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On Some Conjectures about Arithmetic Partial Differential Equations

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Abstract

In this paper, we study the arithmetic partial differential equations $x'_p = ax^n$ and $x'_p = a$. We solve a conjecture of Haukkanen, Merikoski, and Tossavainen (HMT, in short) about the number of solutions (conjectured to be finite) of the equation $x'_p = ax^n$ and improve a theorem of HMT about finding the solutions of the same equation. Furthermore, we also improve another theorem of HMT about the solutions of the equation $x'_p = a$ and discuss one more conjecture of HMT about the number of solutions of $x'_p = a$.

1 Introduction

Let the symbols $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ have their usual meaning. We follow the notation used by Haukkanen, Merikoski, and Tossavainen [1] (HMT, in short), except for $\mathbb{N}$, which here denotes the set of positive integers $\{1, 2, \ldots\}$. We use $\mathbb{P} = \{2, 3, 5, 7, \ldots\}$ for the set of all prime numbers. Let $a \in \mathbb{Q} \setminus \{0\}$. Then there are unique $L \in \mathbb{Z}$ and $M \in \mathbb{Q} \setminus \{0\}$ such that $a = Mp^L$ and $p \nmid M$. The *arithmetic partial derivative* of $a \in \mathbb{Q} \setminus \{0\}$, denoted by a'_p, is defined by HMT [1] as follows:

$$a'_p = MLp^{L-1}.$$

A comprehensive list of references is given in [1] for the readers about the history of the arithmetic derivatives and their several generalizations.

In this paper, we study the arithmetic partial differential equations $x'_p = ax^n$, and $x'_p = a$. In Section 2, we resolve Conjecture 29 of HMT [1] about the finiteness of the number of solutions of the equation $x'_p = ax^n$, and give an efficient algorithm (Theorem 2) to find the solutions of the same equation in Section 3. In Section 4, we improve [1, Theorem 1] concerning the solutions of the equation $x'_p = a$, and give some necessary and sufficient conditions for certain nontrivial solutions in Theorems 3 and 4, respectively. Further, we discuss HMT's Conjecture 27 about the number of solutions of $x'_p = a$ and, based on our findings, we hypothesize that this conjecture is false.

2 Number of solutions of $x'_p = ax^n$

Theorem 1. *The solution set of the equation $x'_p = ax^n$, $a \in \mathbb{Q} \setminus \{0\}$, $p \in \mathbb{P}$, $n \in \mathbb{Z} \setminus \{0, 1\}$, is finite.*

Proof. If we look at the equation, we observe that an obvious solution to the equation is at $x = 0$, provided $n > 0$, for each prime number p. We ignore this solution as a trivial solution and consider only non-zero solutions for the equation. Express x as $x = \beta p^\alpha$, $p \nmid \beta$, $\alpha \in \mathbb{Z} \setminus \{0\}$, p being a prime number. Then $x'_p = \beta\alpha p^{\alpha-1}$. As we have $x'_p = ax^n$, we get $\beta\alpha p^{\alpha-1} = a\beta^n p^{n\alpha}$, which implies

$$\left(\frac{\beta^{n-1}a}{\alpha}\right)(p^{(n-1)\alpha+1}) = 1.$$

Write $a = Mp^L$, $M \in \mathbb{Q} \setminus \{0\}$, $p \nmid M$, $L \in \mathbb{Z}$, and $\alpha = \alpha_0 p^R$, $\alpha_0 \in \mathbb{Z}$, $R \in \mathbb{N}$, $p \nmid \alpha_0$. Then, we get

$$\left(\frac{Mp^L\beta^{n-1}}{\alpha_0 p^R}\right)\left(p^{(n-1)\alpha+1}\right) = 1$$

or

$$\left(\frac{M\beta^{n-1}}{\alpha_0}\right)\left(p^{(n-1)\alpha+1+L-R}\right) = 1.$$

Since $p \nmid \left(\frac{M\beta^{n-1}}{\alpha_0}\right)$, we have

$$(n-1)\alpha + 1 + L - R \;=\; 0, \tag{1}$$

and

$$\frac{M\beta^{n-1}}{\alpha_0} \;=\; 1. \tag{2}$$

Substituting $\alpha = \alpha_0 p^R$ in (1), we get

$$(n-1)\alpha_0 p^R + 1 + L - R \;=\; 0. \tag{3}$$

Equation (2) plays an important role in determining the solution set and proving its finiteness. We first concentrate on the term $\alpha = \alpha_0 p^R$ in the solution $x = \beta p^\alpha$, and prove that only a finite number of values of R are possible for which α forms the solution x of the equation. Then, through equation (2), we conclude that the number of corresponding values of β is also finite, as M is a constant. We consider two separate cases for $R = 0$, and for $R \neq 0$.

