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Fibonacci and Lucas Sedenions

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Abstract

The sedenions form a 16-dimensional non-associative and non-commutative algebra over the set of real numbers. In this paper, we introduce the Fibonacci and Lucas sedenions. We present generating functions and Binet formulas for the Fibonacci and Lucas sedenions, and derive adaptations for some well-known identities of Fibonacci and Lucas numbers.

1 Introduction

Sedenions appear in many areas of science, such as electromagnetic theory and linear gravity. Sedenion algebra, which is usually denoted by $\mathbb{S}$, is a 16-dimensional Cayley–Dickson algebra.

Because of their zero divisors, sedenions do not form a composition algebra or a division algebra. They are hyper-complex numbers, similar to quaternions and octonions. Sedenion algebra is a non-associative, non-commutative, and non-alternative but power-associative Cayley–Dickson algebra over the reals. Throughout this paper, we take the basis elements of $\mathbb{S}$ as $\{e_0, e_1, \ldots, e_{15}\}$, where e_0 is the unit element and $e_1, \ldots, e_{15}$ are imaginaries. A sedenion S can be written as

$$S = \sum_{i=0}^{15} a_i e_i$$

where $a_0, a_1, \cdots, a_{15}$ are reals.

Imaeda and Imaeda [8] defined a sedenion by

$$S = (O_1; O_2) \in \mathbb{S}, \qquad O_1, O_2 \in \mathbb{O}$$

where $\mathbb{O}$ is the octonion algebra over the reals. As a sedenion is an ordered pair of two octonions, the conjugate of a sedenion $S = (O_1; O_2)$ is defined by $\overline{S} = (O_1; -O_2)$. Under the Cayley–Dickson process, the product of two sedenions $S_1 = (O_1; O_2)$ and $S_2 = (O_3; O_4)$ is

$$S_1 S_2 = (O_1 O_3 + \rho \overline{O_4} O_2; O_2 \overline{O_3} + O_4 O_1).$$

After choosing the field parameter $\rho = -1$ and the generator e_8, Imaeda and Imaeda examined the sedenions. By setting $i \equiv e_i$, where $i = 0, 1, \cdots, 15$, Cawagas [3] constructed the following multiplication table for the basis of $\mathbb{S}$.

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	−0	3	−2	5	−4	−7	6	9	−8	−11	10	−13	12	15	−14
2	2	−3	−0	1	6	7	−4	−5	10	11	−8	−9	−14	−15	12	13
3	3	2	−1	−0	7	−6	5	−4	11	−10	9	−8	−15	14	−13	12
4	4	−5	−6	−7	−0	1	2	3	12	13	14	15	−8	−9	−10	−11
5	5	4	−7	6	−1	−0	−3	2	13	−12	15	−14	9	−8	11	−10
6	6	7	4	−5	−2	3	−0	−1	14	−15	−12	13	10	−11	−8	9
7	7	−6	5	4	−3	−2	1	−0	15	14	−13	−12	11	10	−9	−8
8	8	−9	−10	−11	−12	−13	−14	−15	−0	1	2	3	4	5	6	7
9	9	8	−11	10	−13	12	15	−14	−1	−0	−3	2	−5	4	7	−6
10	10	11	8	−9	−14	−15	12	13	−2	3	−0	−1	−6	−7	4	5
11	11	−10	9	8	−15	14	−13	12	−3	−2	1	−0	−7	6	−5	4
12	12	13	14	15	8	−9	−10	−11	−4	5	6	7	−0	−1	−2	−3
13	13	−12	15	−14	9	8	11	−10	−5	−4	7	−6	1	−0	3	−2
14	14	−15	−12	13	10	−11	8	9	−6	−7	−4	5	2	−3	−0	1
15	15	14	−13	−12	11	10	−9	8	−7	6	−5	−4	3	2	−1	−0

Table 1: Multiplication table for the basis of $\mathbb{S}$

Cariow and Cariowa [2] derived an algorithm for the fast multiplication of two sedenions.

