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On the Equation $\sigma(n) = n + \phi(n)$

Douglas E. Iannucci
Towers Condominiums B-16
3600 Contant
St. Thomas, VI 00802
USA
diannuc@gmail.com

Abstract

In this paper we consider the equation $\sigma(n) = n + \phi(n)$, for which $n = 2$ is the only known solution. We provide necessary conditions for the existence of any larger solutions.

1 Introduction

Let $\sigma(n)$ denote the sum of the (positive) divisors of n, and let $\phi(n)$ denote Euler's phi function (the number of positive integers $\leq n$ and relatively prime to it). We remark that $n = 2$ is a solution to the equation

$$\sigma(n) = n + \phi(n) \tag{1}$$

over natural numbers n. By Hardy and Wright [3, Thm. 324], the average value of $\sigma(n)$ is $\frac{1}{6}\pi^2 n$. By [3, Thm. 330], the average value of $n+\phi(n)$ is $(\pi^2+6)n/\pi^2$. Since $\frac{1}{6}\pi^2 \approx 1.645$ and $(\pi^2+6)/\pi^2 \approx 1.608$, it is quite natural to ask if (1) has any solutions $n > 2$. None are known, and the literature seems devoid of any mention to this specific problem. Other, similar problems have been discussed in publications such as Guy [2]; e.g., §B-38, $\phi(m) = \sigma(n)$, and §B-42, $\phi(\sigma(n)) = \sigma(\phi(n))$, $\phi(\sigma(n)) = n$, $\phi(\sigma(n)) = \phi(n)$. Some such problems have been solved and are popular topics of discussion on many internet forums; e.g., $\sigma(n) + \phi(n) = 2n$ if and only if n is a prime or $n = 1$. There are three sequences in *OEIS* (Sloane [5]) related to (1); e.g., A070159 (all n such that $\sigma(n) - n$ divides $\phi(n)$). In fact, all solutions $n > 2$ to (1), if they exist, would belong to sequence A055940, which gives composite n such that $\sigma(n) - n$ divides $\phi(n)$. For, when p is prime then $\phi(p)/(\sigma(p) - p) = p - 1$, hence 2 is the only prime solution to (1). Thus we seek those members n of A055940 where the ratio $\phi(n)/(\sigma(n)-n)) = 1$; none are known. A similar comment applies to A066679, giving n such that $\sigma(n)-n$ is divisible by $\phi(n)$: all members less than 10^{11} are given, the only one of which giving the ratio $(\sigma(n)-n)/\phi(n) = 1$ being $n = 2$. Are there any more?

We conjecture that $n = 2$ is the only solution to (1), and in this paper we obtain necessary conditions for a solution $n > 2$ to exist.

2 Preliminaries

Dividing both sides of (1) by n yields

$$\frac{\sigma(n)}{n} = 1 + \frac{\phi(n)}{n}. \tag{2}$$

The function $\sigma(n)/n$ is multiplicative. For all primes p and natural numbers a,

$$\frac{\sigma(p^a)}{p^a} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^a} \tag{3}$$

is monotonically increasing with a; hence taking the limit of the right hand side of (3) as $a \to \infty$ yields

$$\frac{\sigma(p^a)}{p^a} < \frac{p}{p-1}. \tag{4}$$

If $p < q$ are primes then $q/(q-1) \leq (p+1)/p$, thus by (3) and (4),

$$\frac{\sigma(q^b)}{q^b} < \frac{\sigma(p^a)}{p^a} \tag{5}$$

for all natural numbers a, b. As well,

$$\frac{q^b}{\phi(q^b)} = \frac{q}{q-1} < \frac{p}{p-1} = \frac{p^a}{\phi(p^a)} \tag{6}$$

for all natural numbers a, b. By (4) we have

$$\frac{\sigma(n)}{n} < \prod_{p|n} \frac{p}{p-1} = \frac{n}{\phi(n)}. \tag{7}$$

As usual we let $\alpha = \frac{1}{2}(1+\sqrt{5})$, the golden ratio. Immediately we obtain the following results.