Case 1: $(R = 0)$. From (3) we have that $(n-1)\alpha_0 + 1 + L = 0$, which implies

$$\alpha_0 = -\left(\frac{1+L}{n-1}\right).$$

As α_0 is an integer, we get $(n-1)|(1+L)$. We remark here that if $(n-1) \nmid (1+L)$, then we do not get any solution in this case.

Case 2: $(R \neq 0)$. We rewrite equation (3) as

$$(n-1)\alpha_0 = \frac{R-1-L}{p^R}. \tag{4}$$

Since $n, \alpha_0 \in \mathbb{Z}$, we have $(n-1)\alpha_0 \in \mathbb{Z}$. Moreover, as $R \neq 0$, so $R \in \mathbb{N}$. We further divide this case into the following two subcases.

Case 2.1: $(R = 1+L)$. From equation (4), we get $(n-1)\alpha_0 = 0$. Since $n \neq 1$, hence $\alpha_0 = 0$ implies that $\alpha = 0$. Thus, the only possible value of α is 0.

Case 2.2: $(R \neq 1+L)$. Clearly, if R is not bounded, then there exists an $R_0 \in \mathbb{N}$ such that the right-hand side expression of (4) becomes a fraction for $R \geq R_0$, which is not possible. Hence, R can attain only a finite number of values. So, a necessary condition on R for a solution is $(n-1)|\left(\frac{R-1-L}{p^R}\right)$.

We get a value of $\alpha_0 = \frac{R-1-L}{(n-1)p^R}$ corresponding to every value of R, which satisfies the above condition. We thus obtain finite number of pairs $(\alpha_0,\ R)$ giving finite number of values of $\alpha = \alpha_0 p^R$ at which the solution is possible.

So far, we have analyzed all possible values of R and have come to the conclusion that only finite number of values of R are possible which may form the solution $x = \beta p^\alpha$ with $\alpha = \alpha_0 p^R$. Now, we need to prove that the corresponding values of β also form a finite set.

Clearly, by (2), we can write $\beta = \left(\frac{\alpha_0}{M}\right)^{\frac{1}{n-1}}$. Hence, we conclude that for a given value of α_0, at most two values of β are possible. As $\beta \in \mathbb{Q}$, the quantity $\left(\frac{\alpha_0}{M}\right)^{\frac{1}{n-1}}$ must be a rational number of the form $\left(\frac{E}{F}\right)$, $F \neq 0$, $E, F \in \mathbb{Z}$. So this acts as a filtering condition on α_0 to further qualify for the solution set. So, we get a final condition on α to be satisfied so that α_0 and the corresponding value of R can give us a solution of the equation. This proves that there exist only a finite number of values of β corresponding to every value of α_0 or α, which themselves have finite possible values for the solution set. Hence, $x = \beta p^\alpha$ has only finitely many solutions. $\square$

3 Solutions of $x'_p = ax^n$

In this section, we find all solutions of the equation $x'_p = ax^n$, $a \in \mathbb{Q}\backslash\{0\}$, $p \in \mathbb{P}$, $n \in \mathbb{Z}\backslash\{0,1\}$. The derivation of the solutions following the notation of Section 2 is given below.

Let us recall equation (3) and consider again two separate cases for $R = 0$, and $R \neq 0$.

Case 1: ($R = 0$). We get a solution if $(n-1) \mid (1+L)$, by the argument used in Theorem 1.

