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Congruences Modulo Small Powers of 2 and 3 for Partitions into Odd Designated Summands

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Abstract

Andrews, Lewis and Lovejoy introduced a new class of partitions, partitions with designated summands. Let $\mathrm{PD}(n)$ denote the number of partitions of n with designated summands and $\mathrm{PDO}(n)$ denote the number of partitions of n with designated summands in which all parts are odd. Andrews et al. established many congruences modulo 3 for $\mathrm{PDO}(n)$ by using the theory of modular forms. Baruah and Ojah obtained numerous congruences modulo 3, 4, 8 and 16 for $\mathrm{PDO}(n)$ by using theta function identities. In this paper, we prove several infinite families of congruences modulo 9, 16 and 32 for $\mathrm{PDO}(n)$.

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1 Introduction

A *partition* of a positive integer n is a non increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$, where the λ_i's ($i = 1, 2, \ldots, m$) are called parts of the partition. For example, the partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Let $p(n)$ denote the number of partitions of n. Thus $p(4) = 5$.

Andrews, Lewis and Lovejoy [1] studied *partitions with designated summands*, which are constructed by taking ordinary partitions and tagging exactly one of each part size. Thus the partitions of 4 with designated summands are given by

$$\begin{gathered} 4', \quad 3'+1', \quad 2'+2, \quad 2+2', \quad 2'+1'+1, \quad 2'+1+1', \\ 1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'. \end{gathered}$$

Let $\mathrm{PD}(n)$ denote the number of partitions of n with designated summands and $\mathrm{PDO}(n)$ denote the number of partitions of n with designated summands in which all parts are odd. Thus $\mathrm{PD}(4) = 10$ and $\mathrm{PDO}(4) = 5$.

Recently, Chen et al. [5] obtained the generating functions for $\mathrm{PD}(3n)$, $\mathrm{PD}(3n+1)$ and $\mathrm{PD}(3n+2)$ and gave a combinatorial interpretation of the congruence $\mathrm{PD}(3n+2) \equiv 0 \pmod 3$. Xia [11] proved infinite families of congruences modulo 9 and 27 for $\mathrm{PD}(n)$. For example, for all $n \geq 0$ and $k \geq 1$

$$\mathrm{PD}(2^{18k-1}(12n+1)) \equiv 0 \pmod{27}.$$

Throughout this paper, we use the notation

$$f_k := (q^k; q^k)_\infty \ \ (k = 1, 2, 3, \ldots), \text{ where } \ (a;q)_\infty := \prod_{m=0}^{\infty} (1 - aq^m).$$

The generating function for $\mathrm{PDO}(n)$ satisfies

$$\sum_{n=0}^{\infty} \mathrm{PDO}(n) q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}. \tag{1}$$

Using the theory of q-series and modular forms Andrews et al. [1] derived

$$\sum_{n=0}^{\infty} \mathrm{PDO}(3n) q^n = \frac{f_2^2 f_6^4}{f_1^4 f_{12}^2}, \tag{2}$$

$$\sum_{n=0}^{\infty} \mathrm{PDO}(3n+1) q^n = \frac{f_2^4 f_3^3 f_{12}}{f_1^5 f_4 f_6^2}, \tag{3}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(3n+2)q^n = 2\frac{f_2^3 f_6 f_{12}}{f_1^4 f_4}. \tag{4}$$

They also established, for all $n \geq 0$

$$\mathrm{PDO}(9n+6) \equiv 0 \pmod{3}$$

and

$$\mathrm{PDO}(12n+6) \equiv 0 \pmod{3}.$$

Baruah and Ojah [3] proved several congruences modulo 3, 4, 8 and 16 for $\mathrm{PDO}(n)$. For instance,

$$\mathrm{PDO}(8n+7) \equiv 0 \pmod{8}$$

and

$$\mathrm{PDO}(12n+9) \equiv 0 \pmod{16}.$$

The aim of this paper is to prove several new infinite families of congruences modulo 9, 16 and 32 for $\mathrm{PDO}(n)$. In particular, we prove the following

Theorem 1. *For all nonnegative integers α, β and n, we have*

$$\mathrm{PDO}(2^{\alpha+2}3^{\beta}(72n+66)) \equiv 0 \pmod{144} \tag{5}$$

and

$$\mathrm{PDO}(2^{\alpha+2}3^{\beta}(144n+138)) \equiv 0 \pmod{288}. \tag{6}$$

In Section 2, we list some preliminary results. We prove several infinite families of congruences modulo 9 for $\mathrm{PDO}(n)$ in Section 3, and Theorem 1 and many infinite families of congruences modulo 16 and 32 for $\mathrm{PDO}(n)$ in Section 4.

2 Definitions and preliminaries

We will make use of the following definitions, notation and results.

Let $f(a,b)$ be Ramanujan's general theta function [2, p. 34] given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

In particular,

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \varphi(-q) := f(-q,-q) = \frac{f_1^2}{f_2}, \tag{7}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{f_2^2}{f_1} \tag{8}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = f_1. \tag{9}$$

For any positive integer k, let $k(k+1)/2$ be the k^{th} *triangular number* and $k(3k \pm 1)/2$ be a generalized *pentagonal number*.

Lemma 2. *The following 2-dissections hold:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{10}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{11}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2} \tag{12}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{13}$$

Proof. Lemma 2 is an immediate consequence of dissection formulas of Ramanujan, collected in Berndt's book [2, Entry 25, p. 40]. □

Lemma 3. *The following 2-dissections hold:*

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{14}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{15}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7} \tag{16}$$

and

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}. \tag{17}$$

Proof. Hirschhorn et al. [6] established (14) and (15). Replacing q by $-q$ in (14) and (15), and using the relation

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we obtain (16) and (17). $\square$

Lemma 4. *The following 2-dissections hold:*

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{18}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}} \tag{19}$$

and

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{20}$$

Proof. Baruah and Ojah [4] derived the above identities. $\square$

Lemma 5. *The following 3-dissections hold:*

$$\varphi(-q) = \varphi(-q^9) - 2qf(-q^3,-q^{15}) \tag{21}$$

and

$$\psi(q) = f(q^3,q^6) + q\psi(q^9). \tag{22}$$

Proof. See Berndt's book [2, p. 49] for a proof of (21) and (22). $\square$

Lemma 6. *The following 3-dissection holds:*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - qf_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{23}$$

Proof. Hirschhorn and Sellers [8] have proved the above identity. $\square$

Let t be a positive integer. A partition of n is called a *t-core* partition of n if none of the hook numbers of its associated Ferrers-Young diagram are multiples of t. Let $a_t(n)$ denote the number of *t-core* partitions of n. Then the generating function of $a_t(n)$ satisfies