Theorem 1. *If n is a solution to (1) then*

$$\frac{\sigma(n)}{n} < \alpha < \frac{n}{\phi(n)}.$$

Proof. Setting $x = n/\phi(n)$, by (2) and (7) we obtain the quadratic inequality

$$x^2 - x - 1 > 0,$$

which has the positive solution $x > \alpha$. Thus the right hand inequality is proved. Otherwise by (2)

$$\frac{\sigma(n)}{n} = 1 + \frac{1}{x} < 1 + \frac{1}{\alpha} = \alpha,$$

thus proving the left hand inequality. □

Theorem 2. *If $n > 2$ is a solution to (1) then n is an odd square.*

Proof. If a and b are natural numbers, then $\sigma(a)/a \leq \sigma(b)/b$ whenever $a \mid b$. Thus if $4 \mid n$,

$$\frac{\sigma(n)}{n} \geq \frac{7}{4} > \alpha,$$

whence n is not a solution to (1) by Theorem 1. Thus any even solution greater than 2 has the form $n = 2r$, where $r > 1$ is odd. Thus by (1), since $\phi(r) < r$,

$$3\sigma(r) = 2r + \phi(r) < 2r + r = 3r,$$

a contradiction. Hence n is odd. Since $\phi(n)$ is even, the right hand side of (1) is odd. Hence $\sigma(n)$ is odd. For odd natural numbers n, $\sigma(n)$ is odd if and only if n is square. □

By Theorem 2, $n = m^2$, where $m > 1$ is odd, for all solutions $n > 2$ to (1). Since $\phi(m^2) = m\phi(m)$, it follows that by (1) that

$$\sigma(m^2) = m(m + \phi(m)). \tag{8}$$

Thus we seek necessary conditions for an odd natural number $m > 1$ to satisfy (8).

The cyclotomic polynomials $\Phi_n(x)$ may be defined recursively by $\Phi_1(x) = x - 1$ and

$$x^n - 1 = \prod_{d \mid n} \Phi_d(x)$$

for all $n > 1$. Thus

$$\sigma(p^a) = \prod_{\substack{d \mid a+1 \\ d > 1}} \Phi_d(p) \tag{9}$$

for primes p and natural numbers a. Nagell [4, Thm. 94], has shown

Lemma 3. *Let p and q be distinct odd primes and let $h = e_q(p)$, where $e_q(p)$ denotes the exponent to which p belongs modulo q. Then $q \mid \Phi_n(p)$ if and only if $n = hp^\gamma$ for some $\gamma \geq 0$. If $\gamma > 0$ then $q \| \Phi_{hp^\gamma}(p)$.*

For prime p and natural number n we denote the p-valuation of n by $v_p(n)$. Thus $v_p(n) = k$ if and only if $p^k \| n$. By (9) and Lemma 3,

$$v_q(\sigma(p^a)) = \begin{cases} 0, & \text{if } h \nmid a+1; \\ v_q(a+1), & \text{if } h = 1; \\ v_q(p^h - 1) + v_q(a+1), & \text{if } h > 1,\ h \mid a+1. \end{cases} \tag{10}$$

3 Lower Bounds on $\omega(m)$

The arithmetic function ω counts the number of prime power components of a natural number n; i.e., if n has the unique prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then $\omega(n) = k$.

Theorem 4. *If m is a solution to (8) then $\omega(m) > 2$.*

Proof. That $\omega(m) > 1$ follows from (6) and Theorem 1, since otherwise

$$\frac{m}{\phi(m)} \leq \frac{3}{2} < \alpha,$$

contradicting Theorem 1.

Similarly if $\omega(m) = 2$ then $3 \mid m$, otherwise

$$\frac{m}{\phi(m)} \leq \frac{5}{4} \cdot \frac{7}{6} < \alpha.$$

Thus $m = 3^a p^b$ for some odd prime $p > 3$ and natural numbers a and b. Then $p \geq 11$ (by (5) and Theorem 1), otherwise

$$\frac{\sigma(m^2)}{m^2} \geq \frac{\sigma(3^2)}{3^2} \cdot \frac{\sigma(7^2)}{7^2} = \frac{13}{9} \cdot \frac{57}{49} > \alpha.$$

Then $p \leq 13$ (by (6) and Theorem 1), otherwise

$$\frac{m}{\phi(m)} \leq \frac{3}{2} \cdot \frac{17}{16} < \alpha.$$

Suppose $p = 11$. Thus $m = 3^a 11^b$. Thus by (8),

$$\sigma(m^2) = \sigma(3^{2a} 11^{2b}) = 3^{2a-1} 11^{2b-1} (3 \cdot 11 + \phi(3 \cdot 11)).$$

Since $33 + \phi(33) = 53$, we have $53 \mid \sigma(3^{2a})\sigma(11^{2b})$. By (10), this is impossible since $e_{53}(3) = 52$ and $e_{53}(11) = 26$.