The well-known integer sequences of Fibonacci and Lucas numbers are constructed with the same recurrence relation but different initial conditions. Namely, for $n \geq 2$, Fibonacci numbers satisfy the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, \quad F_1 = 1$$

whereas Lucas numbers satisfy the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2, \quad L_1 = 1.$$

The generating functions for the Fibonacci sequence $\big(F_n\big)_{n\geq 0}$ and Lucas sequence $\big(L_n\big)_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \quad \text{and} \quad \sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2}.$$

The Binet formulas for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n \tag{1}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation of $x^2 - x - 1 = 0$. The positive root α is known as the golden ratio (see [11] for details).

Horadam [6] defined Fibonacci and Lucas quaternions as

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$$

and

$$K_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3$$

respectively, where F_n is the nth classical Fibonacci number and L_n is the nth classical Lucas number. He also defined generalized Fibonacci quaternions by

$$P_n = H_n + H_{n+1}e_1 + H_{n+2}e_2 + H_{n+3}e_3$$

where H_n is the nth generalized Fibonacci number defined by the recursive relation $H_1 = p$, $H_2 = p + q$, $H_n = H_{n-1} + H_{n-2}$ (p and q are arbitrary integers). Iyer [9] described various properties of the Fibonacci quaternions and generalized Fibonacci quaternions. Swammy [12] also studied these two types of quaternions, and obtained some relations between Fibonacci and Lucas numbers. Harman [5] defined complex Fibonacci numbers by the following two-dimensional recurrence relation

$$\begin{aligned} G(n+2,m) &= G(n+1,m) + G(n,m), \\ G(n,m+2) &= G(n,m+1) + G(n,m) \end{aligned}$$

where $G(0,0) = 0$, $G(1,0) = 1$, $G(0,1) = i$, and $G(1,1) = 1+i$. By extending this idea, Horadam [7] defined the recurrence relations

$$\begin{aligned} G(h+2,l,m,n) &= G(h+1,l,m,n) + G(h,l,m,n), \\ G(h,l+2,m,n) &= G(h,l+1,m,n) + G(h,l,m,n), \\ G(h,l,m+2,n) &= G(h,l,m+1,n) + G(h,l,m,n), \\ G(h,l,m,n+2) &= G(h,l,m,n+1) + G(h,l,m,n) \end{aligned}$$

with initial conditions

$$\begin{aligned}
G(0,0,0,0) &= 0, G(1,0,0,0) = 1, G(0,1,0,0) = e_1, G(0,0,1,0) = e_2, G(0,0,0,1) = e_3, \\
G(1,1,0,0) &= 1 + e_1, \ldots, G(0,0,1,1) = e_2 + e_3, \\
G(1,1,1,0) &= 1 + e_1 + e_2, \ldots, G(0,1,1,1) = e_1 + e_2 + e_3, \\
G(1,1,1,1) &= 1 + e_1 + e_2 + e_3
\end{aligned}$$

and examined quaternion recurrence relations. Halici [4] investigated some properties of the Fibonacci and Lucas quaternions, obtaining generating functions, Binet formulas, and certain identities.

Kecilioglu and Akkus [10] introduced the Fibonacci and Lucas octonions. These hypercomplex numbers are defined as

$$Q_n = \sum_{s=0}^{7} F_{n+s} e_s$$

and

$$T_n = \sum_{s=0}^{7} L_{n+s} e_s$$

respectively, where F_n and L_n are the nth classical Fibonacci and Lucas numbers. In another study [1], they defined split Fibonacci and Lucas octonions in a similar manner.