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f_t^t}{f_1}. \tag{24}$$

Many mathematicians have studied arithmetic properties of $a_3(n)$. See for example, Keith [9], and Lin and Wang [10]. Hirschhorn and Sellers [7] obtained an explicit formula for $a_3(n)$ by using elementary methods and proved

Lemma 7. *Let* $3n+1 = \prod_{i=1}^{k} p_i^{\alpha_i} \prod_{j=1}^{m} q_j^{\beta_j}$*, where* $p_i \equiv 1 \pmod 3$ *and* $q_j \equiv 2 \pmod 3$ *with* $\alpha_i, \beta_j \geq 0$ *be the prime factorization of* $3n+1$*. Then*

$$a_3(n) = \begin{cases} \prod_{i=1}^{k} (\alpha_i + 1), & \text{if all } \beta_j \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

3 Congruences modulo 9

In this section, we prove the following infinite families of congruences modulo 9 for $\mathrm{PDO}(n)$.

Theorem 8. *For all nonnegative integers* α, β *and* n*, we have*

$$\mathrm{PDO}(4^{\alpha}(24n+16)) \equiv \mathrm{PDO}(24n+16) \pmod 9, \tag{25}$$

$$\mathrm{PDO}(2^{\alpha}3^{\beta}(24n+24)) \equiv (-1)^{\alpha}\, \mathrm{PDO}(24n+24) \pmod 9, \tag{26}$$

$$\mathrm{PDO}(4^{\alpha}(48n+40)) \equiv 0 \pmod 9 \tag{27}$$

and

$$\mathrm{PDO}(2^{\alpha}3^{\beta}(144n+120)) \equiv 0 \pmod 9. \tag{28}$$

Theorem 9. *For any nonnegative integer* n*, let* $3n+1 = \prod_{i=1}^{k} p_i^{\alpha_i} \prod_{j=1}^{m} q_j^{\beta_j}$*, where* $p_i \equiv 1 \pmod 3$ *and* $q_j \equiv 2 \pmod 3$ *are primes with* $\alpha_i, \beta_j \geq 0$*. Then,*

$$\mathrm{PDO}(48n+16) \equiv \begin{cases} 6 \prod_{i=1}^{k} (\alpha_i + 1) \pmod 9, & \text{if all } \beta_j \text{ are even;} \\ 0 \pmod 9, & \text{otherwise.} \end{cases} \tag{29}$$

and

$$\mathrm{PDO}(72n+24) \equiv \begin{cases} 3 \prod_{i=1}^{k} (\alpha_i + 1) \pmod 9, & \text{if all } \beta_j \text{ are even;} \\ 0 \pmod 9, & \text{otherwise.} \end{cases} \tag{30}$$

Corollary 10. *Let* $p \equiv 2 \pmod 3$ *be a prime. Then for all nonnegative integers* α *and* n *with* $p \nmid n$*, we have*

$$\mathrm{PDO}(48p^{2\alpha+1}n + 16p^{2\alpha+2}) \equiv 0 \pmod 9 \tag{31}$$

and

$$\mathrm{PDO}(72p^{2\alpha+1}n + 24p^{2\alpha+2}) \equiv 0 \pmod 9. \tag{32}$$

Theorem 11. *If* n *cannot be represented as the sum of a triangular number and three times a triangular number, then*

$$\mathrm{PDO}(48n+24) \equiv 0 \pmod 9.$$

Corollary 12. *For any positive integer k, let $p_j \geq 5,\ 1 \leq j \leq k$ be primes. If $(-3/p_j) = -1$ for every j, then for all nonnegative integers n with $p_k \nmid n$ we have*

$$\mathrm{PDO}(48p_1^2p_2^2\cdots p_{k-1}^2p_kn + 24p_1^2p_2^2\cdots p_k^2) \equiv 0 \pmod{9}. \tag{33}$$

By the binomial theorem, it is easy to see that for any positive integer m,

$$f_m^3 \equiv f_{3m} \pmod{3} \tag{34}$$

and

$$f_m^9 \equiv f_{3m}^3 \pmod{9}. \tag{35}$$

Proof of Theorem 8. From (35), it follows that

$$\frac{f_3^3}{f_1^5} \equiv f_1^4 \pmod{9}. \tag{36}$$

In view of (36), we rewrite (3) as

$$\sum_{n=0}^{\infty} \mathrm{PDO}(3n+1)q^n \equiv \frac{f_1^4f_2^4f_{12}}{f_4f_6^2} \pmod{9}. \tag{37}$$

Substituting (12) in (37) and extracting the terms containing odd powers of q, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n+4)q^n \equiv -4\frac{f_1^6f_4^4f_6}{f_2^3f_3^2} \pmod{9}. \tag{38}$$

Employing (14) in (38) and extracting the terms containing even powers of q, we derive

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n+4)q^n \equiv -4\frac{f_2^{10}f_3}{f_1^3f_6^2} \pmod{9}. \tag{39}$$

Substituting (16) in (39) and extracting the terms containing odd powers of q, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+16)q^n \equiv -12\frac{f_1^3f_2^2f_6^2}{f_3} \pmod{9}.$$

From (34),

$$\frac{f_1^3f_2^2f_6^2}{f_3} \equiv \frac{f_6^3}{f_2} \pmod{3}.$$

In view of the above two identities,

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+16)q^n \equiv 6\frac{f_6^3}{f_2} \pmod{9}, \tag{40}$$

which implies that

$$\mathrm{PDO}(48n+40) \equiv 0 \pmod 9 \tag{41}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+16)q^n \equiv 6\frac{f_3^3}{f_1} \pmod 9. \tag{42}$$

Invoking (15) in (42) and extracting the terms containing odd powers of q,

$$\sum_{n=0}^{\infty} \mathrm{PDO}(96n+64)q^n \equiv 6\frac{f_6^3}{f_2} \pmod 9. \tag{43}$$

By (40) and (43),

$$\mathrm{PDO}(96n+64) \equiv \mathrm{PDO}(24n+16) \pmod 9. \tag{44}$$

Congruence (25) follows from (44) and mathematical induction. Congruence (27) follows from (41) and (25).

