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Long and Short Sums of a Twisted Divisor Function

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Abstract

Let $q > 2$ be a prime number and define $\lambda_q := \left(\frac{\tau}{q}\right)$ where $\tau(n)$ is the number of divisors of n and $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol. When $\tau(n)$ is a quadratic residue modulo q, then the convolution $(\lambda_q \star \mathbf{1})(n)$ could be close to the number of divisors of n. The aim of this work is to compare the mean value of the function $\lambda_q \star \mathbf{1}$ to the well known average order of τ. A bound for short sums in the case $q = 5$ is also given, using profound results from the theory of integer points close to certain smooth curves.

1 Introduction and main result

If $\Omega(n)$ stands for the number of total prime factors of n and $\lambda = (-1)^{\Omega}$ is the Liouville function, then

$$L(s, \lambda) = \frac{\zeta(2s)}{\zeta(s)} \quad (\sigma > 1)\,.$$

This implies the convolution identity

$$\sum_{n \leqslant x} (\lambda \star \mathbf{1})(n) = \left\lfloor x^{1/2} \right\rfloor,$$

where, as usual, $F \star G$ is the Dirichlet convolution product of the arithmetic functions F and G given by

$$(F \star G)(n) := \sum_{d \mid n} F(d) G(n/d).$$

Define $\lambda_3 := \left(\frac{\tau}{3}\right)$ where $\tau(n)$ is the number of divisors of n and $\left(\frac{\cdot}{3}\right)$ is the Legendre symbol. Then from Proposition 3 below

$$L(s,\lambda_3) = \frac{\zeta(3s)}{\zeta(s)} \quad (\sigma > 1)\,,$$

implying the convolution identity

$$\sum_{n \leqslant x} \left(\lambda_3 \star \mathbf{1}\right)(n) = \left\lfloor x^{1/3} \right\rfloor .$$

Now let $q > 2$ be a prime number and define $\lambda_q := \left(\frac{\tau}{q}\right)$ where $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol. Our main aim is to investigate the sum

$$\sum_{n \leqslant x} \left(\lambda_q \star \mathbf{1}\right)(n).$$

When $\tau(n)$ is a quadratic residue modulo q, one may wonder if $\left(\lambda_q \star \mathbf{1}\right)(n)$ has a high probability to be equal to the number of divisors of n. Note that this function is multiplicative, and, for any prime p, $\left(\lambda_q \star \mathbf{1}\right)(p) = 1 + \left(\frac{2}{q}\right)$. Consequently, when 2 is a quadratic residue modulo q, then $\left(\lambda_q \star \mathbf{1}\right)(n) = \tau(n)$ for all squarefree numbers n. On the other hand, when 2 is a quadratic nonresidue modulo q, then $\left(\lambda_q \star \mathbf{1}\right)(n) = 0$ unless $n = 1$ or n is squarefull, so that $\sum_{n \leqslant x} \left(\lambda_q \star \mathbf{1}\right)(n) \ll x^{1/2}$ in this case. It then could be interesting to compare this sum to the average order of the function τ, i.e.,

$$\sum_{n \leqslant x} \tau(n) = x\left(\log x + 2\gamma - 1\right) + O\left(x^{\theta+\varepsilon}\right), \tag{1}$$

where

$$\tfrac{1}{4} \leqslant \theta \leqslant \tfrac{131}{416}, \tag{2}$$

the left-hand side being established by Hardy [5], the right-hand side being the best estimate to date due to Huxley [6].

To state our first main result, some specific notation are needed. For any prime $q > 3$, let c_q, respectively d_q, be least positive integer $m \in \{1, \dots, q-2\}$ for which $\left(\frac{m}{q}\right) \neq \left(\frac{m+1}{q}\right)$, respectively $\left(\frac{m}{q}\right) \neq -\left(\frac{m+1}{q}\right)$. Note that c_q and d_q are well-defined, since it is known from [3, p. 75-76] that the number of m for which $\left(\frac{m}{q}\right) = \left(\frac{m+1}{q}\right)$ and $\left(\frac{m}{q}\right) = -\left(\frac{m+1}{q}\right)$ are respectively $\frac{1}{2}(q-3)$ and $\frac{1}{2}(q-1)$. Hence there is at least $\frac{1}{2}(q-3)$ integers m for which $\left(\frac{m}{q}\right) \neq \pm\left(\frac{m+1}{q}\right)$. For convenience, set $d_3 = 3$.

