alpha$, we write $\alpha = \tilde{\alpha}p^\gamma$ where $\gamma, \alpha \in \mathbb{Z}_+$ and $p \nmid \tilde{\alpha}$. Then

$$x'_p = c\alpha p^{\alpha-1} = c\tilde{\alpha}p^\gamma p^{\alpha-1} = c\tilde{\alpha}p^{\gamma+\alpha-1}$$

and

$$x''_{pp} = c\tilde{\alpha}(\gamma + \alpha - 1)p^{\gamma+\alpha-2} = c\alpha p^{-\gamma}(\gamma + \alpha - 1)p^{\gamma+\alpha-2} =$$
$$c\alpha(\gamma + \alpha - 1)p^{\alpha-2} = \alpha(\gamma + \alpha - 1)p^{-2}x.$$

Since

$$\alpha(\gamma + \alpha - 1)p^{-2} \geq p(1 + p - 1)p^{-2} = 1,$$

the theorem follows. $\square$

5 Examples

Example 14. $x'_p = a$, where $p \nmid a \in \mathbb{Q} \setminus \{0\}$.

In (2), $\tilde{a} = a$ and $\alpha = 0$. The only solution of (5) is $x = p$ by (14). Therefore, the only solution is

$$x = ap.$$

Example 15. $x'_p = p$ $(\alpha = 1, n = 0, r = 1)$.

There is no m satisfying (18). If $p \neq 2$, then $r \neq p - 1$. So, by (20),

$$x = \frac{p^2}{2}.$$

If $p = 2$, then $r = p - 1$, and there is no solution.

Example 16. $x'_p = p^{-1}$ $(\alpha = n = -1, r = p - 1)$.

The condition (17) implies that $-1 - m = hp^{(m+1)p-1}$. Writing $h = -k$, where $k \in \mathbb{Z}_+$, this reads

$$1 + m = kp^{(m+1)p-1}.$$

But

$$kp^{(m+1)p-1} \geq 2^{(m+1)\cdot 2-1} = 2^{2m+1} > 2m + 1 \geq m + 1$$

for all $m \in \mathbb{N}$, meaning that there is no solution of type (19). Neither does (20) apply because $r = p - 1$. Thus, there is no solution.

Example 17. $x'_p = p^p$ $(\alpha = p, n = 1, r = 0)$.

The only m satisfying (18) is $m = 0$. Then $h = 1$ in (17) and

$$x = p^p$$

by (19). Since $r \neq p - 1$, also

$$x = \frac{p^{p+1}}{p+1}$$

applies by (20).

Example 18. $x'_p = p^{-p}$ $(\alpha = -p, n = -1, r = 0)$.

By (17),

$$-1 - m = hp^{-mp}.$$

If $m = 0$, then $h = -1$. If $m > 0$, the right-hand side is not an integer (since $p \nmid h$), while the left-hand side is an integer. Therefore, only $m = 0$ and $h = -1$ satisfy (17). Further, (19) gives

$$x = -p^{-p}.$$

Because $r \neq p - 1$, we have another solution

$$x = \frac{p^{1-p}}{1-p}$$

by (20).

Example 19. $x'_2 = 2^{26}$ $(p = 2, \alpha = 26, n = 13, r = 0)$.

The condition (17) is now equivalent to

$$13 - m = 2^{2m} h.$$

If $m = 0$, then $h = 13$. If $m = 1$, then $h = 3$. If $m \geq 2$, then $2^{2m} > 13 - m$; hence $2^{2m} \nmid (13 - m)$ for $2 \leq m \leq 12$, implying that a suitable h does not exist. So, by (19),

$$x = \frac{2^{26}}{13} \quad \text{and} \quad x = \frac{2^{24}}{3}.$$

Again $r \neq p - 1$, and (20) yields the third solution

$$x = \frac{2^{27}}{27}.$$

Example 20. $x'_2 = 2^{-22}$ $(p = 2, \alpha = -22, n = -11, r = 0)$.