Employing (13) in (2),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(3n)q^n = \frac{f_4^{14} f_6^4}{f_2^{12} f_8^4 f_{12}^2} + 4q\frac{f_4^2 f_6^4 f_8^4}{f_2^8 f_{12}^2},$$

which yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n)q^n = \frac{f_2^{14} f_3^4}{f_1^{12} f_4^4 f_6^2} \tag{45}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n+3)q^n = 4\frac{f_2^2 f_3^4 f_4^4}{f_1^8 f_6^2}. \tag{46}$$

Applying (16) and (45),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n)q^n = \frac{f_2^{14}}{f_4^4 f_6^2}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right)^4,$$

which implies that

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n)q^n \equiv \frac{f_2^{20} f_3^{10}}{f_1^{22} f_6^8} \pmod 9.$$

From (35),

$$\frac{f_2^{20} f_3^{10}}{f_1^{22} f_6^8} \equiv \frac{f_2^2 f_3^4}{f_1^4 f_6^2} \pmod 9.$$

In view of the above two identities,

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n)q^n \equiv \frac{f_2^2 f_3^4}{f_1^4 f_6^2} \pmod 9. \tag{47}$$

Substituting (15) and (16) in (47), and extracting the terms containing even powers of q, we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n)q^n \equiv \frac{f_2^9 f_3^3}{f_1^9 f_6^3} + 3q\frac{f_2 f_6^5}{f_1^5 f_3} \pmod{9}. \tag{48}$$

Using (34) and (35) in (48), we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+24)q^n \equiv 3\frac{f_1 f_2 f_6^5}{f_3^3} \pmod{9}. \tag{49}$$

Substituting (17) in (49) and using (34), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO}(24n+24)q^n &\equiv 3\frac{f_2^2 f_4^2 f_{12}^2}{f_6^2} - 3q\frac{f_2^4 f_{12}^6}{f_4^2 f_6^4} \\ &\equiv 3\frac{f_4^2 f_{12}^2}{f_2 f_6} - 3q\frac{f_2 f_4 f_{12}^5}{f_6^3} \pmod{9},\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+24)q^n \equiv 3\psi(q)\psi(q^3) \pmod{9} \tag{50}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+48)q^n \equiv -3\frac{f_1 f_2 f_6^5}{f_3^3} \pmod{9}. \tag{51}$$

By (49) and (51),

$$\mathrm{PDO}(2(24n+24)) \equiv -\mathrm{PDO}(24n+24) \pmod{9}. \tag{52}$$

Employing (23) in (49) and using (34),

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO}(24n+24)q^n &\equiv 3\frac{f_6^6 f_9^4}{f_3^4 f_{18}^2} - 3q\frac{f_6^5 f_9 f_{18}}{f_3^3} - 6q^2\frac{f_6^4 f_{18}^4}{f_3^2 f_9^2} \\ &\equiv 3\frac{f_9^3}{f_3} + 6q\frac{f_{18}^3}{f_6} + 3q^2\frac{f_3 f_6 f_{18}^5}{f_9^3} \pmod{9}\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} \mathrm{PDO}(72n+24)q^n \equiv 3\frac{f_3^3}{f_1} \pmod{9}, \tag{53}$$

$$\sum_{n=0}^{\infty} \mathrm{PDO}(72n+48)q^n \equiv 6\frac{f_6^3}{f_2} \pmod{9} \tag{54}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(72n+72)q^n \equiv 3\frac{f_1 f_2 f_6^5}{f_3^3} \pmod{9}. \tag{55}$$

From (54),

$$\mathrm{PDO}(144n+120) \equiv 0 \pmod{9}. \tag{56}$$

By (49) and (55),

$$\mathrm{PDO}(3(24n+24)) \equiv \mathrm{PDO}(24n+24) \pmod{9}. \tag{57}$$

Congruence (26) follows from (52), (57), and mathematical induction. Congruence (28) follows from (56) and (26). □

Proof of Theorem 9. From (24), (42) and (53), it is clear that for all $n \geq 0$

$$\mathrm{PDO}(48n+16) \equiv 6a_3(n) \pmod{9}, \tag{58}$$

and

$$\mathrm{PDO}(72n+24) \equiv 3a_3(n) \pmod{9}. \tag{59}$$

Congruence (29) follows from Lemma 7 and (58). Congruence (30) follows from Lemma 7 and (59). □

For any prime p and any positive integer N, let $\upsilon_p(N)$ denote the exponent of the highest power of p dividing N.

Proof of Corollary 10. Suppose $\alpha \geq 0$, $p \equiv 2 \pmod{3}$ and $p \nmid n$, then it is clear that

$$\upsilon_p\left(3\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{3}\right)+1\right) = \upsilon_p\left(3p^{2\alpha+1}n + p^{2\alpha+2}\right) = 2\alpha+1. \tag{60}$$

Congruences (31) and (32) follow from (29), (30) and (60). □

Proof of Theorem 11. From (8) and (50), we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+24)q^n \equiv 3\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q^{k(k+1)/2+3m(m+1)/2} \pmod{9}. \tag{61}$$

Theorem 11 follows from (61). □

Proof of Corollary 12. By (61),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+24)q^{48n+24} \equiv 3\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q^{6(2k+1)^2+2(6m+3)^2} \pmod{9},$$

which implies that if $48n+24$ is not of the form $6(2k+1)^2+2(6m+3)^2$, then $\mathrm{PDO}(48n+24) \equiv 0 \pmod 9$. Let $k \geq 1$ be an integer and let $p_i \geq 5,\ 1 \leq i \leq k$ be primes with $(\frac{-3}{p_i}) = -1$. If N is of the form $2x^2+6y^2$, then $v_{p_i}(N)$ is even since $(\frac{-3}{p_i}) = -1$. Let

$$\begin{aligned} N &= 48\left(p_1^2p_2^2\cdots p_{k-1}^2p_kn + \frac{p_1^2p_2^2\cdots p_{k-1}^2p_k^2-1}{2}\right)+24 \\ &= 48p_1^2p_2^2\cdots p_{k-1}^2p_kn + 24p_1^2p_2^2\cdots p_{k-1}^2p_k^2. \end{aligned}$$

If $p_k \nmid n$, then $v_{p_k}(N)$ is an odd number and hence N is not of the form $2x^2+6y^2$. Therefore (33) holds. $\square$

4 Congruences modulo 2^4 and 2^5

In this section, we establish the following infinite families of congruences modulo 16 and 32 for $\mathrm{PDO}(n)$.

