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A Note on Some Recent Results for the Bernoulli Numbers of the Second Kind

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Abstract

In a recent issue of *the Bulletin of the Korean Mathematical Society*, Qi and Zhang discovered an interesting integral representation for the Bernoulli numbers of the second kind (also known as *Gregory's coefficients*, *Cauchy numbers of the first kind*, and the *reciprocal logarithmic numbers*). The same representation also appears in many other sources, either with no references to its author, or with references to various modern researchers. In this short note, we show that this representation is a rediscovery of an old result obtained in the 19th century by Ernst Schröder. We also demonstrate that the same integral representation may be readily derived by means of complex integration. Moreover, we discovered that the asymptotics of these numbers were also the subject of several rediscoveries, including very recent ones. In particular, the first-order asymptotics, which are usually (and erroneously) credited to Johan F. Steffensen, actually date back to the mid-19th century, and probably were known even earlier.

1 Rediscovery of Schröder's integral formula

In a recent article in *the Bulletin of the Korean Mathematical Society* [10], several results concerning the Bernoulli numbers of the second kind were presented.

We recall that these numbers (OEIS A002206 and A002207), which we denote below by G_n, are rational

$$G_1 = +\frac{1}{2}\,, \qquad G_2 = -\frac{1}{12}\,, \qquad G_3 = +\frac{1}{24}\,, \qquad G_4 = -\frac{19}{720}\,,$$
$$G_5 = +\frac{3}{160}\,, \qquad G_6 = -\frac{863}{60480}\,, \qquad G_7 = +\frac{275}{24192}\,, \qquad G_8 = -\frac{33953}{3628800}\,, \qquad \ldots$$

and were introduced by the Scottish mathematician and astronomer James Gregory in 1670 in the context of area's interpolation formula. Subsequently, they were rediscovered by many famous mathematicians, including Gregorio Fontana, Lorenzo Mascheroni, Pierre-Simon Laplace, Augustin-Louis Cauchy, Jacques Binet, Ernst Schröder, Oskar Schlömilch, Charles Hermite and many others. Because of numerous rediscoveries these numbers do not have a standard name, and in the literature they are also referred to as *Gregory's coefficients*, *(reciprocal) logarithmic numbers*, *Bernoulli numbers of the second kind*, normalized *generalized Bernoulli numbers* $B_n^{(n-1)}$ and normalized *Cauchy numbers of the first kind* $C_{1,n}$. Usually, these numbers are defined either via their generating function

$$\frac{u}{\ln(1+u)} = 1 + \sum_{n=1}^{\infty} G_n\, u^n, \qquad |u| < 1\,, \tag{1}$$

or explicitly

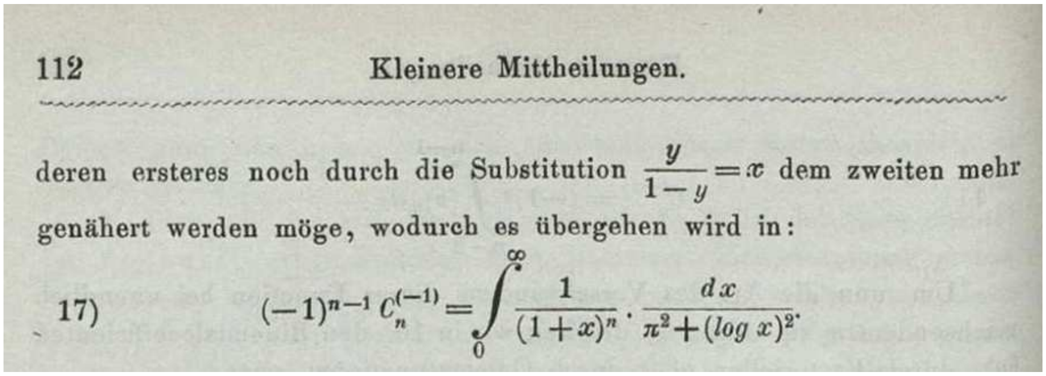
$$G_n = \frac{C_{1,n}}{n!} = \lim_{s\to n} \frac{-B_s^{(s-1)}}{(s-1)\,s!} = \frac{1}{n!}\int_0^1 x\,(x-1)\,(x-2)\cdots(x-n+1)\,dx\,, \qquad n\in\mathbb{N}\,.$$

It is well known that G_n are alternating $G_n = (-1)^{n-1}|G_n|$ and decreasing in absolute value; they behave as $\big(n\ln^2 n\big)^{-1}$ at $n\to\infty$ and may be bounded from below and from above accordingly to formulas (55)–(56) from [3]. For more information about these important numbers, see [3, pp. 410–415], [2, p. 379], and the literature given therein (nearly 50 references).

