Journal of Integer Sequences, Vol. 22 (2019),
Article 19.8.5

A Family of Riordan Group Automorphisms

Ângela Mestre and José Agapito
Centro de Análise Funcional, Estruturas Lineares e Aplicações
Grupo de Estruturas Algébricas, Lineares e Combinatórias
Departamento de Matemática
Faculdade de Ciências, Universidade de Lisboa
1749-016 Lisboa
Portugal
amestre@fc.ul.pt
jaruiz@fc.ul.pt

Abstract

In 2006, Bacher introduced a family of Riordan group automorphisms parametrized by three complex numbers. Bacher's family is a subgroup of the group of automorphisms of the Riordan group and so is the subfamily parametrized only by two real numbers. Here, we study some of the algebraic properties of this subfamily and use the elements to point out isomorphisms between Riordan subgroups. In this context, we prove that the set of Riordan arrays whose row sum sequence is a sequence of partial sums, forms a Riordan subgroup. Moreover, we show that the well-known recursive matrices may be constructed from sequences of images of a Riordan array under automorphisms. Our construction also discloses a correspondence between the recursive matrices and a pair of well-defined Riordan arrays.

1 Introduction

Let $f(t) = \sum_{k=0}^{\infty} f_k t^k$ be a formal power series whose coefficient $f_k = [t^k]f(t)$ is over a field, say, the field of complex numbers $\mathbb{C}$. Recall that $[t^k]$ is the usual coefficient extraction operator. Often, $f(t)$ is called the generating function of the sequence $(f_k)_{k\geq 0}$. Let $d(t)$ and $h(t)$ be two generating functions and consider the infinite matrix $D = (D_{k,n})_{k\geq n\geq 1}$ whose generic entry is given by

$$D_{k,n} = [t^{k-1}]d(t)h(t)^{n-1}. \tag{1}$$

If $h_0 = 0$, then D is an infinite lower triangular matrix, known as a *Riordan array*, usually represented as the pair $D = (d(t), h(t))$. If, in addition, $d_0 \neq 0$ and $h_1 \neq 0$, then D is said to be a *proper* Riordan array. The matrix multiplication rule for Riordan arrays in terms of their generating functions is

$$(d_1(t), h_1(t))(d_2(t), h_2(t)) = (d_1(t)d_2(h_1(t)), h_2(h_1(t))). \tag{2}$$

Also, if $f(t)$ and $g(t)$ are the generating functions of the sequences $(f_k)_{k\geq 0}$ and $(g_k)_{k\geq 0}$, then the following identity is often referred to as the *Fundamental Theorem of Riordan Arrays* (FTRA)

$$g(t) = (d(t), h(t))f(t) = d(t)f(h(t)). \tag{3}$$

The first instances where Riordan arrays were considered in the literature are probably Jabotinsky's work on Bell-type arrays [12, 13, 14], Shapiro's Catalan triangle [23], and Rogers' *renewal arrays* [22]. The formal definition of (proper) Riordan arrays was introduced years later by Shapiro, Getu, Woan, and Woodson [25]. The extra conditions, namely, $d_0 \neq 0$ and $h_1 \neq 0$ allow for the set of proper Riordan arrays to form a group. Here, the Riordan group, denoted by $\mathfrak{Rio}$, is the set $\mathfrak{Rio} = \big\{\big(d(t), h(t)\big) \,|\, d_0 = 1, h_0 = 0, h_1 = 1\big\}$, where the group multiplication is given by formula (2). The group identity is $\mathrm{id}_{\mathfrak{Rio}} = (1, t)$ and the inverse of any element $D = (d(t), h(t))$ is $D^{-1} = (1/d(\bar{h}(t)), \bar{h}(t))$, where $\bar{h}(t)$ denotes the compositional inverse of the generating function $h(t)$, which satisfies $h(\bar{h}(t)) = \bar{h}(h(t)) = t$. Throughtout the paper, unless otherwise stated, by *Riordan array* we actually mean *Riordan group element*.

This paper aims at giving further insight into the Riordan group $\mathfrak{Rio}$ through some of its automorphisms. A Riordan group automorphism ϕ is a 1-1 mapping from $\mathfrak{Rio}$ onto itself such that $\phi(D_1D_2) = \phi(D_1)\phi(D_2)$ for all D_1, D_2 in $\mathfrak{Rio}$. We denote the set of Riordan group automorphisms by $\mathrm{Aut}(\mathfrak{Rio})$. Here, we consider a family of Riordan group automorphisms, denoted by $\phi_{r,s}$, which depend on two real numbers, r and s with $s \neq 0$. The definition is $\phi_{r,s}(d(t), h(t)) = ((h(t)/t)^r d(t)^s, h(t))$. They are a special case of Bacher's automorphisms on three complex numbers introduced in the context of the Lie algebra structure of the Riordan group [5].