As usual in number theory, we adopt Riemann's notation $s = \sigma + it \in \mathbb{C}$ and ζ is the Riemann zeta function, and define the Euler products

$$P_q(s) := \prod_p \left(1 + \sum_{m=c_q}^{q-1} \left(\left(\frac{m+1}{q}\right) - \left(\frac{m}{q}\right)\right) \frac{1}{p^{ms}}\right) \quad \left(\sigma > \tfrac{1}{c_q}\right),$$

and

$$R_q(s) := \prod_p \left(1 + \sum_{m=3}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right)\right) \frac{1}{p^{ms}}\right) \quad \left(\sigma > \tfrac{1}{3}\right).$$

Theorem 1. *Let $q > 3$ be a prime number.*

(a) If $q \equiv \pm 1 \pmod 8$

$$\sum_{n \leqslant x} (\lambda_q \star \mathbf{1})(n) = x\zeta(q)P_q(1)\left(\log x + 2\gamma - 1 + q\frac{\zeta'}{\zeta}(q) + \frac{P_q'}{P_q}(1)\right) + O_{q,\varepsilon}\left(x^{\max(1/c_q,\theta)+\varepsilon}\right),$$

where θ is defined in (1) and (2).

(b) If $q \equiv \pm 11 \pmod{24}$

$$\sum_{n \leqslant x} (\lambda_q \star \mathbf{1})(n) = x^{1/2}\zeta\left(\tfrac{q}{2}\right) R_q\left(\tfrac{1}{2}\right) + O_{q,\varepsilon}\left(x^{1/3+\varepsilon}\right).$$

(c) If $q \equiv \pm 5 \pmod{24}$, there exists $c > 0$ such that

$$\sum_{n \leqslant x} (\lambda_q \star \mathbf{1})(n) \ll_q x^{1/2} e^{-c\left(\log x^{1/4}\right)^{3/5}\left(\log\log x^{1/4}\right)^{-1/5}}.$$

Furthermore, if the Riemann hypothesis is true, then for x sufficiently large

$$\sum_{n \leqslant x} (\lambda_q \star \mathbf{1})(n) \ll_{q,\varepsilon} x^{1/4} e^{\left(\log \sqrt{x}\right)^{1/2}(\log\log\sqrt{x})^{5/2+\varepsilon}}.$$

Example 2.

$$\sum_{n \leqslant x} (\lambda_7 \star \mathbf{1})(n) \doteq 0.454\, x\left(\log x + 2\gamma + 0.784\right) + O_\varepsilon\left(x^{1/2+\varepsilon}\right).$$

$$\sum_{n \leqslant x} (\lambda_{23} \star \mathbf{1})(n) \doteq 0.899\, x\left(\log x + 2\gamma - 0.678\right) + O_\varepsilon\left(x^{131/416+\varepsilon}\right).$$

$$\sum_{n \leqslant x} (\lambda_{13} \star \mathbf{1})(n) \doteq 1.969\, x^{1/2} + O_\varepsilon\left(x^{1/3+\varepsilon}\right).$$

$$\sum_{n \leqslant x} (\lambda_5 \star \mathbf{1})(n) \ll x^{1/2} e^{-c\left(\log x^{1/4}\right)^{3/5}\left(\log\log x^{1/4}\right)^{-1/5}}.$$

2 Notation

In what follows, $x \geqslant e^4$ is a large real number, $\varepsilon \in \left(0, \frac{1}{4}\right)$ is a small real number which does not need to be the same at each occurrence, q always denotes an odd prime number, $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol and

$$\lambda_q := \left(\frac{\tau}{q}\right),$$

where $\tau(n) := \sum_{d|n} 1$. Also, $\mathbf{1}$ is the constant arithmetic function equal to 1.

For any arithmetic functions F and G, $L(s, F)$ is the Dirichlet series of F, and F^{-1} is the Dirichlet convolution inverse of F. If $r \in \mathbb{Z}_{\geqslant 2}$, then

$$a_r(n) := \begin{cases} 1, & \text{if } n = m^r; \\ 0, & \text{otherwise.} \end{cases}$$

For some $c > 0$, set

$$\delta_c(x) := e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} \quad \text{and} \quad \omega(x) := e^{(\log x)^{1/2}(\log\log x)^{5/2+\varepsilon}}.$$

Finally, let $M(x)$ and $L(x)$ be respectively the Mertens function and the summatory function of the Liouville function, i.e.

$$M(x) := \sum_{n \leqslant x} \mu(n) \quad \text{and} \quad L(x) := \sum_{n \leqslant x} \lambda(n).$$

3 The Dirichlet series of λ_q

Proposition 3. *Let $q \geqslant 3$ be a prime number. For any $s \in \mathbb{C}$ such that $\sigma > 1$*

- *If $q \equiv \pm 1 \pmod 8$*

$$L(s, \lambda_q) = \zeta(qs)\zeta(s) \prod_p \left(1 + \sum_{m=c_q}^{q-1} \left(\left(\frac{m+1}{q}\right) - \left(\frac{m}{q}\right)\right) \frac{1}{p^{ms}}\right).$$