Now, the first main result of [10, p. 987] is Theorem 1.[1] It states: *the Bernoulli numbers of the second kind may be represented as follows*

$$G_n = (-1)^{n+1}\int_1^{\infty} \frac{dt}{\big(\ln^2(t-1)+\pi^2\big)\,t^n}\,, \qquad n\in\mathbb{N}\,. \tag{2}$$

[1] Our G_n are exactly b_n from [10] and $\frac{c_{n,1}^{(1)}}{n!}$ from [4, Sect. 5]. Despite a venerable history, these numbers still lack a standard notation and various authors may use different notation for them.


112 Kleinere Mittheilungen.

deren ersteres noch durch die Substitution $\frac{y}{1-y}=x$ dem zweiten mehr genähert werden möge, wodurch es übergehen wird in:

$$17) \qquad (-1)^{n-1}\, C_n^{(-1)} = \int\limits_0^\infty \frac{1}{(1+x)^n}\cdot\frac{dx}{\pi^2+(\log x)^2}.$$

Figure 1: A fragment of p. 112 from Schröder's paper [11]. Schröder's $C_n^{(-1)}$ are exactly our G_n.

The same representation appears in a slightly different form[2]

$$G_n \,=\, (-1)^{n+1}\int\limits_0^\infty \frac{du}{\big(\ln^2 u+\pi^2\big)\,(u+1)^n}\,,\qquad n\in\mathbb{N}\,, \tag{3}$$

in [5, pp. 473–474] and [4, Sect. 5], and is called *Knessl's representation* and *the Qi integral representation* respectively. Furthermore, various internet sources provide the same (or equivalent) formula, either with no references to its author or with references to different modern writers and/or their papers. However, the integral representation in question is not novel and is not due to Knessl nor to Qi and Zhang; in fact, this representation is a rediscovery of an old result. In a little-known paper of the German mathematician Ernst Schröder [11], written in 1879, one may easily find exactly the same integral representation on p. 112; see Fig. 1. Moreover, since this result is not difficult to obtain, it is possible that the same integral representation was obtained even earlier.

2 Simple derivation of Schröder's integral formula by means of the complex integration

Schröder's integral formula [11, p. 112] may, of course, be derived in various ways. Below, we propose a simple derivation of this formula based on the method of contour integration.

If we set $u=-z-1$, then equality (1) may be written as

$$\frac{z+1}{\ln z-\pi i} = -1+\sum_{n=1}^\infty \big|G_n\big|\,(z+1)^n,\qquad |z+1|<1\,.$$

Now considering the following line integral along a contour C (see Fig. 2), where $n\in\mathbb{N}$, and

[2] Put $t=1+u$.

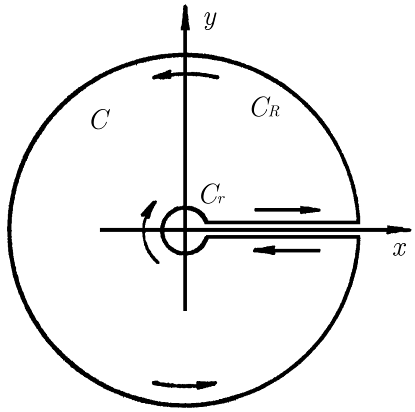


Figure 2: Integration contour C (r and R are radii of the small and big circles respectively, where $r \ll 1$ and $R \gg 1$).

then letting $R \to \infty$, $r \to 0$, we have by the residue theorem

$$\oint_C \frac{dz}{(1+z)^n\,(\ln z - \pi i)} = \int_r^R \ldots\, dz + \int_{C_R} \ldots\, dz + \int_R^r \ldots\, dz + \int_{C_r} \ldots\, dz \overset{\substack{R\to\infty\\ r\to 0}}{=}$$

$$= \int_0^\infty \left\{ \frac{1}{\ln x - \pi i} - \frac{1}{\ln x + \pi i} \right\} \cdot \frac{dx}{(1+x)^n} = 2\pi i \int_0^\infty \frac{1}{(1+x)^n} \cdot \frac{dx}{\ln^2 x + \pi^2} =$$