Case 2: ($R \neq 0$). The basic approach for the derivation is to consider the cases for the values of α_0 such that either $(n-1)\alpha_0 > 0$ or $(n-1)\alpha_0 < 0$ or $(n-1)\alpha_0 = 0$, where n is a constant and the sign of α_0 depends upon the sign of $(n-1)$. The upper and lower bounds for the possible values of R have been derived in all the cases through which we can get corresponding β and can form the solution. The necessary condition to be satisfied by R is that on substituting it in equation (3), α_0 must come out to be an integer. If not, then that value is ignored and we proceed to a next value in the range. This condition acts as a filtering condition for the values of R.

From equation (3), it is clear that $1 + L - R \equiv 0 \pmod{p}$. Since $1 + L$ is a constant, we have $1 + L \equiv 0 \pmod{p}$ implies that $R \equiv 0 \pmod{p}$, and $1 + L \not\equiv 0 \pmod{p}$ implies that $R \not\equiv 0 \pmod{p}$. We can further reduce the solution ranges derived for each cases by examining the above two cases. So, we discuss below each subcase one by one.

Case 2.1: ($(n-1)\alpha_0 > 0$). Clearly, $(n-1)\alpha_0 p^R > 0$ for all R and p. We have $p^R > R$ for all $R \in \mathbb{N}$. Clearly, $(n-1)\alpha_0 \in \mathbb{Z}$. So, $(n-1)\alpha_0 p^R - R > 0$ for all R. By (3),

$$(n-1)\alpha_0 p^R - R \;=\; -1 - L. \tag{5}$$

We get $-1 - L > 0$ or $L < -1$. At $L \geq -1$, this case does not give any solution. Now, there are two possibilities.

Case 2.1.1: ($(n-1)\alpha_0 = 1$). Clearly, $(n-1)\alpha_0 = 1$ implies that $n = 2$, and $\alpha_0 = 1$, as $n \in \mathbb{Z} \setminus \{0,1\}$. Introducing a new variable $K = -1 - L$ and combining it with the equation (5), we get $R + K = p^R$, which implies

$$R = \log_p(R + K). \tag{6}$$

This equation gives us a relation, which also gives a filtering condition on R that

$$R + K \equiv 0 \pmod{p}. \tag{7}$$

We can rewrite equation (6) in the following two ways:

$$R \;=\; \log_p R + \log_p(1 + K/R). \tag{8}$$

$$R = \log_p K + \log_p(1 + R/K). \tag{9}$$

Now, we consider three cases for the values of R and examine in each case the possibility and range for the solution.

Case 2.1.1.1: $(R > K)$. $R > K$ implies that $\log_p(1+K/R) < 1$. So, $R = \log_p R + \log_p(1 + K/R) \Rightarrow R < \log_p R + 1$, which implies that $p^{R-1} < R$. Clearly, this does not hold for any values of p and R. So, we cannot get any solution in this case.

Case 2.1.1.2: $(R = K)$. Substituting $R = K$ in (8) or in (9), we get $R = \log_p K + \log_p 2$ or $R = \log_p(2R)$, which implies $p^R = 2R$. This relation is possible only for $p = 2$ and $R = 1$. So, for $R = K$, we can expect a solution only if $p = 2$ and $R = 1$. In this case, $K + R \equiv 0 \pmod p$ is always satisfied. So, this case may yield a solution when $p = 2$.

Case 2.1.1.3: $(R < K)$. Clearly, $R < K$ implies that $\log_p(1 + R/K) < 1$. So, $R = \log_p K + \log_p(1 + R/K) \Rightarrow R < \log_p K + 1$, which gives $R \in \{1, 2, \ldots, \lceil \log_p K \rceil\}$. So, the feasible values of R at which we may get the solution must lie in the set $\{1, 2, \ldots, \lceil \log_p K \rceil\}$. Further, $R + K \equiv 0 \pmod p$ must be satisfied. So, we take only those values of R which are in the set $\{1, 2, \ldots, \lceil \log_p K \rceil\}$ and satisfy $R + K \equiv 0 \pmod p$.

Case 2.1.2: $((n - 1)\alpha_0 \neq 1)$. Rewrite equation (5) as $(n - 1)\alpha_0 p^R = R + K$. Clearly, $(n - 1)\alpha_0 > 1$ implies that $R + K > p^R$, which implies $R < \log_p(R + K)$.

