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On Particular Families of Hyperquadratic Continued Fractions in Power Series Fields of Odd Characteristic

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Abstract

We discuss the form of certain algebraic continued fractions in the field of power series over $\mathbb{F}_p$, where p is an odd prime number. This leads to giving explicit continued fractions in these fields, satisfying an explicit algebraic equation of arbitrary degree $d \geq 2$ and having an irrationality measure equal to d. Our results are based on a mysterious finite sequence of rational numbers.

1 Introduction

We are concerned with continued fractions in the fields of formal power series over a finite field. For a general account on this matter the reader may consult Schmidt's article [13] and also Thakur's book [14]. Let p be a prime number and $\mathbb{F}_q$ the finite field of characteristic p, having q elements. Given a formal indeterminate T, we consider the ring $\mathbb{F}_q[T]$, the field $\mathbb{F}_q(T)$ and $\mathbb{F}_q((T^{-1}))$, here simply denoted by $\mathbb{F}(q)$, the field of power series in $1/T$ over the finite field $\mathbb{F}_q$. A non-zero element of $\mathbb{F}(q)$ can be written as

$$\alpha = \sum_{i \leq i_0} u_i T^i \quad \text{where} \quad i \in \mathbb{Z}, \quad u_i \in \mathbb{F}_q \quad \text{and} \quad u_{i_0} \neq 0.$$

An ultrametric absolute value is defined over this field by $|0| = 0$ and $|\alpha| = |T|^{i_0}$ where $|T|$ is a fixed real number greater than 1. We also consider the subset $\mathbb{F}(q)^+ = \{\alpha \in \mathbb{F}(q) \quad \text{s.t.} \quad |\alpha| > 1\}$. Note that $\mathbb{F}(q)$ is the completion of the field $\mathbb{F}_q(T)$ for this absolute value.

In power series fields over a general finite field $\mathbb{F}_q$, where q is a power of p, contrarily to the case of real numbers, the continued fraction expansion for many algebraic elements can be explicitly given. This phenomenon is due to the existence of the Frobenius isomorphism in these fields. A particular subset of $\mathbb{F}(q)$, denoted by $\mathcal{H}(q)$, containing certain algebraic elements called hyperquadratic, has been considered by Bluher and the author [2]. Let $t \geq 0$ be an integer and $r = p^t$, an irrational element of $\mathbb{F}(q)$ is called hyperquadratic of order t if it satisfies a non-trivial algebraic equation of the following form:

$$uX^{r+1} + vX^r + wX + z = 0 \quad \text{where} \quad (u, v, w, z) \in (\mathbb{F}_q[T])^4.$$

Note that a hyperquadratic element of order 0 is simply quadratic. Quadratic power series, as quadratic real numbers, have an ultimately periodic continued fraction expansion. However $\mathcal{H}(q)$ contains power series of arbitrary large algebraic degree over $\mathbb{F}_q(T)$ and the continued fraction expansion for various elements in this class has also been given explicitly. Note that if a hyperquadratic element has order t, it also has order kt for all integers $k \geq 1$ [2, p. 258]. The consideration of this subset of algebraic elements was first put forward in the study of Diophantine approximation, beginning with Mahler's article [11]. For more on this topic and more references, the reader may also consult the survey of the author [3].

Let us recall that, for an irrational element $\alpha \in \mathbb{F}(q)$, the irrationality measure is defined by:

$$\nu(\alpha) = \limsup_{|Q| \to \infty} (-\log |\alpha - P/Q| / \log |Q|),$$

where P and Q belong to $\mathbb{F}_q(T)$. Then $\nu(\alpha)$ is a real number greater or equal to 2. By adapting a theorem on rational approximation for real numbers, due to Liouville in the 19th century, Mahler [11] proved that, if α is an algebraic element of degree $d > 1$ over $\mathbb{F}_q(T)$, then we have $\nu(\alpha) \in [2; d]$. Furthermore if α is any irrational number in $\mathbb{F}(q)$, having the infinite continued fraction expansion $\alpha = [a_1, a_2, \dots, a_n, \dots]$, then the irrationality measure is directly connected to the sequence of degrees of the partial quotients [3, p. 214] and we have

$$\nu(\alpha) = 2 + \limsup_{n \geq 1} (\deg(a_{n+1}) / \sum_{1 \leq i \leq n} \deg(a_i)).$$

In this note, we present certain particular algebraic continued fractions in $\mathbb{F}(p)$ with odd p, which can be fully described. In Section 2, we recall several generalities on continued fractions. In Section 3, we describe a large family of algebraic continued fractions and, in Section 4, we exhibit two particular sub-families where the sequence of partial quotients is regularly distributed. The importance of these last continued fractions is highlighted in the last section, bringing families of hyperquadratic elements, in $\mathbb{F}(p)$ with odd p, having a prescribed algebraic degree, an explicit continued fraction expansion and an irrationality measure equal to the algebraic degree.

The reader will observe that this paper contains mainly conjectures based on computer calculations. Some results (see Theorem 4 below) were established in previous works, but the aim of this exposition is to point out several mathematical statements remaining largely mysterious and longing for clearness.

2 Continued fractions

Concerning continued fractions in the area of function fields, we use classical notation, as it can be found for instance in the survey of the author [8, pp. 3–8]. Throughout the paper we are dealing with finite sequences (or words), consequently we recall the following notation on sequences in $\mathbb{F}_q[T]$. Let $W = w_1, w_2, \ldots, w_n$ be such a finite sequence, then we set $|W| = n$ for the length of the word W. If we have two words W_1 and W_2, then W_1, W_2 denotes the word obtained by concatenation.

As usual, we let $[W] = [w_1, \ldots, w_n] \in \mathbb{F}_q(T)$ denote the finite continued fraction $w_1 + 1/(w_2 + 1/(\ldots))$. In this formula the w_i, called the partial quotients, are non-constant polynomials. Still, we will also use the same notation if the w_i are constant and the resulting quantity is in $\mathbb{F}_q$. However in this last case, by writing $[w_1, w_2, \ldots, w_n]$ we assume that this quantity is well defined in $\mathbb{F}_q$, i.e., $w_n \neq 0, [w_{n-1}, w_n] \neq 0, \ldots, [w_2, \ldots, w_n] \neq 0$.

For $n \geq 0$, a continuant X_n is a polynomial, in the n variables $x_1, x_2, \ldots, x_n$, defined recursively by $X_0 = 1$, $X_1 = x_1$ and $X_k = x_k X_{k-1} + X_{k-2}$ for $2 \leq k \leq n$. We use the notation $\langle W \rangle$ for the continuant built from $W = w_1, w_2, \ldots, w_n$. In the sequel the $w_i's$ are in $\mathbb{F}_q[T]$, then the degree in T of $\langle W \rangle$ is clearly equal to the sum of the degrees in T of the $w_i's$.

We let W' (resp., W'') denote the word obtained from W by removing the first (resp., last) letter of W. Hence, we recall that we have $[W] = \langle W \rangle / \langle W' \rangle$. We let $W^* = w_n, w_{n-1}, \ldots, w_1$, be the word W written in reverse order. We have $\langle W^* \rangle = \langle W \rangle$ and also $[W^*] = \langle W \rangle / \langle W'' \rangle$.