The paper is organized as follows. Let $\mathbb{R}$ denote the real numbers and $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Section 2 shows that the set $\{\phi_{r,s}\}_{r\in\mathbb{R}, s\in\mathbb{R}^*}$, has a very rich algebraic structure. We give the involutions, the commutator subgroup of $\{\phi_{r,s}\}_{r\in\mathbb{R}, s\in\mathbb{R}^*}$, and show that the set $\{\phi_{r,\pm 1}\}_{r\in\mathbb{R}}$ is a normal subgroup. Moreover, let $\mathrm{AAut}(\mathfrak{Rio})$ denote the group of automorphisms and anti-automorphisms of the Riordan group $\mathfrak{Rio}$. Let $\psi_{r,s}$ denote the anti-automorphism induced by $\phi_{r,s}$. We give the multiplication and inverse rules for the subgroup of $\mathrm{AAut}(\mathfrak{Rio})$ consisting of $\{\phi_{r,s}, \psi_{r,s}\}_{r\in\mathbb{R}, s\in\mathbb{R}^*}$. We end up this section by posing two questions on the general characterization of the automorphisms of $\mathfrak{Rio}$.

Section 3 determines the images of most of the well-known Riordan subgroups under the automorphisms $\phi_{r,s}$. This points out an infinity of isomorphisms between Riordan subgroups. We use the row sum generating function [24, 11] to show that the image of the stochastic subgroup under the automorphism $\phi_{1,1}$ is the subgroup of Riordan arrays whose row sum sequence is the partial sum of the coefficients of a generating function $h(t)$. We also give a generic closed form for the row-sum sequence of the arrays in the Riordan subgroup obtained by applying $\phi_{1,n}$ to the stochastic subgroup, for any $n \geq 1$.

Furthermore, Section 4 shows that for a given a Riordan array D (*base array*), for very large m, the Riordan array $\phi_{-m,1}(D)$ is an approximation to a bi-infinite triangle which coincides with the recursive matrix of Luzón, Merlini, Morón, and Sprugnoli [17] associated with D. The fact that the automorphism setting does not involve Laurent series is an advantage over the recursive matrices. Moreover, we show that both the east and west side of the bi-infinite triangles can be described by well-defined Riordan arrays. In this context, a bi-infinite triangle turns out to be equivalent to a pair of Riordan arrays. Finally, we observe that the $[-m]$-complementary of a Riordan array D introduced in the former paper (see also [27, 8, 18]), is the image of the inverse matrix D^{-1} under a Bacher automorphism with appropriate parameters.

Throughtout the paper, the integer sequences which are well-known are referred to by their number from the On-Line Encyclopedia of Integer Sequences [26].

2 A family of automorphisms of $\mathfrak{Rio}$

We study some of the algebraic structure of an infinite subgroup of $\mathrm{Aut}(\mathfrak{Rio})$ parametrized by two real numbers. We obtain the involutions, the commutator subgroup, and the normal subgroups of the considered subgroup of $\mathrm{Aut}(\mathfrak{Rio})$. We also raise the question of the general characterization of the automorphisms of the Riordan group.

We consider the following transformation:

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^* &\rightarrow \mathrm{Aut}(\mathfrak{Rio}) \\ (r, s) &\mapsto \phi_{r,s}, \end{aligned}$$

where for any $r \in \mathbb{R}$ and $s \in \mathbb{R}^*$ the mappings $\phi_{r,s}$ are defined by:

$$\begin{aligned} \phi_{r,s} : \mathfrak{Rio} &\rightarrow \mathfrak{Rio} \\ (d(t), h(t)) &\mapsto \left(\left(\frac{h(t)}{t} \right)^r d(t)^s, h(t) \right). \end{aligned} \tag{4}$$

Note that, $\phi_{0,1} = \mathrm{id}_{\mathrm{Aut}(\mathfrak{Rio})}$. It is easy to verify that the mappings $\phi_{r,s}$ are automorphisms of the Riordan group. Moreover, $\phi_{r,s} = \varphi_{s,r,0}$, where for any $\lambda, \mu \in \mathbb{C}$ and $\kappa \in \mathbb{C}^*$, with $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$, the automorphisms $\varphi_{\lambda,\kappa,\mu}$ are defined by Bacher [5] as follows:

$$\begin{aligned} \varphi_{\lambda,\kappa,\mu} : \mathfrak{Rio} &\rightarrow \mathfrak{Rio} \\ D = (d(t), h(t)) &\mapsto \left(\left(\frac{h(t)}{t} \right)^\lambda d(t)^\kappa h'(t)^\mu, h(t) \right), \end{aligned} \tag{5}$$

where $h'(t)$ denotes the derivative of $h(t)$. Note that here the order of the parameters λ and κ is reversed in $\varphi_{\lambda,\kappa,\mu}$ in comparison with [5].