We can rewrite the above inequality in the following two ways:

$$R < \log_p R + \log_p(1 + K/R), \tag{10}$$

$$R < \log_p K + \log_p(1 + R/K). \tag{11}$$

Again proceeding in the same way as in the last case, we take the following three cases:

Case 2.1.2.1: $(R > K)$. Clearly, $R > K$ implies that $\log_p(1 + K/R) < 1$. So, $R < \log_p R + \log_p(1 + K/R) \Rightarrow R < \log_p R + 1$, which implies that $p^{R-1} < R$, which is not possible. So, we do not get any solution in this case.

Case 2.1.2.2: $(R < K)$. Clearly, $R < K$ implies that $\log_p(1 + R/K) < 1$. So, $R < \log_p K + \log_p(1 + R/K) \Rightarrow R < \log_p K + 1$. That is, $R \in \{1, 2, \ldots, \lceil \log_p K \rceil\}$. Further, $R + K \equiv 0 \pmod p$ must be satisfied. So, we only take those values of R which are in the set $\{1, 2, \ldots, \lceil \log_p K \rceil\}$ and satisfy $R + K \equiv 0 \pmod p$.

Case 2.1.2.3: $(R = K)$. Substituting $R = K$ in (10) or in (11), we get $R < \log_p K + \log_p 2$ or $R < \log_p(2R)$, which implies $p^R < 2R$. This inequality cannot be satisfied for any values of R and p in their respective domains.

Thus, we see that if $(n - 1)\alpha_0 > 0$, then we get solutions only for $R < K$ and $R = K$ (provided $p = 2$, and $R = 1$).

Case 2.2: ($(n-1)\alpha_0 < 0$). Rewrite equation (5) as

$$(n-1)\alpha_0 p^R \;\; = \;\; R + K. \tag{12}$$

Then $R-(n-1)\alpha_0 p^R > 0$, because $(n-1)\alpha_0 < 0$. This implies $K < 0$ or $L > -1$. As $(n-1)\alpha_0 < 0$, we have $(n-1)\alpha_0 p^R < 0$. So, we get $R+K < 0$ or $L > R-1$. Thus, we get two conditions: $L > -1$, and $R < 1+L$ for the feasibility of this case.

By introducing two new variables F and W, both of them are positive and such that $(n-1)\alpha_0 = -F$, and $K = -W$, we rewrite equation (12) as

$$R + Fp^R \;\; = \;\; W, \tag{13}$$

where all of W, F, and R are greater than zero.

Clearly, since $W > Fp^R$, we have $W > p^R$ or $R < \log_p W$ or $R < \log_p(-K)$ or equivalently, $R < \log_p(1+L)$.

Thus we get an upper bound for the possible values of R, in the given case $R \in \{1, 2, \ldots, \lceil \log_p K \rceil\}$. Further, $R + K \equiv 0 \pmod{p}$ must be satisfied. So, we take only those values of R which are in the set $\{1, 2, \ldots, \lceil \log_p K \rceil\}$ and satisfy $R + K \equiv 0 \pmod{p}$.

Case 2.3: ($(n-1)\alpha_0 = 0$). Clearly, we have $\alpha_0 = 0$ as $n \neq 1$. So $\alpha = 0$ in this case.

Now that we have the final ranges for the values of R in each case, so, we can find the possible values of $\alpha = \alpha_0 p^R$. First, we find the value of α_0 corresponding to each R. We accept only those values of R which are inside the range and giving an integral value of α_0, otherwise, reject it. This way, we get the possible values of α_0 and R, which are then used to find corresponding α. Then, substituting the value of α_0 in equation (2), we can find the corresponding value of β. If β comes out to be rational, this means solution exists for the given α_0 and $x = \beta p^\alpha$ is the solution of the equation $x'_p = ax^n$. Otherwise, we test the next value of α_0. This is how the algorithm works.