Moreover, if $y \in \mathbb{F}_q^*$, then we define $y \cdot W$ as the following sequence

$$y \cdot W = yw_1, y^{-1}w_2, \ldots, y^{(-1)^{n-1}}w_n.$$

Then, it is also known that $\langle y \cdot W \rangle = y \langle W \rangle$ (resp., $= \langle W \rangle$) if $|W|$ is odd (resp., if $|W|$ is even) and $[y \cdot W] = y[W]$.

If $\alpha \in \mathbb{F}(q)$ is an infinite continued fraction, $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$, we set $x_n = \langle a_1, a_2, \ldots, a_n \rangle$ and $y_n = \langle a_2, \ldots, a_n \rangle$. In this way, we have $x_n/y_n = [a_1, a_2, \ldots, a_n]$, with $x_1 = a_1$, $y_1 = 1$ and by convention $x_0 = 1$, $y_0 = 0$. We introduce $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$ as the tail of the expansion of the complete quotient ($\alpha_1 = \alpha$), and we have

$$\alpha = (x_n \alpha_{n+1} + x_{n-1})/(y_n \alpha_{n+1} + y_{n-1}) \quad \text{for} \quad n \geq 1.$$

We recall the following general result [4, p. 332].

Proposition 1. *Let p be a prime number, $q = p^s$ and $r = p^t$ with $s, t \geq 1$. Let $\ell \geq 1$ be an integer and $(a_1, a_2, \ldots, a_\ell) \in (\mathbb{F}_q[T])^l$, with $\deg(a_i) > 0$ for $1 \leq i \leq \ell$. Let $(P, Q) \in (\mathbb{F}_q[T])^2$ with $\deg(Q) < \deg(P) < r$. Then there exists a unique infinite continued fraction α in $\mathbb{F}(q)^+$ satisfying*

$$\alpha = [a_1, a_2, \ldots, a_\ell, \alpha_{\ell+1}] \quad \text{and} \quad \alpha^r = P\alpha_{\ell+1} + Q. \tag{$*$}$$

This element α is the unique root in $\mathbb{F}(q)^+$ of the following algebraic equation:

$$y_\ell X^{r+1} - x_\ell X^r + (Py_{\ell-1} - Qy_\ell)X - Px_{\ell-1} + Qx_\ell = 0. \tag{$**$}$$

3 P_k-continued fractions

In this note, to avoid unnecessary sophistication in a first stage, the base field will simply be the finite field $\mathbb{F}_p$ having p elements. Moreover, for the subject treated here we need have odd characteristic. Actually, in many areas concerning power series over a finite field, both cases, even and odd characteristic, must be considered separately. Hence, here p is a prime number with $p > 2$ and throughout this note, k is an integer with $1 \leq k < p/2$.

We shall now describe several numbers and polynomials which appeared in earlier papers. These were introduced in a paper by the author [4], to where the reader is invited to refer for the proof of the properties stated below.

In $\mathbb{F}_p[T]$, we define $P_k(T) = (T^2 - 1)^k$. Next, we consider the following euclidean division in $\mathbb{F}_p[T]$:

$$T^p = A_1 P_k + R_k.$$

The polynomial A_1 is the integer part of the rational T^p/P_k and R_k is the remainder. Hence we have $A_1 = [T^p/P_k]$, here the brackets denote the integer part (i.e., polynomial part) of the rational function. Note that we have $\deg(A_1) = p - 2k$ and also $A_1 = T$ if $k = (p-1)/2$. We define

$$\omega_k = (-1)^{k-1} \prod_{1 \leq i \leq k} (1 - 1/2i) \in \mathbb{F}_p^*.$$

In previous works another polynomial closely related to R_k was used. Defining $Q_k = (2k\omega_k)^{-1}R_k$, it was proved that the following holds in $\mathbb{F}_p[T]$:

$$Q_k(T) = \sum_{0 \leq i \leq k-1} (-1)^{k-1-i} \binom{k-1}{i} (2i+1)^{-1} T^{2i+1}.$$

These formulas are the key to obtain the following. In $\mathbb{F}_p(T)$, we have the remarkable continued fraction expansion [4, pp. 338–341]:

$$T^p/P_k = [A_1, w_1T, w_2T, \ldots, w_{2k}T], \tag{1}$$

where the $w_i \in \mathbb{F}_p^*$ are defined by : $w_1 = (2k-1)(2k\omega_k)^{-1}$ and

$$w_{i+1}w_i = (2k-2i-1)(2k-2i+1)(i(2k-i))^{-1} \qquad \text{for} \quad 1 \leq i \leq 2k-1.$$

Note that the w_i are rational numbers which, for all primes p with the condition $2k < p$, exist by reduction in $\mathbb{F}_p^*$. We also have

$$w_{2k+1-i} = -w_i \quad \text{for} \quad 1 \leq i \leq 2k. \tag{2}$$

In the sequel, we consider the finite word $W = w_1T, w_2T, \dots, w_{2k}T$. Hence, with our notation, (1) implies $T^p/P_k = A_1 + 1/[W]$, and we can write

$$P_k/R_k = [w_1T, w_2T, \dots, w_{2k}T] = [W]. \tag{3}$$

Let us do a thorough investigation of the extremal case: $k = (p-1)/2$. Then, we clearly have $A_1 = [T^p/P_k] = T$ and consequently

$$R_k = T^p - T(T^2-1)^k = T^p - T(T^{2k} - kT^{2k-2} + \cdots) = kT^{2k-1} + \cdots$$

Hence, we get $[P_k/R_k] = (1/k)T = w_1T$. Since $2k = -1$, we have $w_1 = 1/k = -2$. We obtain $\omega_k = (2k-1)/(2kw_1) = -2/(-w_1) = -1$. But we also have $w_{i+1}w_i = -4$. It follows that $W = -2T, 2T, -2T, \cdots, 2T$ [4, Corollary 4.2 with a different notation].

Applying Proposition 1, we consider particular algebraic continued fractions in $\mathbb{F}(p)$ defined as follows.

Definition 2. An infinite continued fraction $\alpha \in \mathbb{F}(p)$ is a P_k-expansion if there exist an integer $\ell \geq 1$, an ℓ-tuple, $(\lambda_1, \lambda_2, \dots, \lambda_\ell) \in (\mathbb{F}_p^*)^\ell$ and $(\epsilon_1, \epsilon_2) \in \mathbb{F}_p^* \times \mathbb{F}_p$ such that

$$\alpha = [\lambda_1T, \lambda_2T, \dots, \lambda_\ell T, \alpha_{\ell+1}] \quad \text{and} \quad \alpha^p = \epsilon_1 P_k \alpha_{\ell+1} + \epsilon_2 R_k.$$

For given integers k, p and ℓ, chosen as above, a P_k-expansion is defined by an $(\ell+2)$-tuple: $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell, \epsilon_1, \epsilon_2)$. Moreover, this continued fraction satisfies the algebraic equation $(**)$, introduced in Proposition 1, with $(P, Q) = (\epsilon_1 P_k, \epsilon_2 R_k)$, whose four coefficients in $\mathbb{F}_p[T]$ are only depending on k, p, ℓ and Λ.

These particular continued fraction expansions, introduced by the author [4], have been considered in several articles. They were particularly studied in a more general setting by the author [6]. In previous publications, we used in this definition the polynomial Q_k instead of R_k. This has no consequence, since both polynomials are proportional. Originally, a particular and simple continued fraction expansion in $\mathbb{F}(13)$, solution of an algebraic equation of degree four, was observed by Mills and Robbins [12]. In order to understand this particular and remarkable pattern, we developed this notion of P_k-expansion. So we could show that Mills and Robbins' example was a particular case, defined as above with $p = 13$, $k = 4$ and $\ell = 6$. This implied its full description [5] and a generalization concerning Robbins' quartic [7, 1]. The possibility of describing the sequence of partial quotients for an arbitrary P_k-expansion is yet out of reach. However, using a technical lemma stated below, we will see in the next section that the sequence of partial quotients for certain P_k-expansions can be given explicitly. This lemma is the following (its origin can be found in a paper of the author [4, p. 336 and p. 343]).