The automorphism $\phi_{1,1}$ (as well as its iterations) has recently been studied in the literature. Actually, $\phi_{1,1}$ is the diagonal translation operator used by Luzón, Merlini, Morón, and Sprugnoli [17] (see also [5]). The automorphism $\phi_{1,1}$ is also the mapping $T : \mathfrak{Rio} \to \mathfrak{Rio}$ of He [10], defined by $T(D) = UDU^T$, where $(U_{i,j})_{i,j\geq 1} = (\delta_{i+1,j})_{i,j\geq 1}$.

For any real numbers r and $s \neq 0$, the array $\phi_{r,s}(d(t), h(t))$ has the following semidirect product factorization:

$$\phi_{r,s}(d(t), h(t)) = \big(d(t)^s, t\big)\left(\left(\frac{h(t)}{t}\right)^r, h(t)\right). \tag{6}$$

Clearly, the set $\Phi(\mathbb{R} \times \mathbb{R}^*) = \{\phi_{r,s}\}_{r\in\mathbb{R},s\in\mathbb{R}^*}$ forms a group under composition.

Proposition 1.

$$\Phi(\mathbb{R} \times \mathbb{R}^*) \leq \mathrm{Aut}(\mathfrak{Rio}).$$

Proof. Any mapping $\phi_{r,s}$ is invertible for all $r \in \mathbb{R}$ and $s \in \mathbb{R}^*$, the inverse being given by

$$\phi_{r,s}^{-1} = \phi_{-r/s,1/s}. \tag{7}$$

The set is clearly closed under composition:

$$\phi_{r',s'} \circ \phi_{r,s} = \phi_{r'+rs',ss'}. \tag{8}$$

□

Now, let $\mathbb{Z}$ and $\mathbb{Q}$ denote the integer and rational numbers, respectively. Let $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$ and $\mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$. Observe that whereas $\Phi(\mathbb{Q} \times \mathbb{Q}^*)$ remains a subgroup of $\mathrm{Aut}(\mathfrak{Rio})$, the same does not hold for $\Phi(\mathbb{Z} \times \mathbb{Z}^*)$, except for $\Phi(\mathbb{Z} \times \{\pm 1\})$. Next, we explore some of the algebraic properties of the sets $\Phi(\mathbb{Z} \times \{\pm 1\})$.

Proposition 2. *Any element in* $\Phi(\mathbb{R} \times \{-1\})$ *is an involution.*

Proof. By equation (8), we have $\phi_{r,-1}^2 = \phi_{r,-1} \circ \phi_{r,-1} = \phi_{0,1}$. □

Moreover, it follows from equation (8) that, for all integers $n \geq 0$, the nth iterate of $\phi_{r,s}$ is

$$\phi_{r,s}^n = \phi_{r(1+s+s^2+\cdots+s^{n-1}),s^n}.$$

Hence, there are no elements in $\Phi(\mathbb{R}\times\mathbb{R}^*)$ of order higher than 2, which implies that the set $\Phi(\mathbb{R} \times \{-1\})$ contains all the involutions of $\Phi(\mathbb{R} \times \mathbb{R}^*)$.

Proposition 3. *The set* $\Phi(\mathbb{R} \times \{1\})$ *is the commutator subgroup of* $\Phi(\mathbb{R} \times \mathbb{R}^*)$.

Proof. It is clear that $\Phi(\mathbb{R}\times\{1\})$ is a subgroup of $\Phi(\mathbb{R}\times\mathbb{R}^*)$. Also, by equations (7) and (8), we have

$$\begin{aligned}[\phi_{r',s'},\phi_{r,s}] &= \phi_{r',s'}^{-1}\circ\phi_{r,s}^{-1}\circ\phi_{r',s'}\circ\phi_{r,s}\\ &= \phi_{-r'/s',1/s'}\circ\phi_{-r/s,1/s}\circ\phi_{r',s'}\circ\phi_{r,s}\\ &= \phi_{-r'/s'-r/ss',1/ss'}\circ\phi_{r'+rs',ss'}\\ &= \phi_{(r'-r+rs'-r's)/ss',1}.\end{aligned}$$

Therefore, the commutator of any two elements of the group $\Phi(\mathbb{R}\times\mathbb{R}^*)$ is in $\Phi(\mathbb{R}\times\{1\})$ and so is any finite product of commutators of $\Phi(\mathbb{R}\times\mathbb{R}^*)$. □

Proposition 4. *The set* $\Phi(\mathbb{R}\times\{-1,1\})$ *is a normal subgroup of* $\Phi(\mathbb{R}\times\mathbb{R}^*)$.