- *If $q \equiv \pm 3 \pmod 8$*

$$L(s, \lambda_q) = \frac{\zeta(qs)\zeta(2s)}{\zeta(s)} \prod_p \left(1 + \sum_{m=d_q}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right)\right) \frac{1}{p^{ms}}\right).$$

Proof. Set $\chi_q := \left(\frac{\cdot}{q}\right)$ for convenience. From [8, Lemma 2.1], we have

$$\begin{aligned} L(s,\lambda_q) &= \prod_p \left(1+\sum_{\alpha=1}^{\infty}\frac{\chi_q(\alpha+1)}{p^{s\alpha}}\right) = \prod_p \left(1+p^s\sum_{\alpha=2}^{\infty}\frac{\chi_q(\alpha)}{p^{s\alpha}}\right) \\ &= \prod_p \left(1+p^s\left(\left(1-\frac{1}{p^{qs}}\right)^{-1}\sum_{m=1}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{ms}}-p^{-s}\right)\right) \\ &= \prod_p \left(\left(1-\frac{1}{p^{qs}}\right)^{-1}\sum_{m=1}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{(m-1)s}}\right) \\ &= \zeta(qs)\prod_p\left(1+\sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{(m-1)s}}\right). \end{aligned}$$

If $q \equiv \pm 1 \pmod 8$, then $\left(\frac{2}{q}\right) = 1$ and

$$L(s,\lambda_q) = \zeta(qs)\zeta(s)\prod_p\left(1-\frac{1}{p^s}+\left(1-\frac{1}{p^s}\right)\sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{(m-1)s}}\right),$$

where

$$\begin{aligned} \left(1-\frac{1}{p^s}\right)\sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{(m-1)s}} &= \sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\left(\frac{1}{p^{(m-1)s}}-\frac{1}{p^{ms}}\right) \\ &= \sum_{m=1}^{q-2}\left(\frac{m+1}{q}\right)\frac{1}{p^{ms}}-\sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{ms}} \\ &= \left(\frac{2}{q}\right)\frac{1}{p^s}+\sum_{m=2}^{q-1}\left(\left(\frac{m+1}{q}\right)-\left(\frac{m}{q}\right)\right)\frac{1}{p^{ms}}-\left(\frac{q}{q}\right)\frac{1}{p^{(q-1)s}} \\ &= \sum_{m=2}^{q-1}\left(\left(\frac{m+1}{q}\right)-\left(\frac{m}{q}\right)\right)\frac{1}{p^{ms}}+\frac{1}{p^s}. \end{aligned}$$

Similarly, if $q \equiv \pm 3 \pmod 8$, then $\left(\frac{2}{q}\right) = -1$ and

$$L(s,\lambda_q) = \frac{\zeta(qs)\zeta(2s)}{\zeta(s)}\prod_p\left(1+\frac{1}{p^s}+\left(1+\frac{1}{p^s}\right)\sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{(m-1)s}}\right),$$

where

$$\begin{aligned}
\left(1+\frac{1}{p^s}\right)\sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{(m-1)s}} &= \sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\left(\frac{1}{p^{(m-1)s}}+\frac{1}{p^{ms}}\right)\\
&= \sum_{m=1}^{q-2}\left(\frac{m+1}{q}\right)\frac{1}{p^{ms}}+\sum_{m=2}^{q-1}\left(\frac{m}{q}\right)\frac{1}{p^{ms}}\\
&= \left(\frac{2}{q}\right)\frac{1}{p^s}+\sum_{m=2}^{q-1}\left(\left(\frac{m+1}{q}\right)+\left(\frac{m}{q}\right)\right)\frac{1}{p^{ms}}-\left(\frac{q}{q}\right)\frac{1}{p^{(q-1)s}}\\
&= \sum_{m=2}^{q-1}\left(\left(\frac{m+1}{q}\right)+\left(\frac{m}{q}\right)\right)\frac{1}{p^{ms}}-\frac{1}{p^s}.
\end{aligned}$$

We achieve the proof noting that, if $q \equiv \pm 1 \pmod{24}$, then $\left(\frac{3}{q}\right)-\left(\frac{2}{q}\right)=\left(\frac{4}{q}\right)-\left(\frac{3}{q}\right)=0$ and, similarly, if $q \equiv \pm 11 \pmod{24}$, then $\left(\frac{3}{q}\right)+\left(\frac{2}{q}\right)=0$ whereas $\left(\frac{4}{q}\right)+\left(\frac{3}{q}\right)=2$. $\square$

4 Proof of Theorem 1

4.1 The case $q \equiv \pm 1 \pmod 8$

For $\sigma > 1$, we set

$$G_q(s)=\zeta(qs)\prod_p\left(1+\sum_{m=c_q}^{q-1}\left(\left(\frac{m+1}{q}\right)-\left(\frac{m}{q}\right)\right)\frac{1}{p^{ms}}\right)=\zeta(qs)P_q(s):=\sum_{n=1}^{\infty}\frac{g_q(n)}{n^s}.$$