Lemma 3. *Let $A \in \mathbb{F}_p[T]$, $\delta \in \mathbb{F}_p^*$ and $X \in \mathbb{F}(p)$. Then we have*

$$[A + \delta R_k P_k^{-1}, X] = [A, \delta^{-1} \cdot W, XP_k^{-2} + \delta^{-1} R_k P_k^{-1}].$$

Proof. According to (3), we have $P_k/R_k = [w_1T, w_2T, \ldots, w_{2k}T] = [W]$. By (2), we also have $W^* = -1 \cdot W$. We observe the following links between the polynomials P_k, R_k and the continuants built from W. We have

$$P_k/R_k = [W] = \langle W \rangle / \langle W' \rangle.$$

We note that $\langle W \rangle = \langle w_1T, w_2T, \ldots, w_{2k}T \rangle$ and $P_k = (T^2-1)^k$ have the same degree in T equal to $2k$. Besides, the constant term of P_k is $(-1)^k$ while, since W has even length, the constant term of $\langle W \rangle$ is 1. Consequently we obtain

$$\langle W \rangle = (-1)^k P_k \quad \text{and} \quad \langle W' \rangle = (-1)^k R_k. \tag{4}$$

Hence, we can write

$$A + \delta R_k/P_k = A + 1/(\delta^{-1}[W]) = A + 1/[\delta^{-1} \cdot W] = [A, \delta^{-1} \cdot W].$$

We set $U = A, \delta^{-1} \cdot W = u_1, u_2, \ldots, u_n$. Here we have $n = 2k+1$, $u_1 = A$ and $u_{i+1} = \delta^{(-1)^i} w_i T$ for $1 \leq i \leq 2k$. We set $x_n = \langle U \rangle$ and $y_n = \langle U' \rangle$, so that we have $[U] = x_n/y_n$. Let us consider an arbitrary element $b \in \mathbb{F}(p)$. Since $x_n y_{n-1} - y_n x_{n-1} = (-1)^n$, we can write

$$[U, b] - [U] = (x_n b + x_{n-1})/(y_n b + y_{n-1}) - x_n/y_n = (-1)^{n-1}(y_n(y_n b + y_{n-1}))^{-1}.$$

Then we have $y_n = \langle U' \rangle = \langle \delta^{-1} \cdot W \rangle = \langle W \rangle$, since $|W|$, the length of W, is even. But also $y_{n-1} = \langle (U')'' \rangle = \langle \delta^{-1} \cdot W'' \rangle = \delta^{-1} \langle W'' \rangle$, since $|W''|$ is odd. Due to the properties of continuants and by (2), we have

$$\langle W'' \rangle = \langle w_1T, w_2T, \ldots, w_{2k-1}T \rangle = \langle -w_{2k}T, \ldots, -w_2T \rangle = -\langle W' \rangle.$$

Hence, by (4), we get

$$y_n = \langle W \rangle = (1)^k P_k \quad \text{and} \quad y_{n-1} = -\delta^{-1} \langle W' \rangle = (-1)^{k+1} \delta^{-1} R_k.$$

Therefore, we obtain

$$[U, b] - [U] = (P_k^2 b - \delta^{-1} P_k R_k)^{-1}.$$

Choose $b = XP_k^{-2} + \delta^{-1} R_k P_k^{-1}$ or equivalently $X = P_k^2 b - \delta^{-1} P_k R_k$. The last formula gives $[U, b] = [U] + X^{-1}$ and therefrom, we obtain

$$[A, \delta^{-1} \cdot W, XP_k^{-2} + \delta^{-1} R_k P_k^{-1}] = [U] + X^{-1} = [A + \delta R_k P_k^{-1}, X].$$

□

4 Perfect P_k-continued fractions

Our goal is to describe certain P_k-expansions explicitly. This will be done in two different cases, each corresponding to a particular choice of Λ. Because of some change of notation and to highlight the similarity between both cases, we repeat the results concerning the first case, which have already appeared in previous publications. To describe the partial quotients in these continued fractions, we need to introduce in $\mathbb{F}_p[T]$ two sequences of polynomials $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ as follows [1, p. 134]. Note that all these polynomials are monic. The first sequence is defined by

$$A_0 = T \quad \text{and recursively} \quad A_{n+1} = [A_n^p/P_k] \quad \text{for} \quad n \geq 0.$$

Note that, in agreement to the beginning of the previous section, we have $A_1 = [T^p/P_k]$. While the second sequence is defined by

$$B_0 = A_0 = T \quad \text{and} \quad B_1 = A_1 = [T^p/P_k]$$

and recursively

$$B_{n+1} = B_n^p P_k^{(-1)^{n+1}} \quad \text{for} \quad n \geq 1.$$

We are particularly interested in the degrees of these polynomials. We set $u_n = \deg(A_n)$ and $v_n = \deg(B_n)$. From the recursive definition of these polynomials, we get $u_0 = v_0 = 1$ and also

$$u_{n+1} = pu_n - 2k \quad \text{and} \quad v_{n+1} = pv_n + 2k(-1)^{n+1} \quad \text{for} \quad n \geq 0.$$

Note that the sequence $(u_n)_{n\geq 0}$ is constant if $2k = p - 1$, then we have $A_n = A_0 = T$ for $n \geq 0$. Otherwise, both sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are strictly increasing.

The first case is described in the following theorem.

Theorem 4. *Let p be an odd prime and k, ℓ integers chosen as above. Let $\alpha \in \mathbb{F}(p)$ be a P_k-expansion, depending as above on an $(\ell + 2)$-tuple Λ. We assume that $\epsilon_2 \neq 0$ and Λ satisfies the following equality:*

$$[\lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_2, \lambda_1 - \epsilon_2] = \epsilon_1/(\epsilon_2\omega_k). \qquad \mathcal{C}(A)$$

Then there exist a sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_p^*$ and a sequence $(i(n))_{n\geq 1}$ in $\mathbb{N}$, such that

$$a_n = \lambda_n A_{i(n)} \quad \textit{for} \quad n \geq 1.$$

The proof of this theorem appears in two papers of the author [5, p. 1111] and [6, pp. 256–257]; the second paper corresponds to a much larger context. It also has a remark [6, p. 257] concerning a mistake in the writing of a formula in the first paper [5]. This proof is derived from Lemma 3 presented in Section 3. Both sequences $(\lambda_n)_{n\geq 1}$ and $(i(n))_{n\geq 1}$ have been described in the papers cited above [5, 6], sometimes with different notation. We recall here below the description of these sequences, when the base field is prime, which is the only case we consider here.