Proof. It is straightforward to check that $\Phi(\mathbb{R}\times\{-1,1\})$ is a subgroup of $\Phi(\mathbb{R}\times\mathbb{R}^*)$. The normality property follows from Proposition 3 and from equations (7) and (8):

$$\phi_{r,s}\circ\phi_{r',-1}\circ\phi_{r,s}^{-1}=\phi_{r+r's,-s}\circ\phi_{-r/s,1/s}=\phi_{2r+r's,-1},$$

$$\phi_{r,s}\circ\phi_{r',1}\circ\phi_{r,s}^{-1}=\phi_{r+r's,s}\circ\phi_{-r/s,1/s}=\phi_{r's,1}.$$

□

We define as well a family of anti-automorphisms of $\mathfrak{Rio}$ induced by Φ as follows:

$$\begin{aligned}\Psi:\ \mathbb{R}\times\mathbb{R}^* &\to \mathrm{AAut}(\mathfrak{Rio})\\ (r,s) &\mapsto \psi_{r,s},\end{aligned}$$

where for any $r\in\mathbb{R}$ and $s\in\mathbb{R}^*$, the mapping $\psi_{r,s}$ is defined by:

$$\begin{aligned}\psi_{r,s}:\mathfrak{Rio} &\to \mathfrak{Rio}\\ D &\mapsto \phi_{r,s}(D^{-1})=\left(\left(\frac{\bar h(t)}{t}\right)^r\frac{1}{d(\bar h(t))^s},\bar h(t)\right).\end{aligned}$$

It is easy to see that $\Phi(\mathbb{R}\times\mathbb{R}^*)\cup\Psi(\mathbb{R}\times\mathbb{R}^*)\leq\mathrm{AAut}(\mathfrak{Rio})$. Moreover, the automorphisms $\phi_{r,s}$ commute with the mapping that sends a Riordan group element to its inverse.

Lemma 5. *Let* $\mathrm{Inv}:\mathfrak{Rio}\to\mathfrak{Rio}; D\mapsto D^{-1}$ *be the mapping that sends a Riordan group element* D *to its inverse* D^{-1}. *Then, we have*

$$\phi_{r,s}\circ\mathrm{Inv}=\mathrm{Inv}\circ\phi_{r,s}.$$

The following four equations are immediately derived from the lemma above:

$$\begin{aligned}\psi_{r,s}^{-1} &= \psi_{-r/s,1/s},\\ \psi_{r',s'}\circ\psi_{r,s} &= \phi_{r'+rs',ss'},\\ \phi_{r',s'}\circ\psi_{r,s} &= \psi_{r'+rs',ss'},\\ \psi_{r,s}\circ\phi_{r',s'} &= \psi_{r+r's,ss'}.\end{aligned}$$

Now, let $D_1 = (d_1(t), h_1(t))$ and $D_2 = (d_2(t), h_2(t))$ denote two Riordan matrices. One can partition the set of Riordan arrays into equivalence classes by setting $D_1 \sim D_2$ if and only if $h_1(t) = h_2(t)$. Clearly, the family $\{\phi_{r,s}\}_{r\in\mathbb{R},s\in\mathbb{R}^*}$ (as well as Bacher's larger family) preserves these equivalence classes, something which does not happen, for instance, with the inner automorphisms. Recall that for the Riordan group, the inner automorphism $\phi_{\mathfrak{I}}$ associated with the element $\mathfrak{I} = (g(t), f(t))$ is

$$\begin{aligned}\phi_{\mathfrak{I}} &= (g(t), f(t))(d(t), h(t))(g(t), f(t))^{-1} \\ &= \left(g(t)d(f(t))\frac{1}{g(\bar{f}(h(f(t))))}, \bar{f}(h(f(t)))\right).\end{aligned}$$

We did not find any automorphism, other than the inner automorphisms (or a composition with them), that modifies the second element of the pair $(d(t), h(t))$. Thus, we pose the following two questions:

Question 6. Is there any automorphism in $\mathrm{Aut}(\mathfrak{Rio})$ up to composition with inner automorphisms, that sends a Riordan array $(d_1(t), h_1(t))$ to $(d_2(t), h_2(t))$ with $h_1(t) \neq h_2(t)$?

Due to Bacher's definition of the Riordan group as the semidirect product of the group of formal power series with constant term 1 by the group of formal power series under substitution [5]. We believe that the results of Muckenhoupt [20] and Johnson [16] are likely to be relevant to answer the question above in the negative. Moreover, a more general question in this context is the following:

Question 7. Which is the general characterization of the group of automorphisms of the Riordan group? That is, in terms of the generating functions $d(t)$ and $h(t)$ such that $d_0 = 1, h_0 = 0$, and $h_1 = 1$, which properties must a mapping $(d_1(t), h_1(t)) \mapsto (d_2(t), h_2(t))$ satisfy in order to be a Riordan group automorphism?