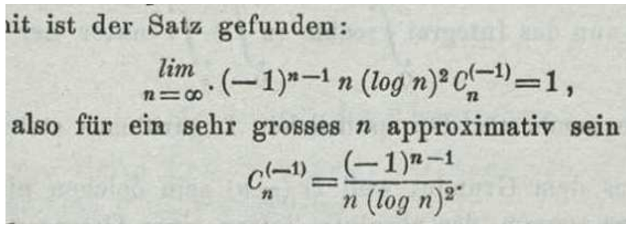
$$= 2\pi i \operatorname*{res}_{z=-1} \frac{1}{(1+z)^n\,(\ln z - \pi i)} = \frac{2\pi i}{n!} \cdot \frac{d^n}{dz^n} \frac{z+1}{\ln z - \pi i}\bigg|_{z=-1} = 2\pi i\,\big|G_n\big|\,,$$

since

$$\left| \int_{C_R} \frac{dz}{(1+z)^n\,(\ln z - \pi i)} \right| = O\left(\frac{1}{R^{n-1}\ln R}\right) = o(1)\,, \qquad R \to \infty\,, \qquad n \geq 1\,,$$

$$\left| \int_{C_r} \frac{dz}{(1+z)^n\,(\ln z - \pi i)} \right| = O\left(\frac{r}{\ln r}\right) = o(1)\,, \qquad r \to 0\,,$$

and because at $z = -1$ the integrand of the contour integral has a pole of the $(n+1)$th order. This completes the proof. Note that above derivations are valid only for $n \geq 1$, and so is Schröder's integral formula, which may also be regarded as one of the generalizations of G_n to the continuous values of n.



it ist der Satz gefunden:

$$\lim_{n=\infty} \cdot (-1)^{n-1}\, n\, (\log n)^2\, C_n^{(-1)} = 1,$$

also für ein sehr grosses n approximativ sein

$$C_n^{(-1)} = \frac{(-1)^{n-1}}{n\,(\log n)^2}.$$

Figure 3: A fragment of p. 115 from Schröder's paper [11].

3 Several remarks on the asymptotics for the Bernoulli numbers of the second kind

The first-order asymptotics $|G_n| \sim \left(n \ln^2 n\right)^{-1}$ at $n \to \infty$ are usually credited to Johan F. Steffensen [12, pp. 2–4], [13, pp. 106–107], [9, p. 29], [7, p. 14, Eq. (14)], [8], who found it in 1924.[3] However, in our recent work [3, p. 415] we noted that exactly the same result appears in Schröder's work written 45 years earlier, see Fig. 3, and the order of the magnitude of the closely related numbers is contained in a work of Jacques Binet dating back to 1839 [1].[4] In 1957 Davis [7, p. 14, Eq. (14)] improved this first-order approximation slightly by showing that $|G_n| \sim \Gamma(1+\xi)\left(n \ln^2 n + n\pi^2\right)^{-1}$ at $n \to \infty$ for some $\xi \in [0,1]$, without noticing that 7 years earlier S. C. Van Veen had already obtained the complete asymptotics for them [14, p. 336], [9, p. 29]. The identical complete asymptotics were recently rediscovered in a slightly different form by Charles Knessl [5, p. 473], and later by Gergő Nemes [8]. An alternative demonstration of the same result was also presented by the author [3, p. 414].

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[3] The same first-order asymptotics also appear in [6, p. 294], but without the source of the formula.

[4] By the "closely related numbers" we mean the so-called *Cauchy numbers of the second kind* (OEIS A002657 and A002790), and numbers $I'(k)$, see [3, pp. 410–415, 428–429]. The comment going just after Eq. (93) [3, p. 429] is based on the statements from [1, pp. 231, 339]. The Cauchy numbers of the second kind $C_{2,n}$ and Gregory's coefficients G_n are related to each other via the relationship $nC_{2,n-1} - C_{2,n} = n!\,|G_n|$, see [3, p. 430].

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