For $n \geq 1$, we set $f(n) = (2k+1)n + \ell - 2k$. At first, according to $\mathcal{C}(A)$, we can define in $\mathbb{F}_p^*$: $z_1 = \lambda_1 - \epsilon_2$ and $z_i = \lambda_i + z_{i-1}^{-1}$ for $2 \leq i \leq \ell$. Then we consider the sequence $(z_n)_{n\geq 1}$ in $\mathbb{F}_p^*$ defined, from the initial values $z_1, z_2, \ldots, z_\ell$, by the recursive formulas

$$z_{f(n)+i} = \theta_i \epsilon_1^{(-1)^{n+i}} z_n^{(-1)^i} \quad \text{for } n \geq 1 \quad \text{and} \quad 0 \leq i \leq 2k,$$

where

$$\theta_0 = -\omega_k \quad \text{and} \quad \theta_i = -iw_i/(2k-2i+1) \quad \text{for} \quad 1 \leq i \leq 2k.$$

Then the sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_p^*$ is defined recursively, from the initial values $\lambda_1, \lambda_2, \ldots, \lambda_\ell$, by the formulas

$$\lambda_{f(n)} = \epsilon_1^{(-1)^n} \lambda_n \quad \text{and} \quad \lambda_{f(n)+i} = \epsilon_1^{(-1)^{n+i}} w_i z_n^{(-1)^i},$$

for $n \geq 1$ and for $1 \leq i \leq 2k$.

It is worth mentioning that the complexity of this sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_p$ has been studied. Indeed, it was proved to be $(2k+1)$-automatic. For a full account on this matter, in the most general setting, the reader is advised to refer to a paper of Yao and the author [10, Section 5].

Concerning the sequence $(i(n))_{n\geq 1}$, we have the following description:

$$i(n) = 0 \quad \text{if} \quad n \notin f(\mathbb{N}^*) \quad \text{and} \quad i(f(n)) = i(n) + 1 \quad \text{for} \quad n \geq 1.$$

We want to compute the irrationality measure for a continued fraction described in Theorem 4. We only need to know the sequence $(i(n))_{n\geq 1}$ and the sequence $(\deg(A_n))_{n\geq 0}$. The proof of the following corollary is based on another equivalent description [1, p. 143] of this sequence $(i(n))_{n\geq 1}$. In the sequel, we will use the following notation. For $n \geq 0$, if we have the word $w, w, \ldots, w$ of length n, then we denote it shortly by $w^{[n]}$ with $w^{[0]} = \emptyset$. In the same way $W^{[n]}$ denotes the word $W, W, \ldots, W$ where W is a finite word repeated n times and $W^{[0]} = \emptyset$. Let $(I_n)_{n\geq 0}$ be the sequence of finite words of integers defined recursively by

$$I_0 = 0 \quad \text{and} \quad I_n = n, I_0^{[2k]}, I_1^{[2k]}, \ldots, I_{n-1}^{[2k]} \quad \text{for } n \geq 1.$$

Then the sequence $I = (i(n))_{n\geq 1}$ in $\mathbb{N}$, introduced in Theorem 4, is given by the infinite word:

$$I = I_0^{[\ell]}, I_1^{[\ell]}, I_2^{[\ell]}, \ldots, I_n^{[\ell]}, \ldots$$

By mean of the last formula, on the irrationality measure, given in the introduction, we get the following result [1, p. 148].

Corollary 5. *Let α be a P_k-expansion, depending on the $(\ell+2)$-tuple Λ satisfying $\mathcal{C}(A)$, then we have*

$$\nu(\alpha) = 2 + (p - 2k - 1)/\ell.$$

Before going on to the second kind of P_k-expansion, we need to point at an exceptional element in $\mathbb{F}(p)$ for all $p \geq 3$ which is quadratic but also a perfect P_k-expansion of the type described in Theorem 4. This element appears in the extremal case $k = (p-1)/2$. We have observed that, in this case, the sequence $(A_n)_{n\geq 0}$ is constant and $A_n = T$ for $n \geq 0$. Consequently, the elements described in Theorem 4.1 will have all partial quotients of the form $a_n = \lambda_n T$ (note that, in agreement to Corollary 5, the irrationality measure of the continued fraction will be equal to 2). In these cases the sequence $(\lambda_n)_{n\geq 1}$ will not generally be periodic and this is why these non-quadratic examples were brought to light [12]. However at the bottom of these examples lies a universal element which is the formal golden mean. In $\mathbb{F}(p)$, for $p \geq 3$, let us consider the following infinite continued fractions

$$\phi(T) = [T, T, \ldots, T, \ldots] \quad \text{and} \quad \rho(T) = [-2T, 2T, \ldots, -2T, 2T, \ldots].$$

Note that we have $\rho(T) = (u/2)\phi(uT)$ with $u^2 = -4$. Elementary computations [4, pp. 331–332] and [9, pp. 267–268] show that we have

$$\rho^p = (-1)^\ell P_k \rho_{\ell+1} - R_k \quad \text{for} \quad k = (p-1)/2 \quad \text{and} \quad \ell \geq 1.$$

Since $\omega_k = -1$, we observe that $\mathcal{C}(A)$ reduces to

$$[(-1)^\ell 2, (-1)^{\ell-1} 2, \ldots, -2 + 1] = (-1)^\ell,$$

which is easily verified by induction. This element ρ will appear again in the last section (Proposition 9).

If we have reported Theorem 4 here, this is due to its somehow mysterious proof and also to the apparent closeness with the following conjecture. There is a second case, corresponding to a different choice of Λ, where a P_k-expansion can be described but only partially and conjecturally.

Conjecture 6. Let p be an odd prime and k, ℓ integers chosen as above. Let $\alpha \in \mathbb{F}(p)$ be a P_k-expansion, depending as above on an $(\ell+2)$-tuple Λ. We assume that Λ satisfies the following equality:

$$[\lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_1 - \epsilon_2] = 0. \qquad \mathcal{C}(B)$$

Then there exist a sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_p^*$ and a sequence $(j(n))_{n\geq 1}$ in $\mathbb{N}$, such that

$$a_n = \lambda_n B_{j(n)} \quad \text{for} \quad n \geq 1.$$

It may be worth noticing that condition $\mathcal{C}(B)$ (and also $\mathcal{C}(A)$) can be stated in a different way. Indeed condition $\mathcal{C}(B)$ is simply equivalent to: $\epsilon_2 = [\lambda_1, \ldots, \lambda_{\ell-1}, \lambda_\ell]$. This conjecture results from broad computer observations, letting the parameters p, k and ℓ vary. We also have a conjectural description of the sequence $(j(n))_{n\geq 1}$ as follows.

Let $(J_n)_{n\geq 1}$ be the sequence of finite words of integers defined recursively by

$$J_0 = 0 \quad \text{and} \quad J_n = n, J_{n-1}^{[2k-1]}, n-1 \quad \text{for } n \geq 1.$$

Then the sequence $J = (j(n))_{n\geq 1}$ in $\mathbb{N}$ is given by the infinite word:

$$J = J_0^{[\ell-1]}, 0, J_1^{[\ell-1]}, 1, \ldots, J_n^{[\ell-1]}, n, \ldots$$

The irrationality measure, for a continued fraction described in Conjecture 6, only depends on the sequences $(j(n))_{n\geq 1}$ and $(\deg(B_n))_{n\geq 0}$. Accordingly, using arguments similar to the ones given in a paper by Ayadi and the author [1, pp. 150–151]), we obtain the following.