Theorem 13. *For all nonnegative integers α and n, we have*

$$\mathrm{PDO}(4^\alpha(12n+8)) \equiv \mathrm{PDO}(12n+8) \pmod{2^4}, \tag{62}$$
$$\mathrm{PDO}(4^\alpha(24n+23)) \equiv 0 \pmod{2^4}, \tag{63}$$
$$\mathrm{PDO}(4^\alpha(48n+14)) \equiv 0 \pmod{2^4} \tag{64}$$

and

$$\mathrm{PDO}(24n+17) \equiv 0 \pmod{2^4}. \tag{65}$$

Theorem 14. *For all nonnegative integers α and n, we have*

$$\mathrm{PDO}(2^\alpha(12n)) \equiv \mathrm{PDO}(12n) \pmod{2^4}, \tag{66}$$
$$\mathrm{PDO}(2^\alpha(72n+42)) \equiv 0 \pmod{2^4} \tag{67}$$

and

$$\mathrm{PDO}(2^\alpha(72n+66)) \equiv 0 \pmod{2^4}. \tag{68}$$

Theorem 15. *For all nonnegative integers α and n, we have*

$$\mathrm{PDO}(4^\alpha(24n) \equiv \mathrm{PDO}(24n) \pmod{2^5}, \tag{69}$$
$$\mathrm{PDO}(9^\alpha(6n+3)) \equiv \mathrm{PDO}(6n+3) \pmod{2^5}, \tag{70}$$
$$\mathrm{PDO}(24n+9) \equiv 0 \pmod{2^5}, \tag{71}$$
$$\mathrm{PDO}(9^\alpha(216n+117)) \equiv 0 \pmod{2^5}, \tag{72}$$
$$\mathrm{PDO}(2^\alpha(72n+69)) \equiv 0 \pmod{2^5}, \tag{73}$$
$$\mathrm{PDO}(3^\alpha(72n+69)) \equiv 0 \pmod{2^5} \tag{74}$$

and

$$\mathrm{PDO}(2^\alpha(144n+42)) \equiv 0 \pmod{2^5}. \tag{75}$$

Theorem 16. *If n cannot be represented as the sum of two triangular numbers, then for all nonnegative integers α and $r \in \{1,6\}$ we have*

$$\mathrm{PDO}(2^\alpha r(12n+3)) \equiv 0 \pmod{2^4}.$$

Corollary 17. *If p is a prime, $p \equiv 3 \pmod{4}$), $1 \le j \le p-1$ and $r \in \{1,6\}$, then for all nonnegative integers α, β and n, we have*

$$\mathrm{PDO}(2^\alpha p^{2\beta+1} r(12pn+12j+3p)) \equiv 0 \pmod{2^4}. \tag{76}$$

For example, taking $p=3$, we deduce that for all $\alpha, \beta, n \ge 0$,

$$\mathrm{PDO}(2^\alpha 3^\beta (216n+126)) \equiv 0 \pmod{2^4} \tag{77}$$

and

$$\mathrm{PDO}(2^\alpha 3^\beta (216n+198)) \equiv 0 \pmod{2^4}. \tag{78}$$

Combining (77) and (67),

$$\mathrm{PDO}(2^\alpha 3^\beta (72n+42)) \equiv 0 \pmod{2^4}. \tag{79}$$

Combining (78) and (68),

$$\mathrm{PDO}(2^\alpha 3^\beta (72n+66)) \equiv 0 \pmod{2^4}. \tag{80}$$

Theorem 18. *If n cannot be represented as the sum of a triangular number and four times a triangular number, then for all nonnegative integers α and $r \in \{1,3\}$ we have*

$$\mathrm{PDO}(2^\alpha r(48n+30)) \equiv 0 \pmod{2^5}.$$

Corollary 19. *If p is any prime with $p \equiv 3 \pmod{4}$), $1 \le j \le p-1$ and $r \in \{1,3\}$, then for all nonnegative integers α, β and n, we have*

$$\mathrm{PDO}(2^\alpha p^{2\beta+1} r(48pn+48j+30p)) \equiv 0 \pmod{2^5}. \tag{81}$$

For example, taking $p=3$ we find that for all $\alpha, n \ge 0$ and $\beta \ge 1$,

$$\mathrm{PDO}(2^\alpha 3^\beta (144n+42)) \equiv 0 \pmod{2^5} \tag{82}$$

and

$$\mathrm{PDO}(2^\alpha 3^\beta (144n+138)) \equiv 0 \pmod{2^5}. \tag{83}$$

Combining (82) and (75), for all $\alpha, \beta, n \ge 0$,

$$\mathrm{PDO}(2^\alpha 3^\beta (144n+42)) \equiv 0 \pmod{2^5} \tag{84}$$

and combining (83), (73) and (74), for all $\alpha, \beta, n \ge 0$,

$$\mathrm{PDO}(2^\alpha 3^\beta (72n+69)) \equiv 0 \pmod{2^5}. \tag{85}$$

Theorem 20. *If n cannot be represented as the sum of twice a pentagonal number and three times a triangular number, then for any nonnegative integer α we have*

$$\mathrm{PDO}(4^{\alpha}(24n+11)) \equiv 0 \pmod{2^4}.$$

Corollary 21. *For any positive integer k, let $p_j \geq 5,\ 1 \leq j \leq k$ be primes. If $(-2/p_j) = -1$ for every j, then for all nonnegative integers α and n with $p_k \nmid n$ we have*

$$\mathrm{PDO}(6 \cdot 4^{\alpha+1} p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 11 \cdot 4^{\alpha} p_1^2 p_2^2 \cdots p_k^2) \equiv 0 \pmod{2^4}.$$

Theorem 22. *If n cannot be represented as the sum of a pentagonal number and six times a triangular number, then for any nonnegative integer α we have*

$$\mathrm{PDO}(4^{\alpha}(48n+38)) \equiv 0 \pmod{2^4}.$$

Corollary 23. *For any positive integer k, let $p_j \geq 5,\ 1 \leq j \leq k$ be primes. If $(-2/p_j) = -1$ for every j, then for all nonnegative integers n with $p_k \nmid n$ we have*

$$\mathrm{PDO}(3 \cdot 4^{\alpha+2} p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 38 \cdot 4^{\alpha} p_1^2 p_2^2 \cdots p_k^2) \equiv 0 \pmod{2^4}.$$

Theorem 24. *If n cannot be represented as the sum of a pentagonal number and four times a pentagonal number, then we have*

$$\mathrm{PDO}(24n+5) \equiv 0 \pmod{2^5}.$$

Corollary 25. *For any positive integer k, let $p_j \geq 5,\ 1 \leq j \leq k$ be primes. If $(-1/p_j) = -1$ for every j, then for all nonnegative integers n with $p_k \nmid n$ we have*

$$\mathrm{PDO}(24 p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 5 p_1^2 p_2^2 \cdots p_k^2) \equiv 0 \pmod{2^5}.$$