Likewise $p = 13$ implies $63 \mid \sigma(3^{2a})\sigma(11^{2b})$ for some natural numbers a and b; thus $3 \mid \sigma(11^{2b})$, which is impossible since $e_3(11) = 2$. $\square$

Theorem 5. *If m is a solution to (8) then $\omega(m) > 3$.*

Proof. By Theorem 4 it suffices to show that the assumption $\omega(m) = 3$ is untenable. By (5), we have $3 \mid m$, since otherwise

$$\frac{m}{\phi(m)} \le \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} < \alpha,$$

contradicting Theorem 1.

Thus $m = 3^a p^b q^c$ for odd primes $3 < p < q$ and natural numbers a, b, and c. Similarly, $5 \nmid m$ and $7 \nmid m$ since otherwise

$$\frac{\sigma(m^2)}{m^2} \ge \frac{13}{9} \cdot \frac{57}{49} > \alpha.$$

Furthermore $p \le 23$ since otherwise

$$\frac{m}{\phi(m)} \le \frac{3}{2} \cdot \frac{29}{28} \cdot \frac{31}{30} < \alpha.$$

Thus $p \in \{11, 13, 17, 19, 23\}$. We shall prove that all the five cases are impossible. Note that by (8),

$$\sigma(3^{2a})\sigma(p^{2b})\sigma(q^{2c}) = 3^{2a-1}p^{2b-1}q^{2c-1}(3pq + 2(p-1)(q-1)). \tag{11}$$

Suppose $p = 23$. Then $q \le 31$ since otherwise

$$\frac{m}{\phi(m)} \le \frac{3}{2} \cdot \frac{23}{22} \cdot \frac{37}{36} < \alpha.$$

Note that $3 \mid \sigma(23^{2b})\sigma(q^{2c})$ by (11), thus by (10) we have $q \equiv 1 \pmod 3$, hence $q = 31$. Furthermore $3 \mid 2c + 1$, hence $\Phi_3(31) \mid 3^{2a} \cdot 23^{2b-1} \cdot 31^{2c-1} \cdot 1153$ by (11). But this is impossible since $331 \mid \Phi_3(31)$. Thus $p \ne 23$.

Suppose $p = 19$. Then $q \le 43$ since otherwise

$$\frac{m}{\phi(m)} \le \frac{3}{2} \cdot \frac{19}{18} \cdot \frac{47}{46} < \alpha.$$

Thus $q \in S$, where $S = \{23, 29, 31, 37, 41, 43\}$. By (11) we have

$$\sigma(3^{2a})\sigma(19^{2b})\sigma(q^{2c}) = 3^{2a}19^{2b-1}q^{2c-1}(31q - 12).$$

We remark that $3 \nmid \sigma(19^{2b})$, since otherwise by (9) and (10) we have $127 \mid 31q - 12$ (as $\Phi_3(19) = 3 \cdot 127$), but this is false for all $q \in S$. Hence $3 \mid \sigma(q^{2c})$, thus by (9) and (10), $q \in \{31, 37, 43\}$ and $\Phi_3(q) \mid 3^{2a}19^{2b-1}(31q - 12)$; this is false. Thus $p \ne 19$.

Suppose $p = 17$. Thus by (11)

$$\sigma(3^{2a})\sigma(17^{2b})\sigma(q^{2c}) = 3^{2a-1}17^{2b-1}q^{2c-1}(83q - 32).$$

By (9) and (10) we have $17 \mid \sigma(q^{2c})$, thus $q \equiv 1 \pmod{17}$, hence $q \geq 103$. Then

$$\frac{m}{\phi(m)} \leq \frac{3}{2} \cdot \frac{17}{16} \cdot \frac{103}{102} < \alpha,$$

contradicting Theorem 1. Thus $p \neq 17$.