Conjecture 7. Let α be a P_k-expansion, depending on the $(\ell+2)$-tuple Λ satisfying $\mathcal{C}(B)$, then we have

$$\nu(\alpha) = 2 + (p-2k+1)(p-1)/(\ell(p+1)-2k).$$

The description of the sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_p^*$ is in general yet out of reach. Note that in the simplest case $\ell = 1$, it can be proved that $J = \mathbb{N}$ and $a_n = \lambda_n B_{n-1}$ where $\lambda_{2n} = \lambda_1 \epsilon_1^{-1}$ and $\lambda_{2n+1} = \lambda_1$ for $n \geq 1$. Moreover, in this case, the irrationality measure is equal to $p+1$ which is the maximal possible value. The sequence of partial quotients has also been fully described [8, pp. 20–21] in the simple case: $k = 1$ and $\ell = 2$.

The reader may wonder why, in the two cases presented above, the P_k-expansions are special. In these cases we say that the expansion is perfect of type A or of type B. First, a computer observation, outside these cases, shows a certain irregularity in the sequence of partial quotients. Moreover the specificity of these perfect expansions of type A and B is pointed out in the next and last section. Finally, the possible connection between these two cases is an open question (regarding the similarity between conditions $\mathcal{C}(A)$ and $\mathcal{C}(B)$).

Before concluding this section, we must make a remark on the origin of these particular continued fractions. As pointed out above, the P_k-expansions appeared from the study of Mills and Robbins quartic over $\mathbb{F}_{13}$ [12, pp. 403–404]. This quartic equation has been generalized [7] for all primes $p \geq 5$. The solution has a continued fraction expansion with a different pattern according to the residue modulo 3 of the prime p. In the case $p \equiv 1 \mod 3$ (particularly for $p = 13$), the pattern of this continued fraction is perfect of type A. In the case $p \equiv 2 \mod 3$, the solution is a hyperquadratic element of order 2, but not of order 1 [2]. The continued fraction for this solution has a different pattern, described by Ayadi and Lasjaunias [1], and the partial quotients are all proportional to elements of the sequence $(B_n)_{n\geq 0}$. This has given rise to the following generalization concerning this sequence $(B_n)_{n\geq 0}$, in connection with hyperquadratic elements of higher order.

Returning to Proposition 1, let us take $r = p^n$ with $n \geq 1$. Let us make the following choice for the pair $(P, Q) \in (\mathbb{F}_p[T])^2$:

$$P = \epsilon_1 P_k^{(p^n+(-1)^{n-1})/(p+1)} \quad \text{and} \quad Q = \epsilon_2 R_k^{p^{n-1}} \quad \text{where} \quad (\epsilon_1, \epsilon_2) \in \mathbb{F}_p^* \times \mathbb{F}_p.$$

We consider the continued fraction α in $\mathbb{F}(p)$ so defined by $(*)$, which is a hyperquadratic element of order n. We observed the following. If the $(\ell+2)$-tuple $\Lambda = (a_1, \cdots, a_\ell, \epsilon_1, \epsilon_2)$ is well chosen, then all partial quotients for α appear to be proportional to certain polynomials B_n as in Conjecture 6 (corresponding to the case $n = 1$). Again this observation is just based on broad computer calculations. Amazingly, this is what happens for the solution of the generalized quartic when $p \equiv 2 \mod 3$, in this case we have $n = 2$ and we may say that the expansion is perfect of type B and order 2. We have described [1, pp. 141–143] a choice of Λ to obtain the above conjecture in the case $n = 2$ and the description of the corresponding analogue of the sequence $(j(n))_{n\geq 1}$.

Here below, to support the observation which has just been made, we have collected a few examples of presumably perfect P_k-expansions of type B and of order 3. To allow easier computer calculations, we have only considered the values $p = 3$, $n = 3$, $k = 1$ and $(\epsilon_1, \epsilon_2) = (1, 1)$.

Conjecture 8. Let $\ell \geq 1$ be an integer. Let $\alpha = [a_1, a_2, \ldots, a_n, \ldots] \in \mathbb{F}(3)$ be the expansion defined by

$$\alpha = [a_1, a_2, \ldots, a_\ell, \alpha_{\ell+1}] \quad \text{and} \quad \alpha^{27} = (T^2 - 1)^7 \alpha_{\ell+1} + T^9$$

where we have $\ell = 3$ and $(a_1, a_2, a_3) = (2T, T, T)$
or $\ell = 4$ and $(a_1, a_2, a_3, a_4) = (T, 2T, 2T, T)$
or $\ell = 4$ and $(a_1, a_2, a_3, a_4) = (2T, T^5 + 2T^3, T, T)$
or $\ell = 5$ and $(a_1, a_2, a_3, a_4, a_5) = (2T, T, T, T, T^5 + 2T^3)$.
Then, in each of these four cases, there exist a sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_3^*$ and a sequence $(j(n))_{n\geq 1}$ in $\mathbb{N}$, such that

$$a_n = \lambda_n B_{j(n)} \quad \text{for} \quad n \geq 1.$$

Let us make a comment about this last statement. There are several ways [7, p. 34] to obtain the first partial quotients of an algebraic element α in $\mathbb{F}(q)$. A natural way is to obtain the beginning of the continued fraction expansion from a rational approximation to α. In the present case (this is valid for all hyperquadratic elements), we can write the algebraic equation (**) satisfied by α in the following way: $\alpha = (a\alpha^r + b)/(c\alpha^r + d) = g(\alpha)$. Then we build rational approximations to α, starting from $R_1 = x_\ell/y_\ell$, by the recursive formula $R_{n+1} = g(R_n)$ for $n \geq 1$. In the cases indicated above, these rational numbers have a special form, showing a particular expansion with partial quotients as wanted. We guess a proof of our conjecture could be derived from the study of the numerators and denominators of R_n.

Finally, we notice the existence of another different generalization for P_k-expansions of type A, also leading to hyperquadratic elements of higher order [6]. In the next and last section, we return to P_k-expansions of order 1 (i.e., $r = p$).

5 Hyperquadratic elements having a low algebraic degree

In this section, we introduce a family of polynomials in the variable X with coefficients in $\mathbb{F}_p[T]$. These polynomials H have the particular form of hyperquadratic type:

$$H(X) = uX^{p+1} + vX^p + wX + z \quad \text{with} \quad u, v, w, z \in \mathbb{F}_p[T].$$

The four coefficients are depending on parameters coming from the finite word W introduced in Section 3. Moreover $H(X) = 0$ will have solutions in $\mathbb{F}(p)$ having a continued fraction expansion of the type discussed above. Under a particular choice of these parameters, H will be reducible and consequently these solutions will have a particular algebraic degree smaller than $p+1$. The consideration of these polynomials comes from a quartic equation, generalizing Mills and Robbins example, introduced by the author [7]. To describe these coefficients, we use some more notation appearing there [7, pp. 30–31].

Here p and k are as above and $W = w_1T, w_2T, \ldots, w_{2k}T$ is the finite word introduced in Section 3.

Let n, t and m be integers with $1 \leq n \leq t \leq m \leq 2k$. Then we introduce the continuant:

$$K_{n,m} = \langle w_nT, w_{n+1}T, \ldots, w_mT \rangle \in \mathbb{F}_p[T].$$

By convention we extend this notation with $K_{n,n-1} = 1$ and $K_{n,n-2} = 0$. From general properties of continuants [8, p. 7], we have the following formula

$$K_{n,m}K_{t,m-1} - K_{n,m-1}K_{t,m} = (-1)^{m-t}K_{n,t-2}. \tag{5}$$

Since, by (2), we have $W^* = -1 \cdot W$, $\langle A^* \rangle = \langle A \rangle$ and $\langle -1 \cdot A \rangle = (-1)^{|A|}\langle A \rangle$ for any finite word A, we also get

$$K_{n,m} = (-1)^{m-n+1}K_{2k+1-m,2k+1-n}. \tag{6}$$

Let us consider the vector $V = (p, k, j, \epsilon)$ where p, k are as above, j is an integer with $1 \leq j \leq 2k-1$ and $\epsilon \in \mathbb{F}_p^*$. To each such vector V, we associate a polynomial H_V in $\mathbb{F}_p[T][X]$ defined by

$$H_V(X) = K_{j+2,2k}X^{p+1} - K_{j+1,2k}X^p + \epsilon(K_{1,j}X + K_{1,j-1}).$$

We have the following proposition.