This Dirichlet series is absolutely convergent in the half-plane $\sigma > \frac{1}{c_q}$, so that

$$\sum_{n\leqslant x}|g_q(n)| \ll_{q,\varepsilon} x^{1/c_q+\varepsilon}.$$

By partial summation, we infer

$$\begin{aligned}
\sum_{n\leqslant x}\frac{g_q(n)}{n} &= \zeta(q)P_q(1)+O\left(x^{-1+1/c_q+\varepsilon}\right),\\
\sum_{n\leqslant x}\frac{g_q(n)}{n}\log\frac{x}{n} &= \zeta(q)P_q(1)\log x+qP_q(1)\zeta'(q)+P_q'(1)\zeta(q)+O\left(x^{-1+1/c_q+\varepsilon}\right).
\end{aligned}$$

From Proposition 3, $\lambda_q \star \mathbf{1} = g_q \star \tau$. Consequently

$$\begin{aligned}
\sum_{n \leqslant x} \left(\lambda_q \star \mathbf{1}\right)(n) &= \sum_{d \leqslant x} g_q(d) \sum_{k \leqslant x/d} \tau(k) \\
&= \sum_{d \leqslant x} g_q(d) \left(\frac{x}{d} \log \frac{x}{d} + (2\gamma - 1)\frac{x}{d} + O\left(\left(\frac{x}{d}\right)^{\theta+\varepsilon} \right) \right) \\
&= x \left(\zeta(q) P_q(1) \log x + q P_q(1) \zeta'(q) + P_q'(1)\zeta(q) + (2\gamma-1)\zeta(q)P_q(1) \right) \\
&\quad + O\left(x^{\max(1/c_q, \theta)+\varepsilon} \right),
\end{aligned}$$

where θ is defined in (1) and where we used

$$x^{-\varepsilon} \sum_{d \leqslant x} \frac{|g_q(d)|}{d^\theta} \ll \begin{cases} x^{1/c_q - \theta}, & \text{if } c_q^{-1} \geqslant \theta\,; \\ 1, & \text{otherwise.} \end{cases}$$

4.2 The case $q \equiv \pm 11 \pmod{24}$

For $\sigma > 1$, we set

$$H_q(s) = \zeta(qs) \prod_p \left(1 + \sum_{m=3}^{q-1} \left(\left(\frac{m+1}{q} \right) + \left(\frac{m}{q} \right) \right) \frac{1}{p^{ms}} \right) = \zeta(qs) R_q(s) := \sum_{n=1}^\infty \frac{h_q(n)}{n^s}.$$

Since $q > 5$, this Dirichlet series is absolutely convergent in the half-plane $\sigma > \frac{1}{3}$, so that

$$\sum_{n \leqslant x} |h_q(n)| \ll_{q,\varepsilon} x^{1/3+\varepsilon}.$$

From Proposition 3, $\lambda_q \star \mathbf{1} = h_q \star a_2$, hence

$$\begin{aligned}
\sum_{n \leqslant x} \left(\lambda_q \star \mathbf{1}\right)(n) &= \sum_{d \leqslant x} h_q(d) \left\lfloor \sqrt{\frac{x}{d}} \right\rfloor \\
&= x^{1/2} \sum_{d \leqslant x} \frac{h_q(d)}{\sqrt{d}} + O\left(x^{1/3+\varepsilon}\right) \\
&= x^{1/2} H_q\left(\tfrac{1}{2}\right) + O\left(x^{1/3+\varepsilon}\right).
\end{aligned}$$

4.3 The case $q \equiv \pm 5 \pmod{24}$

In this case, it is necessary to rewrite $L(s, \lambda_q)$ in the following shape.

Lemma 4. *Assume* $q \equiv \pm 5 \pmod{24}$. *For any* $\sigma > 1$, $L(s,\lambda_q) = \dfrac{K_q(s)}{\zeta(s)\zeta(2s)}$ *with*

$$K_q(s) := \begin{cases} \zeta(5s), & \text{if } q = 5; \\ \zeta(4s)L_q(s), & \text{if } q \equiv \pm 19, \pm 29 \pmod{120}; \\ \dfrac{\mathcal{L}_q(s)}{\zeta(4s)}, & \text{if } q \equiv \pm 43, \pm 53 \pmod{120}; \end{cases}$$

where

$$L_q(s) := \zeta(qs) \prod_p \left(1 + \frac{2\left(p^{2s}+p^s+1\right)}{p^{7s}-p^{5s}} + \frac{p^{2s}+1}{p^{2s}-1} \sum_{m=6}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right) \frac{1}{p^{ms}} \right)$$

and

$$\begin{aligned} \mathcal{L}_q(s) \quad := \quad & \zeta(qs) \prod_p \Bigg(1 - \frac{2p^{2s}-1}{\left(p^{2s}-1\right)^3\left(p^{2s}+1\right)} \\ & + \frac{p^{8s}}{\left(p^{2s}-1\right)^3\left(p^{2s}+1\right)} \sum_{m=6}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right) \frac{1}{p^{ms}} \Bigg). \end{aligned}$$

The Dirichlet series L_q *is absolutely convergent in the half-plane* $\sigma > \frac{1}{5}$, *and the Dirichlet series* $\mathcal{L}_q$ *is absolutely convergent in the half-plane* $\sigma > \frac{1}{6}$.