Theorem 26. *If n cannot be represented as the sum of a pentagonal number and sixteen times a pentagonal number, then we have*

$$\mathrm{PDO}(24n+17) \equiv 0 \pmod{2^5}.$$

Corollary 27. *For any positive integer k, let $p_j \geq 5,\ 1 \leq j \leq k$ be primes. If $(-1/p_j) = -1$ for every j, then for all nonnegative integers n with $p_k \nmid n$ we have*

$$\mathrm{PDO}(24 p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 17 p_1^2 p_2^2 \cdots p_k^2) \equiv 0 \pmod{2^5}.$$

By the binomial theorem, it is easy to see that for all positive integers k and m,

$$f_m^{2^k} \equiv f_{2m}^{2^{k-1}} \pmod{2^k}. \tag{86}$$

Proof of Theorem 13. Using (13), we can rewrite (4) as

$$\sum_{n=0}^{\infty} \mathrm{PDO}(3n+2)q^n = 2\frac{f_4^{13}f_6f_{12}}{f_2^{11}f_8^4} + 8q\frac{f_4f_6f_8^4f_{12}}{f_2^7},$$

which yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n+2)q^n = 2\frac{f_2^{13}f_3f_6}{f_1^{11}f_4^4}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n+5)q^n = 8\frac{f_2f_3f_4^4f_6}{f_1^7}.$$

From (86) with $k = 3$ and $k = 2$, we have

$$\frac{f_2^{13}f_3f_6}{f_1^{11}f_4^4} \equiv \frac{f_2f_3f_6}{f_1^3} \pmod{2^3}$$

and

$$\frac{f_2f_3f_4^4f_6}{f_1^7} \equiv \frac{f_3f_4^4f_6}{f_1^3f_2} \pmod{2^2}.$$

Thus,

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n+2)q^n \equiv 2\frac{f_2f_3f_6}{f_1^3} \pmod{2^4}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n+5)q^n \equiv 8\frac{f_3f_4^4f_6}{f_1^3f_2} \pmod{2^5}.$$

Substituting (16) in the above two congruences and extracting the terms containing even and odd powers of q, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n+2)q^n \equiv 2\frac{f_2^6f_3^4}{f_1^8f_6^2} \equiv 2\frac{f_2^2f_3^4}{f_6^2} \pmod{2^4}, \tag{87}$$

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n+8)q^n \equiv 6\frac{f_2^2f_3^2f_6^2}{f_1^6} \pmod{2^4}, \tag{88}$$

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n+5)q^n \equiv 8\frac{f_2^{10}f_3^4}{f_1^{10}f_6^2} \equiv 8\frac{f_2^6}{f_1^2} \pmod{2^5} \tag{89}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n+11)q^n \equiv 24\frac{f_2^6f_3^2f_6^2}{f_1^8} \equiv 24f_4f_6^3 \pmod{2^4}. \tag{90}$$

It follows from (90) that

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+11)q^n \equiv 8f_2f_3^3 \equiv 8f_2\frac{f_6^2}{f_3} \pmod{2^4} \tag{91}$$

and

$$\mathrm{PDO}(24n+23) \equiv 0 \pmod{2^4}. \tag{92}$$

Employing (12) in (87) and extracting the terms containing odd powers of q,

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+14)q^n \equiv -8q\frac{f_1^2f_{12}^4}{f_6^2} \equiv 8qf_2f_{12}^3 \pmod{2^4},$$

which implies that

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+38)q^n \equiv 8f_1f_6^3 \equiv 8f_1\frac{f_{12}^2}{f_6} \pmod{2^4} \tag{93}$$

and

$$\mathrm{PDO}(48n+14) \equiv 0 \pmod{2^4}. \tag{94}$$

Substituting (13) and (20) in (88), and extracting the terms containing even and odd powers of q, we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+8)q^n \equiv 6\frac{f_2^{18}f_3^3f_6^2}{f_1^{17}f_4^5f_{12}} \equiv 6\frac{f_2^2f_3^3f_6^2}{f_1f_4f_{12}} \pmod{2^4} \tag{95}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO}(24n+20)q^n &\equiv 12\frac{f_2^{15}f_3^4f_{12}}{f_1^{16}f_4^3f_6} + 24\frac{f_2^6f_3^3f_4^3f_6^2}{f_1^{13}f_{12}} \\ &\equiv 12\frac{f_2^7f_6f_{12}}{f_4^3} + 24\frac{f_3^3f_4^3}{f_1} \pmod{2^4}.\end{aligned} \tag{96}$$

Substituting (15) in (95) and (96), and extracting even and odd powers of q, we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+8)q^n \equiv 6\frac{f_2^2f_3^4}{f_6^2} \pmod{2^4}, \tag{97}$$

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+32)q^n \equiv 6\frac{f_1^2f_3^2f_6^2}{f_2^2} \pmod{2^4} \tag{98}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+44)q^n \equiv 8f_2^2f_6^3 \equiv 8\frac{f_4f_{12}^2}{f_6} \pmod{2^4}. \tag{99}$$

Again employing (12) in (97) and extracting the terms containing odd powers of q,

$$\sum_{n=0}^{\infty} \mathrm{PDO}(96n+56)q^n \equiv 8q\frac{f_1^2 f_{12}^4}{f_6^2} \equiv 8qf_2 f_{12}^3 \pmod{2^4}. \tag{100}$$

By (99) and (100),

$$\mathrm{PDO}(96n+92) \equiv 0 \pmod{2^4} \tag{101}$$

and

$$\mathrm{PDO}(192n+56) \equiv 0 \pmod{2^4}. \tag{102}$$

From (86), (88) and (98),

$$\mathrm{PDO}(48n+32) \equiv \mathrm{PDO}(12n+8) \pmod{2^4}. \tag{103}$$

Congruence (62) follows from (103) and mathematical induction. Congruence (63) follows from (92), (101), and (62). Similarly, congruence (64) follows from (94), (102), and (62).

By invoking (11) in (89) and extracting the terms containing even and odd powers of q,

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+5)q^n \equiv 8\frac{f_1 f_4^5}{f_8^2} \equiv 8f_1 f_4 \pmod{2^5} \tag{104}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+17)q^n \equiv 16\frac{f_1 f_2^2 f_8^2}{f_4} \equiv 16 f_1 f_{16} \pmod{2^5}. \tag{105}$$

Congruence (65) follows from (105). □

Proofs of Theorem 14 and Theorem 15. From (86),

$$\frac{f_2^2 f_3^4 f_4^4}{f_1^8 f_6^2} \equiv \frac{f_3^4 f_4^4}{f_2^2 f_6^2} \pmod{2^3}. \tag{106}$$

Using (106), we rewrite (46) as

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n+3)q^n \equiv 4\frac{f_3^4 f_4^4}{f_2^2 f_6^2} \pmod{2^5}. \tag{107}$$

In view of (7) and (8),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n+3)q^n \equiv 4\psi^2(q^2)\phi^2(-q^3) \pmod{2^5}. \tag{108}$$

Substituting (22) in (108) and extracting the terms involving q^{3n+1}, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(18n+9)q^n \equiv 4q\psi^2(q^6)\phi^2(-q) \pmod{2^5}. \tag{109}$$

Using (21) in (109) and extracting the terms involving q^{3n+1},

$$\sum_{n=0}^{\infty} \mathrm{PDO}(54n+27)q^n \equiv 4\psi^2(q^2)\phi^2(-q^3) \pmod{2^5}. \tag{110}$$

Congruence (70) follows from (108), (110), and mathematical induction.