Proposition 9. *Let $V = (p, k, j, \epsilon)$ and $H_V \in \mathbb{F}_p[T][X]$ be defined as above. For each choice of V, there are two infinite continued fractions α and β in $\mathbb{F}(p)^+$, both P_k-expansions, such that we have $H_V(\alpha) = 0$ and $H_V(1/\beta) = 0$. The first one α is defined by*

$$\alpha = [w_{j+1}T, w_{j+2}T, \ldots, w_{2k}T, \alpha_{2k-j+1}]$$

and

$$\alpha^p = \epsilon(-1)^{k+j}(P_k\alpha_{2k-j+1} - R_k).$$

The second one β is defined by

$$\beta = [w_{2k-j+1}T, w_{2k-j+2}T, \ldots, w_{2k}T, \beta_{j+1}]$$

and

$$\beta^p = \epsilon^{-1}(-1)^{k+j}(P_k\beta_{j+1} - R_k).$$

Moreover, corresponding to each triple (p,k,j), we define two different values ϵ_A and ϵ_B in $\mathbb{F}_p^*$, as follows:

$$\epsilon_A = (-1)^{k+j+1}[w_{j+1},\dots,w_{2k},\omega_k] \quad \textit{and} \quad \epsilon_B = (-1)^{k+j+1}[w_{j+1},\dots,w_{2k}].$$

If $\epsilon = \epsilon_A$ then both P_k-expansions α and β are perfect of type A
(i.e., condition $\mathcal{C}(A)$ in Theorem 4 is satisfied).
If $\epsilon = \epsilon_B$ then both P_k-expansions α and β are perfect of type B
(i.e., condition $\mathcal{C}(B)$ in Conjecture 6 is satisfied).
Finally, let $k = (p-1)/2$ and $V = (p,k,j,\epsilon_A)$. Then H_V has a factor P of degree 2. We have $\alpha = \beta$, $P(\alpha) = 0$ and $P(X) = X^2 + 2(-1)^j X + 1$.

Proof. According to (6), we can write

$$K_{1,j} = (-1)^j K_{2k+1-j,2k} \quad \text{and} \quad K_{1,j-1} = (-1)^{j-1} K_{2k+2-j,2k},$$

$$K_{j+1,2k} = (-1)^j K_{1,2k-j} \quad \text{and} \quad K_{j+2,2k} = (-1)^{j-1} K_{1,2k-j-1}.$$

From these formulas, we can write $(-1)^{j-1} H_V(X) =$

$$K_{1,2k-j-1} X^{p+1} + K_{1,2k-j} X^p + \epsilon(-K_{2k+1-j,2k} X + K_{2k+2-j,2k}).$$

Therefore, we get $\epsilon^{-1}(-1)^{j-1} X^{p+1} H_V(1/X) =$

$$K_{2k+2-j,2k} X^{p+1} - K_{2k+1-j,2k} X^p + \epsilon^{-1}(K_{1,2k-j} X + K_{1,2k-j-1}).$$

We set $V^* = (p,k,2k-j,\epsilon^{-1})$. Hence, we have obtained

$$X^{p+1} H_V(1/X) = \epsilon(-1)^{j-1} H_{V^*}(X). \tag{7}$$

By Proposition 1, we know that $\alpha \in \mathbb{F}(p)$ defined by

$$\alpha = [a_1, a_2, \cdots, a_\ell, \alpha_{\ell+1}] \quad \text{and} \quad \alpha^p = P\alpha_{\ell+1} + Q$$

satisfies $I(\alpha) = 0$ where $I \in \mathbb{F}_p[T][X]$ is defined by

$$I(X) = y_\ell X^{p+1} - x_\ell X^p + (Py_{\ell-1} - Qy_\ell)X - Px_{\ell-1} + Qx_\ell.$$

In order to prove that $H_V(\alpha) = 0$, for the element $\alpha \in \mathbb{F}(p)$ defined in this proposition, we will show that $I = H_V$ if

$$(a_1,\dots,a_\ell,P,Q) = (w_{j+1}T,\dots,w_{2k}T,\epsilon_1 P_k, -\epsilon_1 R_k) \quad \text{with} \quad \epsilon_1 = (-1)^{k+j}\epsilon.$$

First, recall that, according to (4) in Section 3, we have

$$P_k = (-1)^k K_{1,2k} \quad \text{and} \quad R_k = (-1)^k K_{2,2k}.$$

Consequently, by (6) and $\epsilon_1 = (-1)^{k+j}\epsilon$, we obtain

$$P = \epsilon(-1)^j K_{1,2k} \quad \text{and} \quad Q = \epsilon(-1)^j K_{1,2k-1}. \tag{8}$$

Since $(a_1, \dots, a_\ell) = (w_{j+1}T, \dots, w_{2k}T)$, we also have

$$x_\ell = K_{j+1,2k}, \quad y_\ell = K_{j+2,2k}, \quad x_{\ell-1} = K_{j+1,2k-1}, \quad y_{\ell-1} = K_{j+2,2k-1}. \tag{9}$$

Hence, by (8), (9) and (5), we get

$$Py_{\ell-1} - Qy_\ell = \epsilon(-1)^j(K_{1,2k}K_{j+2,2k-1} - K_{1,2k-1}K_{j+2,2k}) = \epsilon K_{1,j}$$

and also

$$Qx_\ell - Px_{\ell-1} = \epsilon(-1)^j(K_{1,2k-1}K_{j+1,2k} - K_{1,2k}K_{j+1,2k-1}) = \epsilon K_{1,j-1}.$$

Finally, we obtain the desired outcome:

$$I(X) = K_{j+1,2k}X^{p+1} - K_{j+2,2k}X^p + \epsilon K_{1,j} + \epsilon K_{1,j-1} = H_V(X).$$