Proof. From Proposition 3, we immediately get

$$L(s,\lambda_5) = \frac{\zeta(5s)}{\zeta(s)\zeta(2s)}. \tag{3}$$

Now suppose $q > 5$ and $q \equiv \pm 5 \pmod{24}$. In this case, $\left(\frac{3}{q}\right) + \left(\frac{2}{q}\right) = -2$ and $\left(\frac{4}{q}\right) + \left(\frac{3}{q}\right) = 0$ so that we may write by Proposition 3

$$\begin{aligned} L\left(s,\lambda_q\right) \quad &= \quad \frac{\zeta(qs)\zeta(2s)}{\zeta(s)} \prod_p \left(1 - \frac{2}{p^{2s}} + \sum_{m=4}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right) \frac{1}{p^{ms}} \right) \\ &= \quad \frac{K_q(s)}{\zeta(s)\zeta(2s)} \end{aligned}$$

where

$$K_q(s) := \zeta(qs) \prod_p \left(1 - \frac{1}{\left(p^{2s}-1\right)^2} + \frac{p^{4s}}{\left(p^{2s}-1\right)^2} \sum_{m=4}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right) \frac{1}{p^{ms}} \right).$$

Assume $q \equiv \pm 19, \pm 29 \pmod{120}$. Then

$$\left(\frac{5}{q}\right) + \left(\frac{4}{q}\right) = \left(\frac{6}{q}\right) + \left(\frac{5}{q}\right) = 2.$$

$K_q(s)$ can therefore be written as

$$\begin{aligned}
K_q(s) &= \zeta(qs) \prod_p \left(1 + \frac{p^s + 2}{p^s \left(p^{2s} - 1\right)^2} + \frac{p^{4s}}{\left(p^{2s} - 1\right)^2} \sum_{m=6}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right)\right) \frac{1}{p^{ms}}\right) \\
&= \zeta(qs)\zeta(4s) \prod_p \left(1 + \frac{2\left(p^{2s} + p^s + 1\right)}{p^{7s} - p^{5s}} + \frac{p^{2s} + 1}{p^{2s} - 1} \sum_{m=6}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right)\right) \frac{1}{p^{ms}}\right) \\
&= \zeta(4s) L_q(s).
\end{aligned}$$

Similarly, if $q \equiv \pm 43, \pm 53 \pmod{120}$, then

$$\left(\frac{5}{q}\right) + \left(\frac{4}{q}\right) = \left(\frac{6}{q}\right) + \left(\frac{5}{q}\right) = 0.$$

Hence

$$\begin{aligned}
K_q(s) &:= \zeta(qs) \prod_p \left(1 - \frac{1}{\left(p^{2s} - 1\right)^2} + \frac{p^{4s}}{\left(p^{2s} - 1\right)^2} \sum_{m=6}^{q-1} \left(\left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right)\right) \frac{1}{p^{ms}}\right) \\
&= \frac{\mathcal{L}_q(s)}{\zeta(4s)}.
\end{aligned}$$

The proof is complete. $\square$

We now are in a position to prove Theorem 1 in the case $q \equiv \pm 5 \pmod{24}$.

Assume first that $q \equiv \pm 19, \pm 29 \pmod{120}$ and let $\ell_q(n)$ be the n-th coefficient of the Dirichlet series $L_q(s)$. From Lemma 4, $\lambda_q \star \mathbf{1} = \ell_q \star a_4 \star a_2^{-1}$ and therefore

$$\sum_{n \leqslant x} \left(\lambda_q \star \mathbf{1}\right)(n) = \sum_{d \leqslant x} \ell_q(d) \sum_{m \leqslant (x/d)^{1/4}} M\left(\frac{1}{m^2}\sqrt{\frac{x}{d}}\right) = \sum_{d \leqslant x} \ell_q(d) L\left(\sqrt{\frac{x}{d}}\right).$$