Applying (12) in (107) and using (86),

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO}(6n+3)q^n &\equiv 4\frac{f_4^4 f_{12}^{10}}{f_2^2 f_6^4 f_{24}^4} - 16q^3\frac{f_4^4 f_{24}^4}{f_2^2 f_{12}^2} \\ &\equiv 4\frac{f_4^4 f_{12}^2}{f_2^2 f_6^4} + 16q^3\frac{f_8^2 f_{48}^2}{f_4 f_{24}} \pmod{2^5}\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n+3)q^n \equiv 4\frac{f_2^4 f_6^2}{f_1^2 f_3^4} \pmod{2^5} \tag{111}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n+9)q^n \equiv 16q\frac{f_4^2 f_{24}^2}{f_2 f_{12}} \pmod{2^5}. \tag{112}$$

Congruence (71) follows from (112).

From (112) and (8),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+21)q^n \equiv 16\psi(q)\psi(q^6) \pmod{2^5}. \tag{113}$$

Invoking (22) in (113) and extracting the terms containing q^{3n+2} and q^{3n+1},

$$\mathrm{PDO}(72n+69) \equiv 0 \pmod{2^5} \tag{114}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(72n+45)q^n \equiv 16\psi(q^2)\psi(q^3) \pmod{2^5}. \tag{115}$$

Using (22) in (115) and extracting the terms containing q^{3n+1},

$$\mathrm{PDO}(216n+117) \equiv 0 \pmod{2^5}. \tag{116}$$

Congruence (72) follows from (70) and (116).

Substituting (11) and (13) in (111), and extracting the terms containing odd powers of q, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+15)q^n \equiv 8\frac{f_2^2 f_6^{14} f_8^2}{f_1 f_3^{12} f_4 f_{12}^4} + 16q\frac{f_4^5 f_6^2 f_{12}^4}{f_1 f_3^8 f_8^2}$$
$$\equiv 8\psi(q)\psi(q^4) + 16q\psi(q)\psi(q^{12}) \pmod{2^5}. \tag{117}$$

Employing (22) in (117) and extracting the terms involving q^{3n+2}, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(72n+63)q^n \equiv 16\psi(q^3)\psi(q^4) + 8q\psi(q^3)\psi(q^{12}) \pmod{2^5}. \tag{118}$$

Again, extracting the terms involving q^{3n+2} in (118), we get

$$\mathrm{PDO}(216n+207) \equiv 0 \pmod{2^5}. \tag{119}$$

Congruence (74) follows from (114), (119), and (70).

Substituting (12) and (13) in (45), and extracting the terms containing even powers of q, we find

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n)q^n \equiv \frac{f_2^{38} f_6^{10}}{f_1^{28} f_3^4 f_4^{12} f_{12}^4} \equiv \frac{f_1^4 f_2^6 f_{12}^4}{f_3^4 f_4^4 f_6^6} \pmod{2^4}. \tag{120}$$

Using (12) and (13) in (120), and extracting the terms containing even powers of q, we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n)q^n \equiv \frac{f_1^4 f_2^6 f_6^{18}}{f_3^{20} f_4^4 f_{12}^4} \equiv \frac{f_1^4 f_2^6 f_{12}^4}{f_3^4 f_4^4 f_6^6} \pmod{2^4}. \tag{121}$$

From (120) and (121),

$$\mathrm{PDO}(2(12n)) \equiv \mathrm{PDO}(12n) \pmod{2^4}. \tag{122}$$

Congruence (66) follows from (122) and mathematical induction.

Substituting (20) and (13) in (45), and using (86), we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(6n)q^n = \frac{f_4^{32} f_{12}^4}{f_2^{24} f_8^{10} f_{24}^2} + 16q^2\frac{f_4^8 f_8^6 f_{12}^4}{f_2^{16} f_{24}^2} + 4q^2\frac{f_4^{26} f_6^2 f_{24}^2}{f_2^{22} f_8^6 f_{12}^2}$$
$$+ 64q^4\frac{f_4^2 f_6^2 f_8^{10} f_{24}^2}{f_2^{14} f_{12}^2} + 32q^2\frac{f_4^{17} f_6 f_{12}}{f_2^{19}} + 8q\frac{f_4^{20} f_{12}^4}{f_2^{20} f_8^2 f_{24}^2}$$
$$+ 32q^3\frac{f_4^{14} f_6^2 f_8^2 f_{24}^2}{f_2^{18} f_{12}^2} + 4q\frac{f_4^{29} f_6 f_{12}}{f_2^{23} f_8^8} + 64q^3\frac{f_4^5 f_6 f_8^8 f_{12}}{f_2^{15}}$$
$$\equiv \frac{f_2^8 f_4^{16} f_{12}^4}{f_8^{10} f_{24}^2} + 16q^2 f_8^2 f_{16}^2 + 4q^2\frac{f_4^2 f_6^2 f_8^2 f_{24}^2}{f_2^6 f_{12}^2}$$
$$+ 8q f_4^2 f_8^2 + 4q f_2 f_4 f_6 f_{12} \pmod{2^5},$$

which yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n)q^n \equiv \frac{f_1^8 f_2^{16} f_6^4}{f_4^{10} f_{12}^2} + 16q f_4^2 f_8^2 + 4q \frac{f_2^2 f_3^2 f_4^2 f_{12}^2}{f_1^6 f_6^2} \pmod{2^5} \tag{123}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(12n+6)q^n \equiv 8 f_2^2 f_4^2 + 4 f_1 f_2 f_3 f_6 \pmod{2^5}. \tag{124}$$

Substituting (18) in (124) and extracting the terms containing even and odd powers of q, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+6)q^n \equiv 4 \frac{f_1^2 f_4^2 f_6^4}{f_2^2 f_{12}^2} + 8 f_1^2 f_2^2 \pmod{2^5} \tag{125}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+18)q^n \equiv -4 \frac{f_2^4 f_3^2 f_{12}^2}{f_4^2 f_6^2} \pmod{2^5}. \tag{126}$$