Changing j into $2k-j$ and ϵ into ϵ^{-1}, in the definition for α, we get the definition for β. Since this means changing V into V^*, we have just proved that $H_{V^*}(\beta) = 0$ and according to (7) this gives $H_V(1/\beta) = 0$.
Let us now consider the $(2k-j+2)$-tuple Λ_α defining the P_k-expansion α. We have $\Lambda_\alpha = (w_{j+1}, \dots, w_{2k}, -\epsilon_2, \epsilon_2)$ where $\epsilon_2 = (-1)^{k+j+1}\epsilon$. Hence $\mathcal{C}(A)$ is satisfied if and only if we have $[w_{2k}, w_{2k-1} \dots, w_{j+1} - \epsilon_2] = -1/\omega_k$. This is equivalent to $\epsilon_2 = [w_{j+1}, \dots, w_{2k} + 1/\omega_k]$. Consequently $\mathcal{C}(A)$ is satisfied by Λ_α if and only if we have

$$\epsilon = \epsilon_A = (-1)^{j+k+1}[w_{j+1}, \dots, w_{2k}, \omega_k].$$

Considering the same $(2k-j+2)$-tuple Λ_α, we see in the same way that Λ_α satisfies $\mathcal{C}(B)$ if and only if we have

$$\epsilon = \epsilon_B = (-1)^{j+k+1}[w_{j+1}, \dots, w_{2k}].$$

Now, we need to consider β instead of α. Here the $(j+2)$-tuple Λ_β defining β is $\Lambda_\beta = (w_{2k-j+1}, \dots, w_{2k}, -\epsilon_2, \epsilon_2)$ where $\epsilon_2 = (-1)^{k+j+1}\epsilon^{-1}$. In the same way, we obtain that Λ_β satisfies $\mathcal{C}(A)$ if

$$\epsilon = \overline{\epsilon_A} = (-1)^{j+k+1}[w_{2k-j+1}, \dots, w_{2k}, \omega_k]^{-1}$$

and also that Λ_β satisfies $\mathcal{C}(B)$ if

$$\epsilon = \overline{\epsilon_B} = (-1)^{j+k+1}[w_{2k-j+1}, \dots, w_{2k}]^{-1}.$$

We only need to prove that $\epsilon_A = \overline{\epsilon_A}$ and $\epsilon_B = \overline{\epsilon_B}$. Let us compare ϵ_B and $\overline{\epsilon_B}$. According to (2), we can write $[w_{2k-j+1}, \dots, w_{2k}] = [-w_j, \cdots, -w_1]$. Hence $\epsilon_B = \overline{\epsilon_B}$ is equivalent to

$$[w_{j+1}, \dots, w_{2k}] + [w_j, \dots, w_1]^{-1} = 0.$$

With our notation on continuants, this can be written as

$$(K_{j+1,2k}/K_{j+2,2k})(1) + (K_{1,j-1}/K_{1,j})(1) = 0.$$

This comes from $K_{1,2k}(1) = (-1)^k(1-1)^k = 0$ and the following general formula on continuants [8, p. 7]: $K_{1,2k} = K_{1,j}K_{j+1,2k} + K_{1,j-1}K_{j+2,2k}$. Turning to $\epsilon_A = \overline{\epsilon_A}$, we see, using the same arguments as above that this is equivalent to

$$[w_{j+1}, \dots, w_{2k}, \omega_k] + [w_j, \dots, w_1, -\omega_k]^{-1} = 0.$$

Using the same general formula on continuants, this is again equivalent to

$$\langle -\omega_k, w_1, w_2, \dots, w_{2k}, \omega_k \rangle = 0.$$

The truth of that is derived from the following two equalities:

$$\langle w_2, \dots, w_{2k} \rangle = (-1)^k \quad \text{and} \quad \langle w_2, \dots, w_{2k-1} \rangle = 2(-1)^{k+1}\omega_k.$$

The first one is obtained from $R_k(1) = (-1)^k K_{2,2k}(1) = 1$, while the proof for the second one is more mysterious and it is left to the reader.

We finish the proof of this proposition by considering the extremal case $k = (p-1)/2$. In this particular case, we have seen in Section 3, that we have $W = -2T, 2T, -2T, \dots, 2T$ and $\omega_k = -1$. Hence, for all j, we get

$$\epsilon_A = (-1)^{k+j+1}[2(-1)^{j+1}, \dots, 2, -1] = (-1)^k.$$

Consequently, setting $\ell = 2k - j$, α is defined by

$$\alpha = [2(-1)^{\ell-1}T, \dots, 2T, \alpha_{\ell+1}] \quad \text{and} \quad \alpha^p = (-1)^j(P_k\alpha_{\ell+1} - R_k).$$

Note that the same definition holds for β with $\ell = j$. Let us consider in $\mathbb{F}(p)$ the infinite continued fraction $\gamma = (-1)^j\alpha$. We have $\gamma^p = (-1)^j\alpha^p$ and $\gamma_{\ell+1} = (-1)^j\alpha_{\ell+1}$ for all $\ell \geq 1$. Since $(-1)^j = (-1)^\ell$, we see that γ is defined by

$$\gamma = [-2T, \dots, 2(-1)^\ell T, \gamma_{\ell+1}] \quad \text{and} \quad \gamma^p = (-1)^\ell P_k\gamma_{\ell+1} - R_k.$$

Comparing to the definition of the quadratic continued fraction ρ, derived from the formal golden mean, described in Section 4 and because of the uniqueness in this definition, we obtain $\gamma = \rho$, $\alpha = (-1)^j\rho$ and in the same way $\beta = \alpha$. A basic computation shows that the minimal polynomial of $(-1)^j\rho$ is $P(X) = X^2 + 2(-1)^jTX + 1$. □

Concerning this polynomial H_V, we are interested in its quality of being or not being reducible over $\mathbb{F}_p(T)$. In other words, we are interested in the exact algebraic degree of α and β over $\mathbb{F}_p(T)$. We have checked the reducibility of H_V by computer calculations. In the vector V, we first fix p and k. Then, according to formula (7), we observe that the reducibility of H_V need only be studied for $1 \leq j \leq k$.

To illustrate our purpose, let us consider the simplest, and somehow trivial, case : $p = 3$. Then, we only have to consider $k = j = 1$. Here we have $\omega_1 = 1/2 = -1$ and $W = T, -T$. There are two polynomials H_1 and H_2 corresponding to $V_1 = (3, 1, 1, 1)$ and $V_2 = (3, 1, 1, 2)$:

$$H_1(X) = H_{V_1}(X) = X^4 + TX^3 + TX + 1$$

and

$$H_2(X) = H_{V_2}(X) = X^4 + TX^3 - TX - 1 = (X^2 - 1)(X^2 + TX + 1).$$

Note that reducibility of the polynomials H_V over $\mathbb{F}_p(T)$ has been tested by computer, using `Maple` programming software. The polynomial H_1 is irreducible over $\mathbb{F}_3(T)$. According to Proposition 5.1, in both cases, α is defined by $\alpha = [-T, \alpha_2]$ and $\alpha^3 = \epsilon((T^2 - 1)\alpha_2 - T)$ and we have $\alpha = \beta$ (hence α and $1/\alpha$ are both solutions in $\mathbb{F}(3)$ of H_V). Moreover, we get $\epsilon_A = (-1)^3[2, -1] = -1$ and $\epsilon_B = (-1)^3[2] = 1$. If $\epsilon = 1$, the solution $\alpha \in \mathbb{F}(3)$ of $H_1(X) = 0$ is algebraic of degree 4, we have $\alpha = -[B_0, B_1, \cdots, B_n, \cdots]$ and $\nu(\alpha) = 4$ (see the comment after Conjecture 7). If $\epsilon = -1$, the solution α of $H_2(X) = 0$ satisfies $\alpha^2 + T\alpha + 1 = 0$ and therefrom we get $\alpha = [-T, T, \cdots, -T, T, \cdots]$. Indeed, we have $\alpha = u\phi(uT)$ where $\phi(T) = [T, T, T, \ldots, T, \ldots]$ is the formal golden mean and $u^2 = -1$.

Given the prime p, an easy computation shows that there are $(p^2 - 1)/8$ pairs (j, k) with $1 \le j \le k \le (p - 1)/2$. Hence, for a given odd prime number p, there are $(p^2 - 1)(p - 1)/8$ polynomials H_V to be considered. We have investigated the reducibility of these polynomials in all the possible cases, for the small primes p. The three conjectures presented below are based on computer calculations, with limitation on the size of the prime p. From this observation, we can make the first and main conjecture.