Since $L(z) \ll z\delta_c(z)$ for some $c > 0$

$$\begin{aligned}
\sum_{n \leqslant x} \left(\lambda_q \star \mathbf{1}\right)(n) &\ll x^{1/2} \sum_{d \leqslant x} \frac{|\ell_q(d)|}{\sqrt{d}} \delta_c\left(\sqrt{\frac{x}{d}}\right) \\
&\ll x^{1/2} \left(\sum_{d \leqslant \sqrt{x}} + \sum_{\sqrt{x} < d \leqslant x}\right) \frac{|\ell_q(d)|}{\sqrt{d}} \delta_c\left(\sqrt{\frac{x}{d}}\right) \\
&\ll x^{1/2} \delta_c\left(x^{1/4}\right) + x^{1/2} \sum_{d > \sqrt{x}} \frac{|\ell_q(d)|}{\sqrt{d}}.
\end{aligned}$$

The Dirichlet series $L_q(s) := \sum_{n=1}^{\infty} \ell_q(n) n^{-s}$ is absolutely convergent in the half-plane $\sigma > \frac{1}{5}$, consequently

$$\sum_{d \leqslant z} |\ell_q(d)| \ll_{q,\varepsilon} z^{1/5+\varepsilon}$$

and by partial summation

$$\sum_{d > z} \frac{|\ell_q(d)|}{\sqrt{d}} \ll_{q,\varepsilon} z^{-3/10+\varepsilon}.$$

We infer that

$$\sum_{n \leqslant x} (\lambda_q \star \mathbf{1})(n) \ll x^{1/2} \delta_c\left(x^{1/4}\right) + x^{7/20+\varepsilon} \ll x^{1/2} \delta_c\left(x^{1/4}\right).$$

Now suppose that the Riemann hypothesis is true. By [1], which is a refinement of [9], we know that $M(z) \ll_\varepsilon z^{1/2}\,\omega(z)$. The method of [9, 1] may be adapted to the function L yielding

$$L(z) \ll_\varepsilon z^{1/2}\,\omega(z)\,\log z.$$

Observe that, for any $a \geqslant 2$, $\varepsilon > 0$ and $z \geqslant e^{e^e}$

$$\log z \exp\left(\sqrt{\log z}\,(\log\log z)^a\right) \leqslant \exp\left(\sqrt{\log z}\,(\log\log z)^{a+\varepsilon}\right)$$

so that $L(z) \ll_\varepsilon z^{1/2}\,\omega(z)$ and hence

$$\sum_{n \leqslant x} (\lambda_q \star \mathbf{1})(n) \ll x^{1/4} \sum_{d \leqslant x} \frac{|\ell_q(d)|}{d^{1/4}}\,\omega\left(\sqrt{\frac{x}{d}}\right) \ll x^{1/4} \omega\left(\sqrt{x}\right) \sum_{d \leqslant x} \frac{|\ell_q(d)|}{d^{1/4}} \ll x^{1/4} \omega\left(\sqrt{x}\right)$$

completing the proof in that case. The case $q = 5$ is similar but simpler since $\lambda_5 \star \mathbf{1} = a_5 \star a_2^{-1}$ by (3).

Finally, when $q \equiv \pm 43, \pm 53 \pmod{120}$, we proceed as above. Let $\nu_q(n)$ be the n-th coefficient of the Dirichlet series $\mathcal{L}_q(s)$. Then $\lambda_q \star \mathbf{1} = \nu_q \star a_4^{-1} \star a_2^{-1}$ from Lemma 4, so that

$$\sum_{n \leqslant x} (\lambda_q \star \mathbf{1})(n) = \sum_{d \leqslant x} \nu_q(d) \sum_{m \leqslant (x/d)^{1/4}} \mu(m) M\left(\frac{1}{m^2}\sqrt{\frac{x}{d}}\right)$$

and estimating trivially yields

$$\sum_{n \leqslant x} (\lambda_q \star \mathbf{1})(n) \ll x^{1/2} \sum_{d \leqslant x} \frac{|\nu_q(d)|}{\sqrt{d}} \sum_{m \leqslant (x/d)^{1/4}} \frac{1}{m^2}\,\delta_c\left(\frac{1}{m^2}\sqrt{\frac{x}{d}}\right)$$

and we complete the proof as in the previous case. $\square$

Remark 5. Let us stress that a bound of the shape

$$\sum_{n\leqslant x}\left(\lambda_q\star\mathbf{1}\right)(n)\ll x^{1/4+\varepsilon}$$

for all x sufficiently large and small $\varepsilon>0$, is a necessary and sufficient condition for the Riemann hypothesis. Indeed, if this estimate holds, then by partial summation the series $\sum_{n=1}^{\infty}\left(\lambda_q\star\mathbf{1}\right)(n)n^{-s}$ is absolutely convergent in the half-plane $\sigma>\frac{1}{4}$. Consequently, the function $K_q(s)\zeta(2s)^{-1}$ is analytic in this half-plane. In particular, $\zeta(2s)$ does not vanish in this half-plane, implying the Riemann hypothesis, proving the necessary condition, the sufficiency being established above.