In this paper, following Horadam, Kecilioglu, and Akkus, we define the Fibonacci and Lucas sedenions over the sedenion algebra $\mathbb{S}$. The nth Fibonacci sedenion is

$$\widehat{F}_n = \sum_{s=0}^{15} F_{n+s} e_s \tag{2}$$

and the nth Lucas sedenion is

$$\widehat{L}_n = \sum_{s=0}^{15} L_{n+s} e_s. \tag{3}$$

Using the identities $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$, we immediately have that the generalized Fibonacci and Lucas sedenions with negative indexes are

$$\widehat{F}_{-n} = \sum_{s=0}^{15} (-1)^{n+s+1} F_{n-s} e_s$$

and

$$\widehat{L}_{-n} = \sum_{s=0}^{15} (-1)^{n+s} L_{n-s} e_s.$$

For all integers n, we can easily see that

$$\widehat{F}_n = \widehat{F}_{n-1} + \widehat{F}_{n-2} \quad \text{and} \quad \widehat{L}_n = \widehat{L}_{n-1} + \widehat{L}_{n-2}. \tag{4}$$

Additionally, for any integer n, we have

$$\widehat{L}_n = \widehat{F}_{n-1} + \widehat{F}_{n+1}.$$

2 Generating functions and Binet formulas for the Fibonacci and Lucas sedenions

Generating functions for the Fibonacci and Lucas sedenions are given in the next theorem.

Theorem 1. *The generating functions for the Fibonacci and Lucas sedenions are*

$$\sum_{i=0}^{\infty} \widehat{F}_i x^i = \frac{\widehat{F}_0 + \widehat{F}_{-1} x}{1 - x - x^2} \tag{5}$$

and

$$\sum_{i=0}^{\infty} \widehat{L}_i x^i = \frac{\widehat{L}_0 + \widehat{L}_{-1} x}{1 - x - x^2} \tag{6}$$

respectively.

Proof. Define $f(x) = \sum_{i=0}^{\infty} \widehat{F}_i x^i$. Expanding the first two terms of $f(x)$, we have

$$f(x) = \widehat{F}_0 + \widehat{F}_1 x + \sum_{i=2}^{\infty} \widehat{F}_i x^i. \tag{7}$$

Multiplying both sides of this equation by $-x$ and $-x^2$, we obtain

$$-x f(x) = -\widehat{F}_0 x - \sum_{i=2}^{\infty} \widehat{F}_{i-1} x^i \tag{8}$$

and

$$-x^2 f(x) = -\sum_{i=2}^{\infty} \widehat{F}_{i-2} x^i. \tag{9}$$

Adding Eqs. (7), (8) and (9) with consideration of the first equation in Eq. (4) gives

$$f(x) = \frac{\widehat{F}_0 + (\widehat{F}_1 - \widehat{F}_0) x}{1 - x - x^2}.$$

Using the identity $\widehat{F}_{-1} = \widehat{F}_1 - \widehat{F}_0$, we have Eq. (5). Similarly, we can obtain Eq. (6). □

The next theorem gives the Binet formulas for the Fibonacci and Lucas sedenions.

Theorem 2. *For any integer n, the nth Fibonacci sedenion is*

$$\widehat{F}_n = \frac{\widehat{\alpha} \alpha^n - \widehat{\beta} \beta^n}{\alpha - \beta} \tag{10}$$

and the nth Lucas sedenion is

$$\widehat{L}_n = \widehat{\alpha} \alpha^n + \widehat{\beta} \beta^n \tag{11}$$

where $\widehat{\alpha} = \sum_{s=0}^{15} \alpha^s e_s$ and $\widehat{\beta} = \sum_{s=0}^{15} \beta^s e_s$.

Proof. For the first equation, we have

$$\alpha\widehat{F}_n+\widehat{F}_{n-1}=\sum_{s=0}^{15}\left(\alpha F_{n+s}+F_{n+s-1}\right)e_s.$$