3 Isomorphisms between Riordan subgroups via $\phi_{r,s}$

Since the mappings $\phi_{r,s}$ are automorphisms of the Riordan group $\mathfrak{Rio}$, for any subgroup $\mathfrak{subRio} \leq \mathfrak{Rio}$, the sets $\phi_{r,s}(\mathfrak{subRio})$ are also subgroups of $\mathfrak{Rio}$. In this section, we characterize the images under $\phi_{r,s}$ of several well-known subgroups of the Riordan group. For brevity, we shall write $\mathfrak{subRio}_{r,s} = \phi_{r,s}(\mathfrak{subRio})$. Observe that the groups $\mathfrak{subRio}_{r,s}$ and $\mathfrak{subRio}_{r',s'}$ are isomorphic via $\phi_{r'-rs'/s,s'/s}$ for all pairs (r, s) and (r', s').

3.1 The c-Appell subgroup $\left\{\left(d(t), ct\right) \,\middle|\, c \neq 0\right\}$

Proposition 8. *Let* $c\text{-}\mathfrak{App}_{r,s} = \left\{\left(c^r d^s(t), ct\right) \,\middle|\, c \neq 0\right\}$. *Then, for all* $r \in \mathbb{R}$ *and* $s \in \mathbb{R}^*$*:*

$$c\text{-}\mathfrak{App}_{r,s} \leq \mathfrak{Rio}.$$

The usual Appell subgroup $\mathfrak{App}$ is obtained for $c = 1$ and $s = 1$; that is $1\text{-}\mathfrak{App}_{r,s} = \mathfrak{App}_s$.

3.2 The Lagrange subgroup $\left\{\left(1, h(t)\right)\right\}$

Proposition 9. *Let* $\mathfrak{Lag}_r = \left\{\left(\left(h(t)/t\right)^r, h(t)\right)\right\}$. *Then, for all* $r \in \mathbb{R}$*:*

$$\mathfrak{Lag}_r \leq \mathfrak{Rio}.$$

Note that when $r \neq 0$ the group $\mathfrak{Lag}_{1/r}$ coincides with the r-power-Bell subgroup of Jean-Louis and Nkwanta [15], whose arrays are of the form $((h(t)/t)^{1/r}, h(t))$. When $r = 1$ we recover the well-known isomorphism between the Lagrange and the Bell subgroups given by Jean-Louis and Nkwanta [15] and by He [10]. Furthermore, observe that by equation (6), the group $\{\phi_{r,s}(\mathfrak{Rio})\}_{r\in\mathbb{R},s\in\mathbb{R}^*}$ can be written as the semidirect product of the groups $\mathfrak{App}_s$ and $\mathfrak{Lag}_r$.

3.3 The c-Bell subgroup $\left\{\left(h(t)/t, c\, h(t)\right) \mid c \neq 0\right\}$

Proposition 10. *Let* $c\text{-}\mathfrak{Bell}_{r,s} = \left\{\left(c^r \left(h(t)/t\right)^{r+s}, c\, h(t)\right) \mid c \neq 0\right\}$. *Then, for all* $r \in \mathbb{R}$ *and* $s \in \mathbb{R}^*$*:*

$$c\text{-}\mathfrak{Bell}_{r,s} \leq \mathfrak{Rio}.$$

Here, the Bell subgroup for $c = 1$ is simply denoted by $\mathfrak{Bell}$. We have $\mathfrak{Bell}_{r,s} = \mathfrak{Lag}_{r+s}$.

Example 11. Recall that $c(t) = (1 - \sqrt{1-4t})/2t$ is the generating function of the Catalan numbers $C_n = 1/(n+1)\binom{2n}{n}$, $n \geq 0$ A000108. The function $c(t)$ satisfies the functional equation $c(t) = 1 + tc(t)^2$. There are several well-known Bell matrices in the literature generated by $c(t)$ and accordingly referred to as Catalan triangles, namely,

- Aigner's array $C = (c(t), tc(t))$ of the ballot numbers [1, 2];
- Shapiro's Catalan triangle $B = (c(t)^2, tc(t)^2)$ [23];
- Radoux's triangle of numbers $R = (c(t), tc(t)^2)$ [21].

It is easy to see that $\phi_{r,s}(B) = R$ when $r + s = 1/2$. Also, $\phi_{r,s}(R) = B$ when $2r + s = 2$. We thus have a system of two linear equations with two unknowns r and s. The solution gives the involution $\phi_{3/2,-1}$ such that $\phi_{3/2,-1}(B) = R$ and $\phi_{3/2,-1}(R) = B$.