5 A short interval result for the case $q=5$

5.1 Introduction

This section deals with sums of the shape

$$\sum_{x<n\leqslant x+y}\left(\lambda_5\star\mathbf{1}\right)(n)$$

where $x^{\varepsilon}\leqslant y\leqslant x$. From Theorem 1

$$\sum_{x<n\leqslant x+y}\left(\lambda_5\star\mathbf{1}\right)(n)\ll x^{1/2}e^{-c\left(\log x^{1/4}\right)^{3/5}\left(\log\log x^{1/4}\right)^{-1/5}}$$

and if the Riemann hypothesis is true, then

$$\sum_{x<n\leqslant x+y}\left(\lambda_5\star\mathbf{1}\right)(n)\ll_{\varepsilon} x^{1/4}e^{\left(\log\sqrt{x}\right)^{1/2}\left(\log\log\sqrt{x}\right)^{5/2+\varepsilon}}.$$

The purpose is to improve significantly upon these estimates when $y=o(x)$, by using fine results belonging to the theory of integer points near a suitably chosen smooth curve. To this end, we need the following additional notation. Let $\delta\in\left(0,\frac{1}{4}\right)$, $N\in\mathbb{Z}_{\geqslant 1}$ large, $f:[N,2N]\longrightarrow\mathbb{R}$ be any map, and define $\mathcal{R}(f,N,\delta)$ to be the number of elements of the set of integers $n\in[N,2N]$ such that $\|f(n)\|<\delta$, where $\|x\|$ is the distance from x to its nearest integer. Note that the trivial bound is given by

$$\sum_{x<n\leqslant x+y}\left(\lambda_5\star\mathbf{1}\right)(n)\ll\sum_{x<n\leqslant x+y}\tau(n)\ll y\log x.$$

5.2 Tools from the theory

In what follows, $N \in \mathbb{Z}_{\geqslant 1}$ is large and $\delta \in \left(0, \frac{1}{4}\right)$. The first result is [7, Theorem 5] with $k = 5$. See also [2, Theorem 5.23 (iv)].

Lemma 6 (5th derivative test)**.** *Let $f \in C^5[N, 2N]$ such that there exist $\lambda_4 > 0$ and $\lambda_5 > 0$ satisfying $\lambda_4 = N\lambda_5$ and, for any $x \in [N, 2N]$*

$$\left|f^{(4)}(x)\right| \asymp \lambda_4 \quad \textit{and} \quad \left|f^{(5)}(x)\right| \asymp \lambda_5.$$

Then

$$\mathcal{R}(f, N, \delta) \ll N\lambda_5^{1/15} + N\delta^{1/6} + \left(\delta\lambda_4^{-1}\right)^{1/4} + 1.$$

Remark 7. The basic result of the theory is the following first derivative test (see [2, Theorem 5.6]): *Let $f \in C^1[N, 2N]$ such that there exist $\lambda_1 > 0$ such that $|f'(x)| \asymp \lambda_1$. Then*

$$\mathcal{R}(f, N, \delta) \ll N\lambda_1 + N\delta + \delta\lambda_1^{-1} + 1. \tag{4}$$

This result is essentially a consequence of the mean value theorem.

The second tool is [4, Theorem 7] with $k = 3$.

Lemma 8. *Let $s \in \mathbb{Q}^* \setminus \{\pm 2, \pm 1\}$ and $X > 0$ such that $N \leqslant X^{1/s}$. Then there exists a constant $c_3 := c_3(s) \in \left(0, \frac{1}{4}\right)$ depending only on s such that, if*

$$N^2\delta \leqslant c_3 \tag{5}$$

then

$$\mathcal{R}\left(\frac{X}{n^s}, N, \delta\right) \ll \left(XN^{3-s}\right)^{1/7} + \delta\left(XN^{59-s}\right)^{1/21}.$$

Our last result relates the short sum of $\lambda_5 \star \mathbf{1}$ to a problem of counting integer points near a smooth curve.

Lemma 9. *Let $1 \leqslant y \leqslant x$. Then*

$$\sum_{x < n \leqslant x+y} (\lambda_5 \star \mathbf{1})(n) \ll \max_{(16y^2x^{-1})^{1/5} < N \leqslant (2x)^{1/5}} \mathcal{R}\left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5 x}}\right) \log x + yx^{-1/2} + x^{-1/5}y^{2/5}.$$