From the identity $\alpha F_n+F_{n-1}=\alpha^n$, we obtain

$$\alpha\widehat{F}_n+\widehat{F}_{n-1}=\widehat{\alpha}\alpha^n. \tag{12}$$

Similarly, we have

$$\beta\widehat{F}_n+\widehat{F}_{n-1}=\widehat{\beta}\beta^n. \tag{13}$$

Subtracting Eq. (13) from Eq. (12) gives

$$(\alpha-\beta)\widehat{F}_n=\widehat{\alpha}\alpha^n-\widehat{\beta}\beta^n$$

from which we obtain Eq. (10). Summing Eqs. (12) and (13), we have

$$\widehat{F}_n+2\widehat{F}_{n-1}=\widehat{\alpha}\alpha^n+\widehat{\beta}\beta^n$$

and, with the help of the identity $L_n=F_n+2F_{n-1}$, we obtain Eq. (11). □

When using the Binet formulas to obtain identities for the Fibonacci and Lucas sedenions, we require $\widehat{\alpha}\widehat{\beta}$, $\widehat{\beta}\widehat{\alpha}$, $\widehat{\alpha}^2$, and $\widehat{\beta}^2$. These products are given in the next lemma.

Lemma 3. *We have*

$$\begin{aligned}\widehat{\alpha}\widehat{\beta} &= \widehat{L}_0-\sqrt{5}(\widehat{F}_0-\omega), &(14)\\ \widehat{\beta}\widehat{\alpha} &= \widehat{L}_0+\sqrt{5}(\widehat{F}_0-\omega), &(15)\\ \widehat{\alpha}^2 &= -1505175+\widehat{L}_0+\sqrt{5}(-673134+\widehat{F}_0), &(16)\end{aligned}$$

and

$$\widehat{\beta}^2=-1505175+\widehat{L}_0-\sqrt{5}(-673134+\widehat{F}_0) \tag{17}$$

where

$$\omega=94e_9+94e_{10}+188e_{11}+282e_{12}-188e_{13}+94e_{14}+893e_{15}.$$

Proof. From the definitions of $\widehat{\alpha}$ and $\widehat{\beta}$, and using Table 1, we have

$$\begin{aligned}\widehat{\alpha}\widehat{\beta} &= \left(\sum_{n=0}^{15}\alpha^n e_n\right)\left(\sum_{n=0}^{15}\beta^n e_n\right)\\ &= 2+e_1+3e_2+\cdots+1364e_{15}-\sqrt{5}\Big(e_1+e_2+2e_3+\cdots+21e_8\\ &\qquad -60e_9-39e_{10}-99e_{11}-138e_{12}+421e_{13}+283e_{14}-283e_{15}\Big)\end{aligned}$$

where the final equation gives Eq. (14). The others can be computed similarly. □

This lemma gives us the following useful identity:

$$\widehat{\alpha}\widehat{\beta}+\widehat{\beta}\widehat{\alpha}=2\widehat{L}_0. \tag{18}$$

3 Some identities for the Fibonacci and Lucas sedenions

There are three well-known identities for Fibonacci and Lucas numbers, namely, Catalan's, Cassini's, and d'Ocagne's identities. The proofs of these identities are based on Binet formulas. We can obtain these types of identities for Fibonacci and Lucas sedenions using the Binet formulas derived above. Catalan's identities for Fibonacci and Lucas sedenions are given in the next theorem.

Theorem 4. *For any integers m and n, we have*

$$\widehat{F}_{m+n}\widehat{F}_{m-n}-\widehat{F}_m^2=\big(-1\big)^m F_{-n}\big(F_n\widehat{L}_0-L_n(\widehat{F}_0-\omega)\big) \tag{19}$$

and

$$\widehat{L}_{m+n}\widehat{L}_{m-n}-\widehat{L}_m^2=-5\big(-1\big)^m F_{-n}\big(F_n\widehat{L}_0-L_n(\widehat{F}_0-\omega)\big). \tag{20}$$