Moreover, recall that Shapiro's triangle satisfies the following factorization property $B = CP$, where P is the classical Pascal triangle (see, e.g., [3]). Obviously, $\phi_{r,s}(B) = \phi_{r,s}(C)\phi_{r,s}(P)$. A similar factorization holds for Radoux's triangle. Indeed, for every pair (r, s) such that $r + s = 1/2$ it follows that $R = \phi_{r,s}(C)\phi_{r,s}(P)$. For the involution $\phi_{3/2,-1}$, we have $R = (c(t)^{1/2}, tc(t))(p(t)^{1/2}, tp(t))$ and $\phi_{r,s}(R) = \phi_{r+3s/2,-s}(C)\phi_{r+3s/2,-1}(P)$.

Furthermore, the authors, Petrullo, and Torres [4] showed that the matrices above are particular cases of a wide family of Catalan triangles given by

$$C^{a,b}(q,z) = \left(c(q,z;t)^a,\, t\, c(q,z;t)^b\right), \quad a, b \geq 1,$$

where

$$c(q,z;t)=\frac{1-zt-\sqrt{(1-zt)^2-4qt}}{2qt}\quad \text{with}\quad q,z\geq 0.$$

The family $C^{a,b}(q,z)$ includes:

- He's parametric Catalan triangles $C^{1,1}(q,z-q)$ [9];
- Yang's generalized Catalan triangles $C^{a,b}(q,0)$ [28].

It is easy to check that $\phi_{r,s}(C^{a,b}(q,z))=C^{br+as,b}(q,z)$. Also, note that the action of the automorphisms $\phi_{r,s}$ on the generalized Catalan triangles of He yields $\phi_{r,s}(C^{1,1}(q,z-q))=C^{r+s,1}(q,z-q)$. Hence, the image under $\phi_{r,s}$ of any He's parametric Catalan triangle is not a He's parametric Catalan triangle unless $r+s=1$.

3.4 The checkerboard subgroup $\Big\{\big(d(t),h(t)\big)\,\Big|\,d(t)=d(-t)\,,h(t)=-h(-t)\Big\}$

Proposition 12. *Let* $\mathfrak{Check}_{r,s}=\Big\{\left(\left(h(t)/t\right)^r d(t)^s,h(t)\right)\,\Big|d(t)=d(-t)\,,h(t)=-h(-t)\Big\}$. *Then, for all* $r\in\mathbb{R}$ *and* $s\in\mathbb{R}^*$*:*

$$\mathfrak{Check}_{r,s}\leq\mathfrak{Rio}.$$

Jean-Louis and Nkwanta [15] proved that the checkerboard subgroup is the centralizer of the element $(1,-t)$. We note that the checkerboard subgroup $\mathfrak{Check}$ is also the centralizer of $\phi_{r,s}(1,-t)$ for every pair (r,s).

3.5 The derivative subgroup $\Big\{\big(h'(t),h(t)\big)\Big\}$

Proposition 13. *Let* $\mathfrak{Der}_{r,s}=\Big\{\left(\left(h(t)/t\right)^r h'(t)^s,h(t)\right)\Big\}$. *Then, for all* $r\in\mathbb{R}$ *and* $s\in\mathbb{R}^*$*:*

$$\mathfrak{Der}_{r,s}\leq\mathfrak{Rio}.$$

The derivative subgroup is isomorphic to the Lagrange subgroup via Bacher's automorphism $\varphi_{0,1,-1}:(h'(t),h(t))\mapsto(1,h(t))$. The analogous result for the subgroups $\mathfrak{Der}_{r,s}$ and $\mathfrak{Lag}_r$ follows via $\varphi_{0,1,-s}:\left(\left(h(t)/t\right)^r h'(t)^s,h(t)\right)\mapsto\left(\left(h(t)/t\right)^r,h(t)\right)$. Note that the subgroup $\mathfrak{Der}_{r,s}$ was previously introduced in [17].

3.6 The hitting time subgroup $\Big\{\big(t\,h'(t)/h(t),h(t)\big)\Big\}$

Proposition 14. *Let* $\mathfrak{Hit}_{r,s}=\Big\{\Big(\left(h(t)/t\right)^{r-s}h'(t)^s,h(t)\Big)\Big\}$. *Then, for all* $r\in\mathbb{R}$ *and* $s\in\mathbb{R}^*$*:*

$$\mathfrak{Hit}_{r,s}\leq\mathfrak{Rio}.$$

The traditional hitting time subgroup corresponds to the case $r=-1$ and $s=1$. As a direct consequence of the above definition, we have $\mathfrak{Hit}_{r,s}=\mathfrak{Der}_{r-s,s}$. Also, the groups $\mathfrak{Hit}_{r,s}$ and $\mathfrak{Lag}_{r-s}$ are isomorphic via Bacher's automorphism $\varphi_{1,1,-s}$.