Now (17) reads

$$-11 - m = 2^{2m} h.$$

If $m = 0$, then $h = -11$. If $m = 1$, then $h = -3$. Proceeding as above, we see that no $m \geq 2$ works, and we obtain three solutions

$$x = -\frac{2^{-22}}{11}, \quad x = -\frac{2^{-24}}{3}, \quad x = -\frac{2^{-21}}{21}.$$

Example 21. (Due to Jori Mäntysalo.) $x'_2 = 2^{2058}$ $(p = 2, \alpha = 2058, n = 1029, r = 0)$.

Computer experiments suggest $m = 0$, $m = 1$, and $m = 5$. Then, respectively, $h = 1029$, $h = 257$, and $h = 1$, giving us three solutions of type (19). Also, (20) applies. All in all, we have four solutions

$$x = \frac{2^{2058}}{1029}, \quad x = \frac{2^{2056}}{257}, \quad x = 2^{2048}, \quad x = \frac{2^{2059}}{2059}.$$

Example 22. $x'_p = -px^n$, $n \in \mathbb{Z} \setminus \{0\}$.

If $n = 1$, then, by Theorem 3, $x = cp^{-p^2}$ where $p \nmid c \in \mathbb{Q}$. To obtain nontrivial solutions in the case of $n \neq 1$, we see as in the proof of Corollary 7 that necessarily $n = 2$, $n = 3$, or $n = -1$; then $\tilde{\xi} = -2$ (for $p \neq 2$), $\tilde{\xi} = -1$, and $\tilde{\xi} = 1$, respectively. Further, $\xi = \tilde{\xi}$. Again (23) holds, but now (ii) in Theorem 5 is violated for $n = -1$. Thus, the nontrivial solution is, for $n = 3$,

$$x = \left(\frac{-1}{-p^2}\right)^{\frac{1}{2}} = \frac{1}{p}$$

and, for $n = 2$, $p \neq 2$,

$$x = \frac{-2}{-p^2} = \frac{2}{p^2}.$$

Example 23. $x'_p = p^{-1}x^n$, $n \in \mathbb{Z} \setminus \{0\}$.

If $n = 1$, then, again by Theorem 3,

$$x = cp,$$

where $p \nmid c \in \mathbb{Q}$. If $n \neq 1$, we have, by applying (22) and using the notation of the proof of Corollary 6,

$$\gamma = -1 + (n-1)\xi + 1 = (n-1)\tilde{\xi}p^{\gamma}.$$

As in that proof, we see that no $\gamma \in \mathbb{N}$ works. Consequently, there is only the trivial solution for $n \geq 2$ and no solution for $n \leq -1$.

Example 24. $x'_p = p^{1-(n-1)p^2}x^n$, $n \in \mathbb{Z} \setminus \{0, 1\}$.

By (22),

$$\gamma = 1 - (n-1)p^2 + (n-1)\xi + 1 = 2 - (n-1)p^2 + (n-1)\tilde{\xi}p^{\gamma},$$

i.e.,

$$\tilde{\xi} = \frac{\gamma + (n-1)p^2 - 2}{(n-1)p^{\gamma}}.$$

We proceed as in the proof of Corollary 7.

Case 1. $\gamma = 0$. Again, $\tilde{\xi} \in \mathbb{Z}$ if and only if $n - 1 = \pm 1$ or $n - 1 = \pm 2$; hence necessarily $n = 2$, $n = 3$, or $n = -1$. We have $\tilde{\xi} = p^2 - 2$ (for $p \neq 2$) and $\tilde{\xi} = p^2 \mp 1$, respectively. Also $\xi = \tilde{\xi}$. But $\delta = \nu_q(p^2 \mp 1)$ is odd for at least one $q \in \mathbb{P} \setminus \{p\}$. (If not, then $p^2 \mp 1 = b^2$ for some $b \in \mathbb{Z}$, so $(b-p)(b+p) = \mp 1$, implying a contradiction.) Then

$$\frac{\nu_q(\xi) - \nu_q(a)}{n-1} = \frac{\nu_q(p^2 \pm 1) - \nu_q(p^{1-(n-1)p^2})}{\pm 2} = \pm\frac{\delta}{2} \notin \mathbb{Z},$$

violating (23). So, only the subcase $n = 2$, $p \neq 2$, $\xi = p^2 - 2$ remains, which yields

$$x = \left(\frac{\xi}{ap}\right)^{\frac{1}{n-1}} = \frac{p^2 - 2}{p^{1-(2-1)p^2}p} = (p^2 - 2)p^{p^2-2}.$$