Suppose $p = 13$. Then $q \geq 37$ since otherwise

$$\frac{\sigma(m^2)}{m^2} \geq \frac{13}{9} \cdot \frac{183}{169} \cdot \frac{993}{961} > \alpha.$$

Furthermore $v_3(m) \leq 2$ since otherwise

$$\frac{\sigma(m^2)}{m^2} \geq \frac{1093}{729} \cdot \frac{183}{169} > \alpha.$$

Hence we consider the two cases, $v_3(m) = 2$ and $v_3(m) = 1$.

In the former case we have $v_{13}(m) \leq 2$ since otherwise

$$\frac{\sigma(m^2)}{m^2} \geq \frac{121}{81} \cdot \frac{5229043}{4826809} > \alpha,$$

leaving two subcases, $v_{13}(m) = 2$ or $v_{13}(m) = 1$. The first subcase yields by (11),

$$11^2 \cdot 30941\sigma(q^{2c}) = 3^4 \cdot 13^3 q^{2c-1}(21q - 8),$$

implying both that $q = 30941$ and $11 \mid 21q-8$, a contradiction. Similarly, the second subcase yields

$$11^2 \cdot 61\sigma(q^{2c}) = 3^3 \cdot 13^3 q^{2c-1}(21q - 8),$$

implying both that $q = 61$ and $11 \mid 21q - 8$, a contradiction.

In the latter case, viz., $v_3(m) = 1$, we have by (11)

$$\sigma(13^{2b})\sigma(q^{2c}) = 3^2 \cdot 13^{2b-2} q^{2c-1}(21q - 8). \tag{12}$$

Note that

$$b > 1, \tag{13}$$

otherwise (12) yields $61\sigma(q^{2c}) = 3q^{2c-1}(21q - 8)$ whence $q = 61$. Thus $\sigma(61^{2c}) = 3 \cdot 61^{2c-2} \cdot 19 \cdot 67$, thus $c = 1$, thus the contradiction $13 \cdot 97 = 19 \cdot 67$. Since $b > 1$, it follows from (12) that $13 \mid \sigma(q^{2c})$.

We show here $q \not\equiv 1 \pmod{13}$. Otherwise by (12) and (10) $2b - 2 = v_{13}(\sigma(q^{2c})) = v_{13}(2c+1)$, hence $2c+1 \geq 13^{2b-2}$, hence $2c-1 > 10^b$. Thus (since (12) implies $q^{2c-1} \mid \sigma(13^{2b})$) we have $\sigma(13^{2b}) \geq q^{2c-1} > q^{10^b} > 13^{10^b} > 13^{2b+1} > \sigma(13^{2b})$, a contradiction. Hence $q \not\equiv 1 \pmod{13}$; therefore by (10),

$$c \equiv 1 \pmod{3}; \tag{14}$$

furthermore

$$2b-2=v_{13}(\sigma(q^{2c})=v_{13}(q^2+q+1)+v_{13}(2c+1). \tag{15}$$

Let $t=v_{13}(2c+1)$. Suppose $t>0$. Since $v_{13}(q+2+q+1)<3\log_{13}q$, from (15) it follows that

$$2b-2<3\log_{13}q+t. \tag{16}$$

Since $13^t \mid 2c+1$, we have $2c-1\geq 13^t-2$, hence by (12)

$$13^{2b+1}>\sigma(13^{2b})\geq q^{2c-1}\geq q^{13^t-2},$$

hence

$$2b-2>(13^t-2)\log_{13}q-3. \tag{17}$$

By (16) and (17) it follows (since $\log_{13}(q)>1$) that

$$t>(13^t-5)\log_{13}q-3>13^t-8,$$

a contradiction if $t\geq 1$. Therefore $t=0$. Thus by (10), (12),

$$2b-2=v_{13}(\sigma(q^{2c}))=v_{13}(q^2+q+1). \tag{18}$$

Since $q^2+q+1<q^3$, from (12) and (18) it follows that

$$\begin{aligned}13^{2b-1}>\sigma(13^{2b})\geq q^{2c-1}=(q^3)^{\frac{2c-1}{3}}>(q^2+q+1)^{\frac{2c-1}{3}}\geq(13^{2b-2})^{\frac{2c-1}{3}}\\ =13^{\frac{2}{3}(b-1)(c-2)}.\end{aligned}$$