3.7 The Jean-Louis–Nkwanta subgroup $\Big\{\big(f(t)/f(h(t)),h(t)\big)\Big\}$

Recall that the stochastic Riordan subgroup is the stabilizer of the column vector $(1,1,1,\ldots)$. More generally, let $f(t)=f_0+f_1t+f_2t^2+\cdots$ be the generating function of the sequence $(f_0,f_1,f_2,\ldots)$. By using the fundamental theorem of Riordan arrays, Jean-Louis and Nkwanta [15] showed that the set of Riordan matrices of the form $(f(t)/f(h(t)),h(t))$ is the stabilizer of the column vector $(f_0,f_1,f_2,\ldots)$. In particular, the elements of the stochastic Riordan subgroup are of the form $\big((1-h(t))/(1-t),h(t)\big)$.

Proposition 15. *Let* $\mathfrak{JLN}_{r,s}=\Big\{\Big(\big(h(t)/t\big)^r\big(f(t)/f(h(t))\big)^s,h(t)\Big)\Big\}$. *Then, for all* $r\in\mathbb{R}$ *and* $s\in\mathbb{R}^*$*:*

$$\mathfrak{JLN}_{r,s}\leq\mathfrak{Rio}.$$

3.7.1 The stochastic subgroup

Let $\mathfrak{D}^h=\phi_{1,1}\big((1-h(t))/(1-t),h(t)\big)$. Note that the row sum sequence of $\mathfrak{D}^h$, here denoted $\mathfrak{r}^{\mathfrak{D}^h}$, is the partial sum of the coefficients of the generating function $h(t)$. That is, $\mathfrak{r}_n^{\mathfrak{D}^h}=\sum_{i=1}^n h_i$. This follows from observing that the generating function for the row sum of a Riordan array is given by $d(t)/(1-h(t))$ [24, 11]. Therefore, the Riordan subgroup $\mathfrak{Stoch}_{1,1}=\{\mathfrak{D}^h\}$ is the set of Riordan arrays whose row sum sequence is the partial sum of the coefficients of the generating function $h(t)$.

Example 16. Let $h(t)/t=p(t),c(t),c^2(t)$, where $p(t)=1/(1-t)$ and $c(t)$ the generating function of the Catalan numbers. The elements of the group $\mathfrak{Stoch}_{1,1}$ related to these functions are:

$$\mathfrak{D}^{tp}=\left(\frac{p(t)-tp^2(t)}{1-t},tp(t)\right)=\begin{bmatrix}1&0&0&0&0&0&\cdots\\1&1&0&0&0&0&\cdots\\0&2&1&0&0&0&\cdots\\-2&2&3&1&0&0&\cdots\\-5&0&5&4&1&0&\cdots\\-9&-5&5&9&5&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix},$$

$$\mathfrak{D}^{tc} = \Big(\frac{c(t)-tc^2(t)}{1-t}, tc(t)\Big) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 9 & 8 & 4 & 1 & 0 & \cdots \\ 1 & 23 & 22 & 13 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\mathfrak{D}^{tc^2} = \Big(\frac{c^2(t)-tc^4(t)}{1-t}, tc^2(t)\Big) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 12 & 6 & 1 & 0 & 0 & \cdots \\ -3 & 33 & 25 & 8 & 1 & 0 & \cdots \\ -36 & 88 & 91 & 42 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Their row sum sequence numbers are A000027, A014137, and A014138. Obviously, we have $\mathfrak{D}^{tc}\mathfrak{D}^{tp} = \mathfrak{D}^{tc^2}$.

The above result fixes $s = 1$. Restricting s to integers greater than 1 gives the next result, which is a closed form for the row-sum sequences of the the arrays in the image of the Stochastic subgroup under the mapping $\phi_{1,n}$.

Proposition 17. *Let $\mathfrak{D}^h_n = \phi_{1,n}\big((1-h(t))/(1-t), h(t)\big)$, where $h(t) = h_1t + h_2t^2 + h_3t^3 + \cdots$ with $h_1 \neq 0$ is any power series over the complex numbers. Then, for all $n \geq 1$ the Riordan subgroup $\mathfrak{Stoch}_{1,n} = \{\mathfrak{D}^h_n\}$ is the set of Riordan arrays whose row sum sequence is given by:*

$$\begin{aligned} \mathfrak{r}^{\mathfrak{D}^h_n} &= \sum_{j=0}^{\infty}\sum_{i=0}^{j}\sum_{k=0}^{n-1}\sum_{i_1+i_2+\cdots+i_{k+1}=i} h_{i_1+1}\ldots h_{i_{k+1}+1} \times \\ &\quad (-1)^k\binom{n+j-i-1}{n-1}\binom{n-1}{k}t^{j+k}, \end{aligned}$$

where the third sum runs over all sequences of non-negative integers $(i_1, \ldots, i_{k+1})$ satisfying $i_1 + i_2 + \cdots + i_{k+1} = i$.