By (86) and (8),

$$\frac{f_2^4 f_3^2 f_{12}^2}{f_4^2 f_6^2} \equiv \psi^2(q^3) \pmod{2^2}$$

and

$$4 \frac{f_1^2 f_4^2 f_6^4}{f_2^2 f_{12}^2} + 8 f_1^2 f_2^2 \equiv -4\psi^2(q) \pmod{2^4}.$$

In view of above identities,

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+6)q^n \equiv -4\psi^2(q) \pmod{2^4} \tag{127}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(24n+18)q^n \equiv -4\psi^2(q^3) \pmod{2^4}. \tag{128}$$

It follows from (128) that

$$\mathrm{PDO}(72n+42) \equiv 0 \pmod{2^4} \tag{129}$$

and

$$\mathrm{PDO}(72n+66) \equiv 0 \pmod{2^4} \tag{130}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(72n+18)q^n \equiv -4\psi^2(q) \pmod{2^4}. \tag{131}$$

Employing (10) in (125) and (126) and extracting the terms containing odd powers of q, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+30)q^n \equiv -8\frac{f_2^2 f_3^4 f_8^2}{f_1 f_4 f_6^2} + 16\frac{f_1^3 f_8^2}{f_4} \equiv 8\psi(q)\psi(q^4) \pmod{2^5} \tag{132}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+42)q^n \equiv 8q\frac{f_1^4 f_6^2 f_{24}^2}{f_2^2 f_3 f_{12}} \equiv 8q\psi(q^3)\psi(q^{12}) \pmod{2^5}. \tag{133}$$

In view of (133), we have

$$\mathrm{PDO}(144n+42) \equiv 0 \pmod{2^5} \tag{134}$$

and

$$\mathrm{PDO}(144n+138) \equiv 0 \pmod{2^5} \tag{135}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(144n+90)q^n \equiv 8\psi(q)\psi(q^4) \pmod{2^5}. \tag{136}$$

Substituting (12), (13), and (20) in (123), and extracting the terms containing even and odd powers of q, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO}(24n)q^n \equiv& \frac{f_1^{12} f_2^{10} f_3^4}{f_4^8 f_6^2} + 16q\frac{f_1^{20} f_3^4 f_4^8}{f_2^{14} f_6^2} + 16q\frac{f_2^8 f_4^3 f_6^4}{f_1^{13} f_3 f_{12}} + 8q\frac{f_2^{17} f_6 f_{12}}{f_1^{16} f_4^3} \\ \equiv& \frac{f_3^4 f_4^8}{f_1^{20} f_2^6 f_6^2} + 16q f_4^6 + 16q\frac{f_1^3 f_4^3 f_{12}}{f_3} + 8q f_2 f_4 f_6 f_{12} \pmod{2^5}\end{aligned} \tag{137}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO}(24n+12)q^n &\equiv -8\frac{f_1^{16} f_3^4}{f_2^2 f_6^2} + 4\frac{f_2^{20} f_6^4}{f_1^{17} f_3 f_4^5 f_{12}} + 16 f_2^2 f_4^2 \\ &\equiv 4\frac{f_2^4 f_6^4}{f_1 f_3 f_4 f_{12}} + 8 f_2^2 f_4^2 \pmod{2^5}.\end{aligned} \tag{138}$$

Employing (19) in (138) and extracting the terms involving even and odd powers of q, we get

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+12)q^n \equiv 4\frac{f_1^2 f_4^2 f_6^4}{f_2^2 f_{12}^2} + 8 f_1^2 f_2^2 \pmod{2^5} \tag{139}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}(48n+36)q^n \equiv 4\frac{f_2^4 f_3^2 f_{12}^2}{f_4^2 f_6^2} \pmod{2^5}. \tag{140}$$

In view of (125), (126), (139), and (140), we have

$$\mathrm{PDO}(2(24n+6)) \equiv \mathrm{PDO}(24n+6) \pmod{2^5} \tag{141}$$

and

$$\mathrm{PDO}(2(24n+18)) \equiv -\mathrm{PDO}(24n+18) \pmod{2^5}. \tag{142}$$

Congruence (67) follows from (129), (142), and (66). Congruence (68) follows from (130), (142), and (66).

Substituting (13), (14), and (20) in (137), and extracting even and odd powers of q, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PDO}(48n)q^n &\equiv \frac{f_2^{72}f_6^4}{f_1^{72}f_4^{18}f_{12}^2}+4q\frac{f_2^{66}f_3^2f_{12}^2}{f_1^{70}f_4^{14}f_6^2}-48q\frac{f_1^2f_2^2f_6^4}{f_3^2}\\ &\equiv \frac{f_2^8f_6^4}{f_1^8f_4^2f_{12}^2}+4q\frac{f_2^2f_3^2f_4^2f_{12}^2}{f_1^6f_6^2}+16qf_2f_4f_6f_{12} \pmod{2^5}\end{aligned} \tag{143}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PDO}(48n+24)q^n &\equiv 4\frac{f_2^{69}f_3f_6}{f_1^{71}f_4^{16}}+16\frac{f_2^{60}f_6^4}{f_1^{68}f_4^{10}f_{12}^2}+8f_1f_2f_3f_6\\ &\equiv 12f_1f_2f_3f_6+16f_2^2f_4^2 \pmod{2^5}.\end{aligned} \tag{144}$$

Substituting (18) in (144) and extracting the terms containing even and odd powers of q, we get

$$\sum_{n=0}^{\infty}\mathrm{PDO}(96n+24)q^n \equiv -4\frac{f_1^2f_4^2f_6^4}{f_2^2f_{12}^2}+16f_1^2f_2^2 \pmod{2^5} \tag{145}$$

and

$$\sum_{n=0}^{\infty}\mathrm{PDO}(96n+72)q^n \equiv -12\frac{f_2^4f_3^2f_{12}^2}{f_4^2f_6^4} \pmod{2^5}. \tag{146}$$

From (126) and (146), we have

$$\mathrm{PDO}(2^2(24n+18)) \equiv 3\,\mathrm{PDO}(24n+18) \pmod{2^5}. \tag{147}$$

Employing (10) in (145) and extracting the terms containing odd powers of q, we get

$$\sum_{n=0}^{\infty}\mathrm{PDO}(192n+120)q^n \equiv 8\frac{f_2^2f_3^4f_8^2}{f_1f_4f_6^2} \equiv 8\psi(q)\psi(q^4) \pmod{2^5}. \tag{148}$$