Thus $2b-1>\frac{2}{3}(b-1)(2c-1)$; equivalently, $(b-1)(c-2)<\frac{9}{4}$. Therefore $(b-1)(c-2)\leq 2$. By (13) it follows that

$$c\leq\frac{2b}{b-1}. \tag{19}$$

By (14) we have $c=1$ if $b\geq 3$. Suppose $b=2$. By (14), (19), $c=4$ or $c=1$. But $c=4$ implies, by (12),

$$\sigma(13^4)\sigma(q^8)=3^2\cdot 13^2q^7(21q-8),$$

hence by (9) $q^7\mid\Phi_5(13)=30941$, a contradiction. Therefore $c=1$ by (13). Hence by (12),

$$\sigma(13^{2b})(q^2+q+1)=3^2\cdot 13^{2b-2}q(21q-8). \tag{20}$$

By (13), (20), we have $13^2\mid q^2+q+1$, hence $q\geq 191$.

We remark here that (20) is equivalent to

$$q=\frac{13^{2b+1}-1}{13^{2b-2}(71q-3061)+(q+1)}.$$

Therefore
$$q < \frac{13^{2b+1}}{13^{2b-2}(71q-3061)} = \frac{2197}{71q-3061},$$
a contradiction since $q \geq 191$. Therefore $p \neq 13$.

Finally, suppose $p = 11$. By (3) and (5), we have $3 \| m$, otherwise
$$\frac{\sigma(m^2)}{m^2} \geq \frac{121}{81} \cdot \frac{133}{121} > \alpha,$$
contradicting Theorem 1. Thus by (11),
$$13\sigma(11^{2b})\sigma(q^{2c}) = 3 \cdot 11^{2b-1} q^{2c-1}(53q-20). \tag{21}$$

We show here $q \not\equiv 1 \pmod{11}$. Otherwise by (10), (21), $2b-1 = v_{11}(\sigma(q^{2c})) = v_{11}(2c+1)$, hence $2c+1 \geq 11^{2b-1}$. Thus by (21),
$$11^{2b+1} > \sigma(11^{2b}) \geq q^{2c-1} \geq q^{11^{2b-1}-2},$$
thus $2b+1 > (11^{2b-1}-2)\log_{11} q > 11^{2b-1}-2$, a contradiction.

Therefore $q \not\equiv 1 \pmod{11}$. By (10), $e_{11}(q) = 5$ (whence $q > 13$) and thus
$$5 \mid 2c+1; \tag{22}$$
furthermore by (21)
$$2b-1 = v_{11}(\sigma(q^{2c})) = v_{11}\left(\Phi_5(q)\right) + v_{11}(2c+1). \tag{23}$$
Let $t = v_{11}(2c+1)$. Suppose $t > 0$. Since $\Phi_5(q) < q^5$, we have by (23)
$$2b-1 < 5\log_{11} q + t. \tag{24}$$
On the other hand, since $2c+1 \geq 11^t$, (21) yields $11^{2b+1} > \sigma(11^{2b}) \geq q^{2c-1} \geq q^{11^t-2}$, hence
$$2b+1 > (11^t-2)\log_{11} q. \tag{25}$$
Then (24), (25), together imply
$$t > (11^t-7)\log_{11} q - 2 > 11^t - 9,$$
which is impossible for $t \geq 1$. Therefore $t = 0$. Hence by (23)
$$2b-1 = v_{11}\left(\Phi_5(q)\right). \tag{26}$$
Since $\Phi_5(q) < q^5$, we have by (21), (26),
$$11^{2b+1} > \sigma(11^{2b}) \geq q^{2c-1} = (q^5)^{\frac{2c-1}{5}} > (\Phi_5(q))^{\frac{2c-1}{5}} \geq (11^{2b-1})^{\frac{2c-1}{5}}.$$

Thus $2b+1 > \frac{1}{5}(2b-1)(2c+1)$; equivalently,

$$c < \frac{6b+2}{2b-1}. \tag{27}$$

By (10), $3 \nmid \sigma(11^{2b})$ since $e_3(11) = 2$. Thus by (21), we have $3 \mid \sigma(q^{2c})$, hence by (10) $3 \mid 2c+1$. Thus by (22), $15 \mid 2c+1$, hence $c \geq 7$. Thus by (27) we have $b = 1$, $c = 7$. Thus by (21)

$$7 \cdot 13 \cdot 19\sigma(q^{14}) = 3 \cdot 11 q^{13}(53q - 20),$$

implying $q^{13} \mid 1729$, which is impossible. Therefore $p \neq 11$, thus completing the proof. □

4 Results on the shape of m

By the *shape* of a natural number, we mean the nature of the exponents of the prime factors in the unique prime factorization of that number. We have already seen in Theorem 2 that a solution $n > 2$ to (1) must be an odd square, $n = m^2$, as in (8). We seek necessary conditions on the shape of m.