We summarize above discussion in the following:

Theorem 2. *The equation $x'_p = ax^n$, where $p \in \mathbb{P}$, $a \in \mathbb{Q}\setminus\{0\}$ with $a = Mp^L$, $M \in \mathbb{Q}\setminus\{0\}$, $p \nmid M$, $L \in \mathbb{Z}$ has a nontrivial solution $(0 \neq)x = \beta p^\alpha$, $p \nmid \beta$, $\alpha \in \mathbb{Z}\setminus\{0\}$ with $\alpha = \alpha_0 p^R$, $\alpha_0 \in \mathbb{Z}$, $R \in \mathbb{N}$, $p \nmid \alpha_0$ if and only if any one of the following conditions hold*

1. *$(n-1) \mid (1+L), \alpha = -\frac{1+L}{n-1}$, and $\beta = (\frac{\alpha}{M})^{\frac{1}{n-1}} \in \mathbb{Q}$.*
2. *$(-2 \neq)L < -1$, $R \in \{1, 2, \ldots, \lceil \log_p(-1-L) \rceil\}$ with $R - 1 - L \equiv 0 \pmod{p}$ such that $\alpha_0 = \frac{R-1-L}{(n-1)p^R} \in \mathbb{Z}$, and $\beta = (\frac{\alpha}{M})^{\frac{1}{n-1}} \in \mathbb{Q}$.*
3. *$L = -2$, $p = 2$, $R = 1$, $\alpha_0(n-1) = 1$, $\beta = (\frac{\alpha}{M})^{\frac{1}{n-1}} \in \mathbb{Q}$.*
4. *$L > -1$, $R \in \{1, 2, \ldots, \lceil \log_p(1+L) \rceil\}$ with $R - 1 - L \equiv 0 \pmod{p}$ such that $\alpha_0 = \frac{R-1-L}{(n-1)p^R} \in \mathbb{Z}$, and $\beta = (\frac{\alpha}{M})^{\frac{1}{n-1}} \in \mathbb{Q}$.*

Furthermore, all solutions are found in this way.

4 Solutions of $x'_p = a$

In this section, we discuss the solutions of $x'_p = a$. Let us express a in the form Mp^L with $p \nmid M$, $M \in \mathbb{Q}$, $L \in \mathbb{Z}$. We improve Theorem 1 of [1] and give a better bound for the solution range. An "alternate step" approach has been introduced to reach the solution even faster.

Let $y = x/M$, so that $y'_p = p^L$. Following Theorem 1 of [1], we start with the sets I_0 and I exactly same as in Theorem 1 of [1] depending on whether $L > 0$, $L < 0$ or $L = 0$. We then improve the set I_0 and hence improve the set I, which is the candidate for the solutions.

Case 1: ($L > 0$). Let $I_0 = \{0, 1, 2, \ldots, L-1\}$, and $I = \{i \in I_0 : p^{i+1} || (L-i)\}$. Then Theorem 1 of [1] implies that $y = \frac{p^{L+1}}{L-i}$ is a solution of the equation $y'_p = p^L$ for each $i \in I$. Besides this, there is one more possibility of a solution at when $p \nmid (L+1)$, giving $y = \frac{p^{L+1}}{L+1}$ as a solution. We concentrate only on positive values of i in I_0. We can test separately for the possibilities at $i = 0$ and at when $p \nmid (L+1)$.

We first derive a necessary condition for the existence of at least one solution of the equation for the positive values of i. Suppose that there exists a solution at i and $p^{i+1} || (L-i)$. Let us write $L - i = Cp^{i+1}$, where $p \nmid C$, $C \in \mathbb{N}$. Then

$$i + Cp^{i+1} \quad = \quad L. \tag{14}$$

Since $C \geq 1$, and $i > 0$, we have $L > Cp^{i+1} > p^{i+1}$. This implies that $i + 1 < \log_p L$ or $i < \log_p L - 1$.

So, we get a new upper limit for the value of i in I_0, which is $\lceil \log_p L \rceil - 2$. So, now we replace I_0 by a much smaller set $\{1, 2, \ldots, \lceil \log_p L \rceil - 2\}$. The new upper bound is of logarithmic order of L and thus it will be much easier to work with. Also a necessary condition for the existence of a solution for given p and L is

$$L \geq 1 + p^2,$$

which follows from equation (14). If $L < 1 + p^2$, then we do not get any solution for $i > 0$. So, we can have at most two solutions for the given equation: one for $i = 0$, and another in the case $p \nmid (L+1)$.