Proof. From the Binet formula for Fibonacci sedenions, we have

$$\begin{aligned}\widehat{F}_{m+n}\widehat{F}_{m-n}-\widehat{F}_m^2 &= \frac{1}{5}\Big(\big(\widehat{\alpha}\alpha^{m+n}-\widehat{\beta}\beta^{m+n}\big)\big(\widehat{\alpha}\alpha^{m-n}-\widehat{\beta}\beta^{m-n}\big)\\ &\qquad -\big(\widehat{\alpha}\alpha^{m}-\widehat{\beta}\beta^{m}\big)^2\Big)\\ &= \frac{1}{5}\Big((-1)^{m-n+1}\big(\widehat{\alpha}\widehat{\beta}\alpha^{2n}+\widehat{\beta}\widehat{\alpha}\beta^{2n}\big)\\ &\qquad +\big(-1\big)^m 2\widehat{L}_0\Big).\end{aligned}$$

We require Eqs. (14) and (15). Using this equation, we obtain

$$\begin{aligned}\widehat{F}_{m+n}\widehat{F}_{m-n}-\widehat{F}_m^2 &= \frac{1}{5}\Big((-1)^{m-n+1}\big(\big(\widehat{L}_0-\sqrt{5}(\widehat{F}_0-\omega)\big)\alpha^{2n}\\ &\qquad +\big(\widehat{L}_0+\sqrt{5}(\widehat{F}_0-\omega)\big)\beta^{2n}\big)+2(-1)^m\widehat{L}_0\Big)\\ &= \frac{1}{5}\big((-1)^{m-n+1}\big(\widehat{L}_0L_{2n}-5(\widehat{F}_0-\omega)F_{2n}\big)\\ &\qquad +2(-1)^m\widehat{L}_0\big)\\ &= \frac{1}{5}(-1)^{m-n+1}\widehat{L}_0\big(L_{2n}-2(-1)^n\big)\\ &\qquad -(-1)^{m-n+1}F_{2n}(\widehat{F}_0-\omega).\end{aligned}$$

Using the identity $5F_n^2=L_{2n}-2(-1)^n$ gives

$$\widehat{F}_{m+n}\widehat{F}_{m-n}-\widehat{F}_m^2=(-1)^{m+n+1}\big(F_n^2\widehat{L}_0-F_{2n}(\widehat{F}_0-\omega)\big).$$

With the help of the identity $F_{2n}=F_nL_n$, we have Eq. (19). Similarly, we can obtain Eq. (20). □

Taking $n=1$ in this theorem and using the identity $L_n - F_n = 2F_{n-1}$, we obtain Cassini's identities for Fibonacci and Lucas sedenions.

Corollary 5. *For any integer m, we have*

$$\widehat{F}_{m+1}\widehat{F}_{m-1} - \widehat{F}_m^2 = \big(-1\big)^m \big(2\widehat{F}_{-1} + \omega\big) \tag{21}$$

and

$$\widehat{L}_{m+1}\widehat{L}_{m-1} - \widehat{L}_m^2 = -5\big(-1\big)^m \big(2\widehat{F}_{-1} + \omega\big). \tag{22}$$

The following theorem gives d'Ocagne's identities for Fibonacci and Lucas sedenions.

Theorem 6. *For any integers n and m, we have*

$$\widehat{F}_m\widehat{F}_{n+1} - \widehat{F}_{m+1}\widehat{F}_n = (-1)^n \big(F_{m-n}\widehat{L}_0 + L_{m-n}(\widehat{F}_0 - \omega)\big)$$

and

$$\widehat{L}_m\widehat{L}_{n+1} - \widehat{L}_{m+1}\widehat{L}_n = -5(-1)^n \big(F_{m-n}\widehat{L}_0 + L_{m-n}(\widehat{F}_0 - \omega)\big).$$

Proof. Using the Binet formula for the Fibonacci sedenions gives

$$\begin{aligned}
\widehat{F}_m\widehat{F}_{n+1} - \widehat{F}_{m+1}\widehat{F}_n &= \frac{1}{5}\big(\widehat{\alpha}\alpha^m - \widehat{\beta}\beta^m\big)\big(\widehat{\alpha}\alpha^{n+1} - \widehat{\beta}\beta^{n+1}\big) \\
&\quad -\frac{1}{5}\big(\widehat{\alpha}\alpha^{m+1} - \widehat{\beta}\beta^{m+1}\big)\big(\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n\big) \\
&= \frac{\sqrt{5}}{5}(-1)^n \big(\widehat{\alpha}\widehat{\beta}\alpha^{m-n} - \widehat{\beta}\widehat{\alpha}\beta^{m-n}\big).
\end{aligned}$$