Substituting (13) and (20) in (143), and extracting the terms containing even and odd power of q, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PDO}(96n)q^n &\equiv \frac{f_2^{26}f_3^4}{f_1^{20}f_4^8f_6^2}+16q\frac{f_2^2f_3^4f_4^8}{f_1^{12}f_6^2}+16q\frac{f_2^8f_4^3f_6^4}{f_1^{13}f_3f_{12}}+8q\frac{f_2^{17}f_6f_{12}}{f_1^{16}f_4^3}\\ &\equiv \frac{f_3^4f_4^8}{f_1^{20}f_2^6f_6^2}+16qf_4^6+16q\frac{f_1^3f_4^3f_{12}}{f_3}+8qf_2f_4f_6f_{12} \pmod{2^5}\end{aligned} \tag{149}$$

and

$$\sum_{n=0}^{\infty}\mathrm{PDO}(96n+48)q^n \equiv 8\frac{f_3^4f_2^{14}}{f_1^{16}f_6^2}+4\frac{f_2^{20}f_6^4}{f_1^{17}f_3f_4^5f_{12}}+16f_1f_2f_3f_6$$

$$\equiv 8f_2^2f_4^2+4\frac{f_2^4f_6^4}{f_1f_3f_4f_{12}}+16f_1f_2f_3f_6 \pmod{2^5}. \tag{150}$$

By (149) and (137)

$$\mathrm{PDO}(4(24n)) \equiv \mathrm{PDO}(24n) \pmod{2^5}. \tag{151}$$

Congruence (69) follows from (151) and mathematical induction.

By substituting (19) and (18) in (150), and extracting the terms containing even and odd powers of q,

$$\sum_{n=0}^{\infty}\mathrm{PDO}(192n+48)q^n \equiv 8f_1^2f_2^2+20\frac{f_1^2f_4^2f_6^4}{f_2^2f_{12}^2} \pmod{2^5} \tag{152}$$

and

$$\sum_{n=0}^{\infty}\mathrm{PDO}(192n+144)q^n \equiv -12\frac{f_2^4f_3^2f_{12}^2}{f_4^2f_6^2} \pmod{2^5}. \tag{153}$$

From (125) and (152),

$$\mathrm{PDO}(2^3(24n+6)) \equiv -3\,\mathrm{PDO}(24n+6) \pmod{2^5}. \tag{154}$$

In view of (126) and (153),

$$\mathrm{PDO}(2^3(24n+18)) \equiv 3\,\mathrm{PDO}(24n+18) \pmod{2^5}. \tag{155}$$

Congruence (73) follows from (114), (135), (142), (147), (155), and (69). Congruence (75) follows from (134), (142), (147), (155), and (69).

□

Proof of Theorem 16. Using (86) and (8) in (111),

$$\sum_{n=0}^{\infty}\mathrm{PDO}(12n+3)q^n \equiv 4\psi^2(q) \pmod{2^4}. \tag{156}$$

From (156), (127), (141) and (66),

$$\sum_{n=0}^{\infty}\mathrm{PDO}(2^\alpha(24n+6))q^n \equiv -4\psi^2(q) \pmod{2^4} \tag{157}$$

From (131), (141) and (66),

$$\sum_{n=0}^{\infty}\mathrm{PDO}(2^\alpha(72n+18))q^n \equiv \begin{cases}-4\psi^2(q) \pmod{2^4}, & \text{if } \alpha=0;\\ 4\psi^2(q) \pmod{2^4}, & \text{if } \alpha\neq 0.\end{cases} \tag{158}$$

Combining (156), (157), and (158),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(2^{\alpha} r(12n+3))q^n \equiv \begin{cases} (-1)^{r+1} 4 \sum\limits_{k,m=0}^{\infty} q^{k(k+1)/2+m(m+1)/2} \pmod{2^4}, & \text{if } \alpha = 0; \\ (-1)^{r} 4 \sum\limits_{k,m=0}^{\infty} q^{k(k+1)/2+m(m+1)/2} \pmod{2^4}, & \text{if } \alpha \neq 0. \end{cases} \tag{159}$$

Theorem 16 follows from (159). □

Proof of Theorem 18. From (132), (141), (148), (154) and (69), we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}(2^{\alpha}(48n+30))q^n \equiv 8\psi(q)\psi(q^4) \pmod{2^5}. \tag{160}$$

From (136), (142), (147), (155) and (69), we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}(2^{\alpha}(144n+90))q^n \equiv \begin{cases} 8\psi(q)\psi(q^4) \pmod{2^5}, & \text{if } \alpha = 0; \\ -8\psi(q)\psi(q^4) \pmod{2^5}, & \text{if } \alpha \neq 0. \end{cases} \tag{161}$$

Theorem 18 follows from (160) and (161). □

Proofs of Theorems 20, 22, 24 and 26. From (99),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(96n+44)q^n \equiv 8f_2\frac{f_6^2}{f_3} \pmod{2^4}. \tag{162}$$

Replacing n by $8n+3$ in (62), we see that for all $\alpha, n \geq 0$,

$$\mathrm{PDO}(4^{\alpha}(96n+44)) \equiv \mathrm{PDO}(96n+44) \pmod{2^4}. \tag{163}$$

In view of (8), (91), (162) and (163),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(4^{\alpha}(24n+11)q^n \equiv 8f_2\psi(q^3) \pmod{2^4}. \tag{164}$$

Theorem 20 follows from (164). From (100),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(192n+152)q^n \equiv 8f_1\frac{f_{12}^2}{f_6} \pmod{2^4}. \tag{165}$$

Replacing n by $16n+12$ in (62), we see that for all $\alpha, n \geq 0$,

$$\mathrm{PDO}(4^{\alpha}(192n+152)) \equiv \mathrm{PDO}(192n+152) \pmod{2^4}. \tag{166}$$

In view of (8), (93), (165) and (166),

$$\sum_{n=0}^{\infty} \mathrm{PDO}(4^{\alpha}(48n+38)q^{n} \equiv 8f_1\psi(q^6) \pmod{2^4}. \tag{167}$$

Theorem 22 follows from (167). Theorem 24 follows from (9) and (104). Theorem 26 follows from (9) and (105). □

The proofs of Corollaries 17, 19, 21, 23, 25 and 27 are similar to the proof of Corollary 12; hence we omit the details.

Proof of Theorem 1. Replacing n by $2n+1$ in (28),

$$\mathrm{PDO}(2^{\alpha+2}3^{\beta}(72n+66)) \equiv 0 \pmod{9}. \tag{168}$$

Congruence (5) follows readily from (80) and (168). Replacing n by $4n+3$ in (28),

$$\mathrm{PDO}(2^{\alpha+2}3^{\beta}(144n+138)) \equiv 0 \pmod{9}. \tag{169}$$

Congruence (6) follows readily from (83) and (169). □

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