Theorem 6. *If m is a solution to (8) then m is neither squarefree nor the square of a squarefree natural number.*

Proof. First suppose that m is squarefree. By (9), $\sigma(m^2)$ is the product of cyclotomic polynomials $\Phi_3(p)$ evaluated at the prime divisors p of m. Thus by Lemma 3, if q is a prime dividing $\sigma(m^2)$, then either $q = 3$ or $q \equiv 1 \pmod 3$. Thus by (8), since $m \mid \sigma(m^2)$,

$$m = 3^{\epsilon} p_1 p_2 \cdots p_k,$$

where $\epsilon \in \{0, 1\}$ and $p_1 < p_2 < \cdots < p_k$ are primes such that $p_j \equiv 1 \pmod 3$, $1 \leq j \leq k$. By Lemma 3 we have $3 \| \Phi_3(p_j)$, $1 \leq j \leq k$, hence $v_3(\sigma(m^2)) = k$. Since $k \geq 3$ by Theorem 5, we have $3^2 \mid \phi(m)$, hence $3^{\epsilon} \| m + \phi(m)$, hence

$$v_3(m(m + \phi(m)) = 2\epsilon.$$

Thus by (8), $k = 2\epsilon$, a contradiction since $k \geq 3$ and $\epsilon \leq 1$. Hence m cannot be squarefree.

Now suppose that m is the square of a squarefree natural number. As above, $\sigma(m^2)$ is the product of cyclotomic polynomials $\Phi_5(p)$ evaluated at the prime divisors p of m, hence if q is a prime dividing $\sigma(m^2)$, then either $q = 5$ or $q \equiv 1 \pmod 5$; consequently

$$m = 5^{\epsilon} p_1^2 p_2^2 \cdots p_k^2,$$

where $\epsilon \in \{0, 2\}$ and $p_1 < p_2 < \cdots < p_k$ are primes such that $p_j \equiv 1 \pmod 5$, $1 \leq j \leq k$. Again, as above, $v_5(\sigma(m^2)) = k$. Since $k \geq 3$ by Theorem 5, we have $5^3 \mid \phi(m)$, hence $5^{\epsilon} \| m + \phi(m)$, hence

$$v_5(m(m + \phi(m)) = 2\epsilon.$$

Thus $k = 2\epsilon$, hence $k = 4$. Thus by (6)

$$\frac{m}{\phi(m)} \leq \frac{5}{4} \cdot \frac{11}{10} \cdot \frac{31}{30} \cdot \frac{41}{40} \cdot \frac{61}{60} < \alpha,$$

contradicting Theorem 1. Hence m is not the square of a squarefree natural number. □

5 Concluding Remarks

A natural number n is said to be *perfect* if $\sigma(n) = 2n$. No odd perfect numbers are known, and the question regarding their existence remains what is widely believed to be the oldest unsolved problem in all of mathematics. It seems the question of whether (1) has any solutions $n > 2$ parallels that regarding the existence of odd perfect numbers. For example, the only even solution to (1) is 2, while the even solutions to $\sigma(n) = 2n$ are in one-to-one correspondence with the set of Mersenne primes, and are thus characterized completely. On the other hand, no odd solutions to (1), or to $\sigma(n) = 2n$, are known. In either case, the only results known are those stating necessary conditions for such solutions to exist. Considering the difficulty involved in proving Theorem 5, it is clear that the problem regarding solutions $n > 2$ to (1) is no less difficult than that of the existence of odd perfect numbers.

All calculations for this paper were done by hand. This was in fact quite easy, since the bulk of the work comprised comparing rational numbers to the golden ratio, α. This entailed computing the first few convergents of the rational number in question until a partial quotient exceeded 1, a task easily done by hand. Otherwise, some of the factorizations of cyclotomic polynomials given were found in Brillhart et al. [1].

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