We require Eqs. (14) and (15). Substituting these into the previous equation, we have

$$\begin{aligned}
\widehat{F}_m\widehat{F}_{n+1} - \widehat{F}_{m+1}\widehat{F}_n &= \frac{\sqrt{5}}{5}(-1)^n \Big(\big(\widehat{L}_0 - \sqrt{5}(\widehat{F}_0 - \omega)\big)\alpha^{m-n} \\
&\quad -\big(\widehat{L}_0 + \sqrt{5}(\widehat{F}_0 - \omega)\big)\beta^{m-n}\Big) \\
&= (-1)^n \big(F_{m-n}\widehat{L}_0 - L_{m-n}(\widehat{F}_0 - \omega)\big).
\end{aligned}$$

The second identity in the theorem, i.e., d'Ocagne's identity for the Lucas sedenions, can be proved similarly. □

After deriving these three famous identities, we present some other identities for the Fibonacci and Lucas sedenions.

Theorem 7. *For any integer n, we have*

$$\widehat{L}_n^2 - \widehat{F}_n^2 = \frac{4}{5}(-1505175 + \widehat{L}_0)L_{2n} + 4(-673134 + \widehat{F}_0)F_{2n} + \frac{12}{5}(-1)^n\widehat{L}_0 \tag{23}$$

and

$$\widehat{L}_n^2 + \widehat{F}_n^2 = \frac{6}{5}(-1505175 + \widehat{L}_0)L_{2n} + 6(-673134 + \widehat{F}_0)F_{2n} + \frac{8}{5}(-1)^n\widehat{L}_0. \tag{24}$$

Proof. Using the Binet formulas for the Fibonacci and Lucas sedenions, we obtain

$$\begin{aligned}\widehat{L}_n^2 - \widehat{F}_n^2 &= \big(\widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n\big)^2 - \frac{1}{5}\big(\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n\big)^2 \\ &= \big(\widehat{\alpha}^2\alpha^{2n} + \widehat{\beta}^2\beta^{2n} + \widehat{\alpha}\widehat{\beta}\alpha^n\beta^n + \widehat{\beta}\widehat{\alpha}\alpha^n\beta^n\big) \\ &\qquad -\frac{1}{5}\big(\widehat{\alpha}^2\alpha^{2n} + \widehat{\beta}^2\beta^{2n} - \widehat{\alpha}\widehat{\beta}\alpha^n\beta^n - \widehat{\beta}\widehat{\alpha}\alpha^n\beta^n\big).\end{aligned}$$

Substituting Eqs. (14) and (15) into the last equation, we have

$$\widehat{L}_n^2 - \widehat{F}_n^2 = \frac{4}{5}\big(\widehat{\alpha}^2\alpha^{2n} + \widehat{\beta}^2\beta^{2n}\big) + \frac{12}{5}(-1)^n\widehat{L}_0. \tag{25}$$

Using Eqs. (16) and (17), we obtain

$$\big(\widehat{\alpha}^2\alpha^{2n} + \widehat{\beta}^2\beta^{2n}\big) = (-1505175 + \widehat{L}_0)L_{2n} + 5(-673134 + \widehat{F}_0)F_{2n}. \tag{26}$$

Substituting Eq. (26) into Eq. (25) gives Eq. (23). Equation (24) can be proved similarly. □

Theorem 8. *For any integers n, r and s, we have*

$$\widehat{L}_{n+r}\widehat{F}_{n+s} - \widehat{L}_{n+s}\widehat{F}_{n+r} = 2\,(-1)^{n+r}\,\widehat{L}_0 F_{s-r}. \tag{27}$$