Case 2. $\gamma = 1$. Then

$$\tilde{\xi} = \frac{1 + (n-1)p^2 - 2}{(n-1)p} = p - \frac{1}{(n-1)p} \notin \mathbb{Z}.$$

Case 3. $\gamma = 2$. Then

$$\tilde{\xi} = \frac{2 + (n-1)p^2 - 2}{(n-1)p^2} = 1;$$

so $\xi = p^2$. Since (23) is satisfied, we obtain

$$x = \left(\frac{p^2}{p^{1-(n-1)p^2}p}\right)^{\frac{1}{n-1}} = p^{p^2}.$$

Case 4. $\gamma \geq 3$. Then

$$|\tilde{\xi}| = \frac{|\gamma + (n-1)p^2 - 2|}{|n-1|p^\gamma} \leq \frac{|n-1|p^2 + \gamma}{|n-1|p^\gamma} \leq \frac{p^2+\gamma}{p^\gamma} = \frac{1}{p^{\gamma-2}} + \frac{\gamma}{p^\gamma} \leq \frac{1}{p} + \frac{3}{p^3} \leq \frac{1}{2} + \frac{3}{8} < 1.$$

Thus, $x = p^{p^2}$ is a nontrivial solution for all $n \neq 0, 1$. Also $(p^2-2)p^{p^2-2}$ is such a solution for $n = 2$, $p \neq 2$. These are the only nontrivial solutions.

Example 25. $x''_{pp} = px$.

By (38), an equivalent condition is that $x = 0$ or

$$\tilde{\xi}p^\gamma(\tilde{\xi}p^\gamma + \gamma - 1) = p^3.$$

As in the proof of Theorem 11, this reduces to

$$p^\gamma[p^\gamma \pm (\gamma - 1)] = \pm p^3.$$

It is easy to see that no $\gamma \in \mathbb{N}$ applies. So, there are no other solutions than the trivial one.

Example 26. $x''_{pp} = 0$.

By (38), this is equivalent to $x = 0$ or $\tilde{\xi} = 0$ (accepted here) or $\tilde{\xi}p^\gamma + \gamma - 1 = 0$. It is easy to see that $\gamma = 0$ and $\gamma = 1$ apply; the first case gives us $\tilde{\xi} = 1$ and the second one $\tilde{\xi} = 0$. So, $x = 0$ or $\xi = 0$ or $\xi = p$, which implies that $x = cp^d$, where $p \nmid c$ and $d \in \{0, 1\}$.

6 Notes on solution sets

Examples 14–21 and further computer experiments encourage us to state the following conjecture.

Conjecture 27. Let $p \in \mathbb{P}$ and $a \in \mathbb{Q} \setminus \{0\}$. The equation

$$x'_p = a$$

has at most four solutions if $p = 2$, and at most two solutions otherwise.

What about the number of solutions of $x'_p = ax^n$? The case $n = 1$ is easy.

Theorem 28. *Let $p \in \mathbb{P}$ and $a \in \mathbb{Q}$. The number of nontrivial solutions of the equation*

$$x'_p = ax$$

is either zero or infinite.

Proof. Apply Theorem 3. $\square$

The case $n \geq 2$ is difficult. Because we have not done thorough computer experiments, we only record the following speculation.

Conjecture 29. Let $p \in \mathbb{P}$, $a \in \mathbb{Q} \setminus \{0\}$, and $n \in \mathbb{Z} \setminus \{0, 1\}$. The equation

$$x'_p = ax^n$$

has only finitely many solutions.

Our last topic concerns the density of infinite solution sets. We begin by verifying two auxiliary results.

Lemma 30. *Let $r, s \in \mathbb{P}$, $r \neq s$. The set*

$$T = \{\pm r^m s^n : m, n \in \mathbb{Z}\} \tag{40}$$

is dense in $\mathbb{Q}$.