Beginning with $i = 1$, we start testing whether it is included in the set I. Let $i = i_0$ be some value of i that satisfies the condition for inclusion in the set I. Now, we derive the condition for the possibility of getting an alternate solution for a value of i, higher than that of i_0 and the step size from the initial value i_0 at which we can get another i, so that we do not have to traverse each and every value of $i \in I$ till $\lceil \log_p L \rceil - 2$. Sometimes, even $\log_p L$ may be large. In such cases, the step size method described below helps in reducing the work greatly.

Since we get a solution at $i = i_0$, we have

$$i_0 + C_0 p^{i_0+1} \quad = \quad L, \tag{15}$$

where $p \nmid C_0$. Let the alternate solution exist at $i = i_1$. So, we have

$$i_1 + C_1 p^{i_1+1} \quad = \quad L, \tag{16}$$

where $p \nmid C_1, i_1 > i_0$. From equation (15), we have

$$C_0 < \frac{L}{p^{i_0+1}}. \tag{17}$$

From equations (15) and (16), we get

$$i_0 + C_0 p^{i_0+1} = i_1 + C_1 p^{i_1+1}$$

$$\Rightarrow i_1 = i_0 + p^{i_0+1}(C_0 - C_1 p^{i_1-i_0}). \tag{18}$$

We have $p \mid C_1 p^{i_1-i_0}$, $p \nmid C_0$. Hence, $p \nmid (C_0 - C_1 p^{i_1-i_0})$. Let $K = C_0 - C_1 p^{i_1-i_0}$. Then

$$i_1 = i_0 + K p^{i_0+1}, \quad p \nmid K. \tag{19}$$

So, we conclude that the candidate of $i \in I$ for the alternate solution is in the form of (19). The step size is Kp^{i_0+1}, where $K > 0$ and not divisible by p.

From equation (19), $i_1 - i_0 = Kp^{i_0+1}$. Now, since $i_1 > i_0$, $i_1 - i_0 > 0$, we have $K > 0$ or $C_0 - C_1 p^{i_1-i_0} > 0$. This gives $C_1 < \frac{C_0}{p^{i_1-i_0}}$. Hence, $C_1 \geq 1 \Rightarrow \frac{C_0}{p^{i_1-i_0}} > 1$, which implies

$$i_1 - i_0 < \log_p C_0. \tag{20}$$

Combining (17), (19), and (20), we get

$$K < \frac{1}{p^{i_0+1}} \log_p \left(\frac{L}{p^{i_0+1}} \right). \tag{21}$$

We get an upper bound for the number of steps in terms of Kp^{i_0+1}, within which we can expect an alternate solution of the equation, once we get an initial solution. Starting from $K = 1$, we traverse till the upper bound in (21). As $p \nmid K$, we also exclude all those values which are divisible by p. Here, we introduce a new set called *Alternate Step Range Set* or *ASR*, in short, containing the possible values of K for a given i_0. Let $U = \frac{1}{p^{i_0+1}} \log_p \left(\frac{L}{p^{i_0+1}} \right)$. Then

$$ASR = \{1, 2, \ldots, \lfloor U \rfloor\} \setminus \{p, 2p, \ldots\}.$$

By iterating through the ASR set, we can get the alternate solution of the equation within very few steps. Once we reach the alternate solution, say at $i = i_1$, we repeat the same steps and form the ASR range using $i = i_1$, which will then be used to get next higher value of i. We stop this process when we do not get an alternate solution. Under computational limits, this method is highly efficient in reaching all the solutions.

We now derive a necessary condition for the existence of an alternate solution, given that a solution exists at $i = i_0$. If an alternate solution exists, the minimum value of K must be 1. So, we get

$$1 < \tfrac{1}{p^{i_0+1}} \log_p \left(\tfrac{L}{p^{i_0+1}}\right).$$

From the above relation we get a necessary condition for the existence of an alternate solution for i greater than the given initial value i_0 as

$$L > p^{\left(p^{i_0+1}+i_0+1\right)}. \tag{22}$$

Thus, we get a new condition for the existence of the alternate solution for $i_0 \in \mathbb{N}$. If inequality (22) is not satisfied, this means that there exists no solution for $i > i_0$. Moreover, $i_0 \geq 1$, so putting i_0 in (22), we conclude that if $L \leq p^{(p^2+2)}$, then we cannot have more than one solution for the positive values of i. In such a situation, we can get at most three solutions of the partial differential equation, one in this range and the other two for $i = 0$, and for $p \nmid (L+1)$.