Proof. The Binet formulas for the Fibonacci and Lucas sedenions give

$$\begin{aligned}\widehat{L}_{n+r}\widehat{F}_{n+s} - \widehat{L}_{n+s}\widehat{F}_{n+r} &= \frac{1}{\sqrt{5}}\Big(\big(\widehat{\alpha}\alpha^{n+r} + \widehat{\beta}\beta^{n+r}\big)\big(\widehat{\alpha}\alpha^{n+s} - \widehat{\beta}\beta^{n+s}\big) \\ &\qquad -\big(\widehat{\alpha}\alpha^{n+s} + \widehat{\beta}\beta^{n+s}\big)\big(\widehat{\alpha}\alpha^{n+r} - \widehat{\beta}\beta^{n+r}\big)\Big) \\ &= \frac{1}{\sqrt{5}}\big(-\widehat{\alpha}\widehat{\beta}\alpha^{n+r}\beta^{n+s} + \widehat{\beta}\widehat{\alpha}\beta^{n+r}\alpha^{n+s} \\ &\qquad +\widehat{\alpha}\widehat{\beta}\alpha^{n+s}\beta^{n+r} - \widehat{\beta}\widehat{\alpha}\alpha^{n+r}\beta^{n+s}\big).\end{aligned}$$

Using Eqs. (14) and (15), we have

$$\begin{aligned}\widehat{L}_{n+r}\widehat{F}_{n+s} - \widehat{L}_{n+s}\widehat{F}_{n+r} &= \frac{2}{\sqrt{5}}\big(-\alpha^{n+r}\beta^{n+s}\widehat{L}_0 \\ &\qquad +\alpha^{n+s}\beta^{n+r}\widehat{L}_0\big) \\ &= \frac{2}{\sqrt{5}}\widehat{L}_0\Big(\alpha^{n+r}\beta^{n+r}\left(\alpha^{s-r} - \beta^{s-r}\right)\Big) \\ &= 2\,(-1)^{n+r}\,\widehat{L}_0 F_{s-r}.\end{aligned}$$

□

Some identities for Fibonacci and Lucas sedenions are given without proof in the next theorem.

Theorem 9. *For any integers m and n, we have*

$$\widehat{F}_{m+n} + (-1)^n \widehat{F}_{m-n} = \widehat{F}_m L_n,$$

$$\widehat{F}_m \widehat{L}_n - \widehat{L}_n \widehat{F}_m = -2(-1)^m \widehat{L}_0 F_{n-m},$$

$$\widehat{F}_m \widehat{L}_n - \widehat{L}_m \widehat{F}_n = -2(-1)^m \Big(\widehat{L}_0 F_{n-m} + (\widehat{F}_0 - \omega) L_{n-m}\Big),$$

$$\widehat{F}_n \widehat{F}_m - \widehat{F}_m \widehat{F}_n = -2(-1)^m (\widehat{F}_0 - \omega) F_{n-m},$$

and

$$\widehat{L}_n \widehat{L}_m - \widehat{L}_m \widehat{L}_n = 10(-1)^m (\widehat{F}_0 - \omega) F_{n-m}.$$

The following interesting identities for Fibonacci and Lucas sedenions come directly from their definitions.

Corollary 10. *Fibonacci and Lucas sedenions satisfy*

$$\widehat{F}_{n+r} F_{n+r} - \widehat{F}_{n-r} F_{n-r} = \widehat{F}_{2n} F_{2r},$$

$$\widehat{F}_{n+r} L_{n+r} + \widehat{F}_{n-r} L_{n-r} = \widehat{F}_{2n} L_{2r} + 2(-1)^{n+r} \widehat{F}_0,$$

$$\widehat{L}_{n+r} L_{n+r} - \widehat{L}_{n-r} L_{n-r} = 5 \widehat{F}_{2n} F_{2r},$$

and

$$\widehat{L}_{n+r} L_{n+r} + \widehat{L}_{n-r} L_{n-r} = L_{2r} \widehat{L}_{2n} + 2(-1)^{n+r} \widehat{L}_0.$$

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