Proof. An equivalent claim is that the set

$$\{r^m s^n : m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{Q}_+$. By taking logarithms of the elements of this set, we get another equivalent claim stating that the set

$$\{m \ln r + n \ln s : m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. This, is turn, is equivalent to the density of

$$T' = \Big\{m + n \frac{\ln s}{\ln r} : m, n \in \mathbb{Z}\Big\}.$$

Because $\ln s / \ln r$ is irrational, T' is dense in $\mathbb{R}$ by Dirichlet's theorem [2, Theorem 1.23]. $\square$

Lemma 31. *Let $p, q \in \mathbb{P}$, $p \neq q$. The sets*

$$S_p = \{c \in \mathbb{Q} \setminus \{0\} : p \nmid c\}$$

and

$$S_{pq} = \{c \in \mathbb{Q} \setminus \{0\} : p, q \nmid c\}$$

are dense in $\mathbb{Q}$.

Proof. For $r, s \in \mathbb{P} \setminus \{p, q\}$, $r \neq s$, let T be as in (40). Since $T \subset S_{pq} \subset S_p$, the claim follows from Lemma 30. $\square$

Theorem 32. *Let $p, q \in \mathbb{P}$, $p \neq q$, and $a \in \mathbb{Q}$, $ap \in \mathbb{Z}$. The solution sets of the equations*

$$x'_p = ax, \quad x'_p = x'_q, \quad x''_{pq} = x''_{qp}, \quad x''_{pp} = x, \quad \text{and} \quad x''_{pp} + x''_{qq} = x$$

are dense in $\mathbb{Q}$.

Proof. Apply Theorems 3, 8, 10, 11, 12 and Lemma 31. $\square$

In fact, we can say more about the second and third equations.

Theorem 33. *Let $p, q \in \mathbb{P}$, $p \neq q$. The solution sets of the equations*

$$x'_p = x'_q = 0, \quad x''_{pq} = x''_{qp} = 0, \quad x''_{pp} = 0, \quad \textit{and} \quad x''_{pp} + x''_{qq} = 0$$

are dense in $\mathbb{Q}$.

Proof. The set S_{pq}, which is dense by Lemma 31, is contained in the solution set of each equation. □

This theorem raises the following

Conjecture 34. Let $p, q \in \mathbb{P}$, $p \neq q$, and $a \in \mathbb{Q}$. The solution sets of the equations

$$x''_{pq} = x''_{qp} = a, \quad x''_{pp} = a, \quad \textit{and} \quad x''_{pp} + x''_{qq} = a$$

are dense in $\mathbb{Q}$.

7 Concluding remarks

Ufnarovski and Åhlander [6] studied certain arithmetic differential equations. For example, they proved [6, Corollary 3] that if the equation

$$x' = a \quad (x \in \mathbb{Z}_+,\, a \in \mathbb{Z}_+ \setminus \{1\})$$

has a solution, then it has infinitely many solutions. The equation $x'_p = a$ ($x \in \mathbb{Q}$, $a \in \mathbb{Q} \setminus \{0\}$), in contrast to that, has always only finitely many solutions, as we saw in the proof of Theorem 1. For another example, they showed [6, Theorem 6] that

$$x' = x \quad (x \in \mathbb{Z}_+)$$

if and only if $x = p^p$ for some $p \in \mathbb{P}$. A special case of Theorem 3 gives a somewhat parallel result, stating that $x_p = x$ ($x \in \mathbb{Z}_+$) if and only $x = cp^p$ where $p \nmid c \in \mathbb{Z}_+$.

Kovič [4] studied certain arithmetic partial differential equations. We pursued this topic a few steps further, yet there is still much work to do with both arithmetic derivative and arithmetic partial derivatives. Connections between different derivations are of special interest. For example, the *logarithmic derivative* of $a \in \mathbb{Q} \setminus \{0\}$ defined [6, p. 13] by

$$\operatorname{ld} a = \frac{a'}{a}$$

can also be expressed as

$$\operatorname{ld} a = \sum_{p \in \mathbb{P}} \frac{a'_p}{a}.$$

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