Case 2: $(L < 0)$. Let $I_0 = \mathbb{N} \cup \{0\}$, and I is same as in Case 1. One can test separately at $i = 0$, and for $p \nmid (L+1)$. So, we take only the positive values of i. Let $L = -Q$. Then $p^{i+1} || (L-i)$ or $p^{i+1} || (Q+i)$. Write

$$Q + i \;\; = \;\; Cp^{i+1},\; p \nmid C, \quad C > 0. \tag{23}$$

We now derive the condition for the existence of a solution for this range. Rewrite equation (23) as

$$\frac{Q}{p^{i+1}} + \frac{i}{p^{i+1}} \;\; = \;\; C. \tag{24}$$

Since $0 < \frac{i}{p^{i+1}} < 1$, we have $\frac{Q}{p^{i+1}} > C - 1$. This implies that $\frac{Q}{C-1} > p^{i+1}$.

Here, $(C-1)$ is in the denominator, so, one can test separately at $C = 1$ and for the rest of the cases, we assume $C > 1$. At $C > 1$, $\frac{Q}{C-1} < Q$. So, we get $p^{i+1} < Q$, which gives

$$i < \log_p Q - 1. \tag{25}$$

Thus, we get an upper bound on the value of i, which is $\lceil \log_p Q \rceil - 2$. So, the infinite set I_0 has now been reduced to $I_0 = \{1, 2, \ldots, \lceil \log_p Q \rceil - 2\}$. Also, $Q > (C-1)p^{i+1}$, so for the existence of a solution at $C > 1$, $Q > p^{i+1}$. The minimum value of i may be 1, so a necessary condition for the existence of a solution is $Q > p^2$ or $L < -p^2$.

Now, we examine the range where an alternate solution is possible and also derive the possibility of an alternate solution.

Let there exists a solution at $i = i_1$. Here, we consider i_1 to be the highest value of i at which solution is possible and consider the alternate solution at some smaller value of i, unlike the previous case, where we considered alternate solution for the higher value of i and started with a smaller value of i. So, let an alternate solution exist at $i = i_2$. Hence, we have the following two equations.

$$Q + i_1 \;\; = \;\; C_1 p^{i_1+1}, \quad p \nmid C_1,$$

and

$$Q + i_2 = C_2 p^{i_2+1}, \quad p \nmid C_2.$$

Hence,

$$i_1 - i_2 \;\; = \;\; p^{i_2+1}(p^{i_1-i_2}C_1 - C_2).$$

Put $K = p^{i_1-i_2}C_1 - C_2$ with $K \geq 1$. We get $i_1 - i_2 = Kp^{i_2+1}$, $p \nmid K$, which implies $Kp^{i_2+1} < i_1$, $K \geq 1$. Hence,

$$i_2 < \log_p i_1 - 1. \tag{26}$$

So, we get a relation that for a given i_1, which forms the solution, an alternate solution to it exists somewhere between 0 and the upper bound $\log_p i_1 - 1$, which depends on the value of i_1 itself. Thus, this reduces the search for the alternate solution. We repeat the same algorithm for getting the next alternate solution and so on, till the range of i permits.

We also derive a necessary condition for the existence of an alternate solution once we have a solution at $i = i_1$, for $C > 1$. Considering the inequality (26), we put $i_2 = 1$, as this would be the minimum value of i_2, in case it exists. So, we get $1 < \log_p i_1 - 1$ or $i_1 > p^2$.

So if $i \leq p^2$, we terminate the process as there will not be any alternate solution at a smaller value of i.