Conjecture 10. Let p be an odd prime number. Let $V = (p, k, j, \epsilon)$ and H_V be defined as above. If $\epsilon \ne \epsilon_A$ and $\epsilon \ne \epsilon_B$, then the polynomial H_V is irreducible over $\mathbb{F}_p(T)$.

The irreducibility of H_V, outside the two values for ϵ given in the proposition, may be surprising. Another surprise is that the reducibility may sometimes happen, for a particular value of ϵ, but only if the triple (p, k, j) is well chosen. A last surprise is the following: if the reducibility appears for a certain triple (p, k, j) and a particular choice of ϵ, then this is most of the times if $\epsilon = \epsilon_A$, and sometimes if $\epsilon = \epsilon_A$ or $\epsilon = \epsilon_B$, but apparently never if $\epsilon = \epsilon_B$ alone. To illustrate this, we give below three tables where we have collected the data obtained by computer for $p = 7$, $p = 11$ and $p = 13$. Each cell (j, k) with $j \le k$, corresponds to $p - 1$ tested polynomials. In each cell, we have written Ir. if the polynomials H_V are irreducible for all ϵ. If H_V is reducible (for $\epsilon = \epsilon_A$ or $\epsilon = \epsilon_B$), we indicate the degrees of its irreducible factors. For instance, $A : 5^2.2$ means 2 factors of degree 5 and 1 of degree 2, if $\epsilon = \epsilon_A$ and $B : 8.4$ means 1 factor of degree 8 and 1 of degree 4, if $\epsilon = \epsilon_B$. Note that, in all cases, the sum of these degrees is equal to $p + 1$.

Reducibility of H_V: $p = 7$

	1	2	3
3	$A : 3^2.2$	$A : 3^2.2$	$A : 2^3.1^2$
2	$A : 4^2$	$A : 3^2.1^2$ $B : 4^2$	
1	$A : 6.1^2$		

Reducibility of H_V: $p = 11$

	1	2	3	4	5
5	$A : 5^2.2$	$A : 5^2.2$	$A : 5^2.2$	$A : 5^2.2$	$A : 2^5.1^2$
4	Ir.	Ir.	Ir.	$A : 5^2.1^2$ $B : 6.3^2$	
3	Ir.	Ir.	$A : 10.1^2$ $B : 8.4$		
2	Ir.	$A : 5^2.1^2$ $B : 6^2$			
1	$A : 10.1^2$				

Reducibility of H_V: $p = 13$

	1	2	3	4	5	6
6	$A : 6^2.2$	$A : 6.3^2.2$	$A : 4^2.2^3$	$A : 6.3^2.2$	$A : 6^2.2$	$A : 2^6.1^2$ $B : 7^2$
5	Ir.	Ir.	Ir.	Ir.	$A : 12.1^2$	
4	$A : 6.4^2$	$A : 6.4^2$	Ir.	$A : 3^4.1^2$ $B : 7^2$		
3	$A : 8.6$	Ir.	$A : 4^3.1^2$			
2	Ir.	$A : 6^2.1^2$ $B : 7^2$				
1	$A : 12.1^2$					

Our second conjecture is the following. Note that the irrationality measure for α, given in both conjectures below, can be directly derived applying Corollary 5, with $\ell = 2k - j$.

Conjecture 11. Let $k \geq 1$ and $m \geq 2$ be integers. Let p be a prime number such that $p = km + 1$. Let $V = (p, k, k, \epsilon_A)$ and H_V be defined as above. Then the polynomial H_V has an irreducible factor P in $\mathbb{F}_p[T][X]$ such that $P(\alpha) = 0$, $\deg_X(P) = m$ and $\nu(\alpha) = m$. More precisely, we have

$$P(X) = \sum_{0 \leq i \leq m} (-1)^{ki} \binom{m}{i} T^{(1-(-1)^i)/2} X^{m-i}.$$

Finally, we give a last conjecture allowing us to return to the quartic equation which has been the starting point of this investigation.

Conjecture 12. Let p be a prime number such that $p \equiv 1 \mod 3$ and $k = (p-1)/3$. Let $V = (p, k, k/2, \epsilon_A)$ and H_V be defined as above. Then the polynomial H_V has an irreducible factor P in $\mathbb{F}_p[T][X]$ such that $P(\alpha) = 0$, $\deg_X(P) = 4$ and $\nu(\alpha) = 8/3$. More precisely, with $j = k/2$, we have

$$P(X) = X^4 - w_{j+1}TX^3 - (w_{j+1}/w_{j+2})X^2 - (w_{j+1}/w_{j+2})^2/12.$$

If $V = (13, 4, 2, -1)$ then we have $P = X^4 + 6TX^3 + 2X^2 + 4$. By elementary transformations on P and α [5, pp. 1114–1115], we return to the quartic equation $X^4 + X^2 - TX + 1 = 0$, having a solution in $\mathbb{F}(13)$, with predictable continued fraction expansion, introduced by Mills and Robbins [12, p. 403].

Lastly, we want to add that the interested reader can willingly obtain the code for the programs supporting these conjectures, by writing to the author.

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References

[1] Kh. Ayadi and A. Lasjaunias, On a quartic equation and two families of hyperquadratic continued fractions in power series fields, *Mosc. J. Comb. Number Theory* **6** (2016), 132–155.

[2] A. Bluher and A. Lasjaunias, Hyperquadratic power series of degree four, *Acta Arith.* **124** (2006), 257–268.

[3] A. Lasjaunias, A survey of Diophantine approximation in fields of power series, *Monatsh. Math.* **130** (2000), 211–229.

[4] A. Lasjaunias, Continued fractions for hyperquadratic power series over a finite field, *Finite Fields Appl.* **14** (2008), 329–350.

[5] A. Lasjaunias, On Robbins' example of a continued fraction for a quartic power series over $\mathbb{F}_{13}$, *J. Number Theory* **128** (2008), 1109–1115.

[6] A. Lasjaunias, Algebraic continued fractions in $\mathbb{F}_q((T^{-1}))$ and recurrent sequences in $\mathbb{F}_q$, *Acta Arith.* **133** (2008), 251–265.

[7] A. Lasjaunias, On the continued fraction of the unique root in $\mathbb{F}(p)$ of the equation $x^4 + x^2 - Tx - 1/12 = 0$ and other related hyperquadratic expansions, *Finite Fields Appl.* **18** (2012), 26–34.

[8] A. Lasjaunias, Continued fractions, preprint, 2017. Available at `https://arxiv.org/abs/1711.11276`.

[9] A. Lasjaunias and J.-Y. Yao, Hyperquadratic continued fractions in odd characteristic with partial quotients of degree one, *J. Number Theory* **149** (2015), 259–284.

[10] A. Lasjaunias and J.-Y. Yao, On certain recurrent and automatic sequences in finite fields, *J. Algebra* **478** (2017), 133–152.

[11] K. Mahler, On a theorem of Liouville in fields of positive characteristic, *Canadian J. Math.* **1** (1949), 397–400.

[12] W. Mills and D. P. Robbins, Continued fractions for certain algebraic power series, *J. Number Theory* **23** (1986), 388–404.

[13] W. Schmidt, On continued fractions and diophantine approximation in power series fields, *Acta Arith.* **95** (2000), 139–166.

[14] D. Thakur, *Function Field Arithmetic*, World Scientific, 2004.

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