Proof. Using (3), we get

$$\sum_{n \leqslant x} (\lambda_5 \star \mathbf{1})(n) = \sum_{d \leqslant \sqrt{x}} \mu(d) \left\lfloor \left(\frac{x}{d^2}\right)^{1/5} \right\rfloor$$

so that

$$\begin{aligned}
\sum_{x<n\leqslant x+y} (\lambda_5 \star \mathbf{1})(n) &= \sum_{d\leqslant\sqrt{x}} \mu(d) \left(\left\lfloor \left(\frac{x+y}{d^2}\right)^{1/5} \right\rfloor - \left\lfloor \left(\frac{x}{d^2}\right)^{1/5} \right\rfloor \right) + \sum_{\sqrt{x}<d\leqslant\sqrt{x+y}} \mu(d) \\
&\ll \sum_{d\leqslant\sqrt{x}} \left(\left\lfloor \left(\frac{x+y}{d^2}\right)^{1/5} \right\rfloor - \left\lfloor \left(\frac{x}{d^2}\right)^{1/5} \right\rfloor \right) + yx^{-1/2} \\
&\ll \sum_{d\leqslant\sqrt{x}} \sum_{x<d^2n^5\leqslant x+y} 1 + yx^{-1/2} \\
&\ll \sum_{n\leqslant(2x)^{1/5}} \sum_{\left(\frac{x}{n^5}\right)^{1/2}<d\leqslant\left(\frac{x+y}{n^5}\right)^{1/2}} 1 + yx^{-1/2} \\
&\ll \sum_{(16y^2x^{-1})^{1/5}<n\leqslant(2x)^{1/5}} \left(\left\lfloor \sqrt{\frac{x+y}{n^5}} \right\rfloor - \left\lfloor \sqrt{\frac{x}{n^5}} \right\rfloor \right) + x^{-1/5}y^{2/5} + yx^{-1/2}
\end{aligned}$$

and for any integers $N \in \left] (16y^2x^{-1})^{1/5}, (2x)^{1/5} \right]$ and $n \in [N, 2N]$

$$\sqrt{\frac{x+y}{n^5}} - \sqrt{\frac{x}{n^5}} < \frac{y}{\sqrt{N^5x}} < \frac{1}{4}$$

so that the sum does not exceed

$$\ll \max_{(16y^2x^{-1})^{1/5}<N\leqslant(2x)^{1/5}} \mathcal{R}\left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5x}}\right) \log x + x^{-1/5}y^{2/5} + yx^{-1/2}$$

as asserted. $\square$

5.3 The main result

Theorem 10. *Assume $y \leqslant c_3\, x^{11/20}$ where $c_3 := c_3\left(\frac{5}{2}\right)$ is given in (5). Then*

$$\sum_{x<n\leqslant x+y} (\lambda_5 \star \mathbf{1})(n) \ll \left(x^{1/12} + yx^{-4/9}\right) \log x.$$

Furthermore, if $y \leqslant c_3\, x^{19/36}$

$$\sum_{x<n\leqslant x+y} (\lambda_5 \star \mathbf{1})(n) \ll x^{1/12} \log x.$$

Proof. We split the first term in Lemma 9 into three parts, according to the ranges

$$\left(16y^2x^{-1}\right)^{1/5} < N \leqslant 2x^{1/10}, \quad 2x^{1/10} < N \leqslant 2x^{1/6} \quad \text{and} \quad 2x^{1/6} < N \leqslant (2x)^{1/5}.$$

In the first case, we use Lemma 6 with $\lambda_4 = (xN^{-13})^{1/2}$ and $\lambda_5 = (xN^{-15})^{1/2}$ which yields

$$\max_{(16y^2x^{-1})^{1/5} < N \leqslant 2x^{1/10}} \mathcal{R}\left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5 x}}\right) \ll x^{1/12} + x^{-1/40}y^{1/6} + x^{-3/20}y^{1/4}.$$

For the second range, we use Lemma 8 with $X = x^{1/2}$, $s = \frac{5}{2}$ and $\delta = y\,(N^5x)^{-1/2}$. Notice that the conditions $N > 2x^{1/10}$ and $y \leqslant c_3\,x^{11/20}$ ensure that $\delta < \frac{1}{4}$ and $N^2\delta \leqslant c_3$. We get

$$\max_{2x^{1/10} < N \leqslant 2x^{1/6}} \mathcal{R}\left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5 x}}\right) \ll x^{1/12} + yx^{-4/9}.$$

The last range is easily treated with (4), giving

$$\max_{2x^{1/6} < N \leqslant (2x)^{1/5}} \mathcal{R}\left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5 x}}\right) \ll x^{1/12} + yx^{-3/4}.$$

Using Lemma 9, we finally get

$$\sum_{x < n \leqslant x+y} (\lambda_5 \star \mathbf{1})\,(n) \ll \left(x^{1/12} + x^{-1/40}y^{1/6} + x^{-3/20}y^{1/4} + yx^{-4/9}\right)\log x + x^{-1/5}y^{2/5}$$

and note that $x^{-1/40}y^{1/6} + x^{-3/20}y^{1/4} + x^{-1/5}y^{2/5} \ll x^{1/12}$ as soon as $y \leqslant x^{13/20}$. This completes the proof of the first estimate, the second one being obvious. □

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