Now, we derive a necessary condition for the existence of at least two solutions for $C > 1$. Considering inequality (26), we put $i_2 = 1$, as this would be the minimum value of i_2, in case it exists, and for i_1, we substitute $i_1 < \log_p Q - 1$. We get

$$1 < \log_p i_1 - 1 \Rightarrow \qquad 2 < \log_p(\log_p Q - 1) \Rightarrow p^2 + 1 < \log_p Q \Rightarrow \qquad Q > p^{(p^2+1)}.$$

This gives a necessary condition to have at least two solutions for $C > 1$.

Case 3: $(L = 0)$. Clearly, $y'_p = 1 \Rightarrow y = p$ is the only solution.

Now, we have the values of i for which we have the solution for $y_p = p^L$. We can get the corresponding solution of the equation $x'_p = a$, by multiplying M to the solution obtained through the above methods, since $y = \frac{x}{M} \Rightarrow x = My$, we have $x'_p = My'_p$.

Now we restate the improved version of [1, Theorem 1] and give another theorem (using the notation used in the discussion) about the nature of solutions of $x'_p = p^L$, which is the outcome of the above discussion.

Theorem 3. *Let $p \in \mathbb{P}$ and $L \in \mathbb{Z}$. Further, let $I_0 = \{1, 2, \ldots, \lceil \log_p L \rceil - 2\}$ for $L > 0$, $I_0 = \{1, 2, \ldots, \lceil \log_p(-L) \rceil - 2\}$ for $L < 0$, and $I_0 = \emptyset$ for $L = 0$. Let also $I = \{i \in I_0 : p^{i+1} || (L - i)\}$. Then $x = \frac{p^{L+1}}{L-i}$ is a solution of $x'_p = p^L$ for each $i \in I$. If $p \nmid (L+1)$, then also $x = \frac{p^{L+1}}{L+1}$ is a solution. All solutions are obtained in this way. The only solution of $x'_p = 1$ is $x = p$. The equation $x'_p = 0$ holds if and only if $p \nmid x$.*

Theorem 4. *1. Let $L > 0$.*

(i) A necessary and sufficient condition for the existence of a solution of $x'_p = p^L$ in the case $i > 0$, where $i \in I$, is $L \geq 1 + p^2$.

(ii) A necessary and sufficient condition for the existence of at least two solutions of $x'_p = p^L$ in the case $i > 0$, where $i \in I$, is $L > p^{p^{i_0+1}+i_0+1}$ provided the first solution is obtained at $i_0 \in I$.

2. Let $L < 0$.

(i) A necessary and sufficient condition for the existence of a solution of $x'_p = p^L$ in the case $i > 0$, where $i \in I$, is $-L > p^2$.

(ii) A necessary and sufficient condition for the existence of at least two solutions of $x'_p = p^L$ in the case $i > 0$, where $i \in I$, is $-L > p^{p^2+1}$.

In the remark given below, we discuss about the possibilities of the number of solutions of $x'_p = a$. Through this discussion, we have a strong belief that Conjecture 27 of [1] is false.

Remark 5. The maximum number of possible solutions may be greater than four, as is evident from the algorithm that on increasing the value of L, we have a higher range with more number of testing steps in the alternating sequence range. Two solutions are possible at $i = 0$, and when $p \nmid (L + 1)$. Then, for the positive values of i, we have derived the minimum positive value or maximum negative value for L, so as to have at least one solution and an alternate solution. The possibility of two solutions exists for any value of L, except at $L = 0$, where only one solution is possible. At negative values of L, we have one more case, namely, $C = 1$. So, for negative L, we already get a possibility of existence of three solutions. We concentrate on the positive values of i for further possibilities.

For $p = 2$, the minimum value of L must be 5 in order for of three or more solutions to exist. If, L is negative, its maximum value must be -5, in order for three or more solutions to exist. Further, $L > 2^{(2^2+2)} = 64$, for the possible existence of at least one alternate solution, given that $L > 0$, which will also form the fourth solution. New solutions are possible, if we further increase the value of L.

Similarly, for $p > 2$, we can easily test for first three solutions, but for the alternate solution, the minimum value of L is 3^{11} for $p = 3$, and 5^{26} for $p = 5$, and so on. Due to such a high value, it is difficult to investigate for further solutions at $p > 2$, but it is quite possible to get more than three solutions if we increase the limit drastically beyond the given values.

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References

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