Proof. The proof is a straightforward computation of the product of the power series $h(1-h(t))^{n-1}/t$ and $1/(1-t)^n = \sum_{i=0}^{\infty}\binom{n+i-1}{n-1}t^i$. $\square$

Example 18. We consider again $h(t)/t = p(t), c(t), c^2(t)$. The row sum sequences associated with the elements $\mathfrak{D}^{tp}_2$, $\mathfrak{D}^{tc}_2$, and $\mathfrak{D}^{tc^2}_2$ of the group $\mathfrak{Stoch}_{1,2}$ are:

$$\begin{aligned} \big(\mathfrak{r}^{\mathfrak{D}^{tp}_2}\big) &= 1, 2, 2, 0, -5, -14, -28, -48, -75, -110, -154, -208, -273, -350, -440, \ldots \\ &\quad (\textit{A005586} \text{ from the fourth term on}), \\ \big(\mathfrak{r}^{\mathfrak{D}^{tc}_2}\big) &= 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \ldots (\textit{A000027}), \\ \big(\mathfrak{r}^{\mathfrak{D}^{tc^2}_2}\big) &= 1, 3, 6, 9, 6, -30, -209, -960, -3921, -15280, -58293, -220170, -827787, \ldots. \end{aligned}$$

4 Bi-infinite triangles defined via automorphisms

This section uses automorphisms to introduce bi-infinite triangles. These bi-infinite triangles coincide with the recursive matrices of Luzón, Merlini, Morón, and Sprugnoli [17]. Recall that a recursive matrix $(d_{k',n'})_{k'\geq n'\in\mathbb{Z}} = \chi(d(t), h(t))$ is defined by

$$d_{k',n'} = [t^{k'}]d(t)h(t)^{n'} \quad \forall k', n' \in \mathbb{Z}, \quad k' \geq n',$$

where $d(t)$ and $h(t)$ are formal power series as in the definition of Riordan arrays. Also, $d(t)h(t)^{n'} = \sum_{i=n'}^{\infty} l_i t^i$ is a formal Laurent series (which yields a formal power series if $n' \geq 0$).

Let m be a non-negative integer. Let D be any Riordan array. We examine the matrix $\phi_{-m,1}(D)$ for large m. Though, for simplicity, we fix the second parameter of the automorphism equal to 1, our results generalize straightforwardly for $\phi_{-m,s}(D)$, with $s \in \mathbb{R}^*$, by replacing $d(t)$ by $d(t)^s$.

For any non-negative integer m, the submatrix of $\phi_{-m,1}(D)$ that results from deleting the first m columns and the first m rows is the base Riordan array D. Moreover, any array $\phi_{-m,1}(D)$ may be obtained from $\phi_{-(m+1),1}(D)$ by removing the first row and the first column. That is to say that the array sequence $(\phi_{-m,1}(D))_{m\geq 1}$ is an ascending chain of matrices, in the north west direction. Thus, every array in the sequence $(\phi_{-m,1}(D))_{m\geq 1}$ is a submatrix of a bi-infinite triangle, denoted by $\phi_{-\infty,1}(D)$, constructed as follows. Take the matrix D and place on the right of D, the first column of each of the matrices $\phi_{-m,1}(D)$ (from right to left). Then, add infinite rows of zeros on the top of each column (including the columns of D) to yield bi-infinite columns of a bi-infinite triangle. Then, for any pair of integers (k', n') the entries of $\phi_{-\infty,1}(D)$ are given by

$$\left(\phi_{-\infty,1}(D)\right)_{k'\geq n'} = \begin{cases} D_{k',n'}, & \text{if } n' \geq 1; \\ \left(\phi_{n'-1,1}(D)\right)_{k'-n'+1,1}, & \text{otherwise,} \end{cases}$$

where $\left(\phi_{n'-1,1}(D)\right)_{k'-n'+1,1} = [t^{k'-1}]d(t)h(t)^{n'-1}$. Therefore, our construction of the bi-infinite triangle $\phi_{-\infty,1}(D)$ leads to the definition of the recursive array $\chi(d(t), h(t))$ of Luzón, Merlini, Morón, and Sprugnoli [17], up to a column phase shift of one unit. Observe that the phase shift is due to the fact that in Equation (1) the rows and columns, indexed by k and n, respectively, are numbered from 1 on, while in [17] they go from 0 on.

The array $\phi_{-\infty,1}(D)$ can be visualized as follows:

$$\phi_{-\infty,1}(D) = \left[\begin{array}{cc} \mathfrak{L}_D & \mathfrak{R}_D \end{array} \right],$$

where $\mathfrak{R}_D = \left[\quad D \; \right]^T$ and the matrix $\mathfrak{L}_D$ is a lower-triangular array which grows to the west region of $\phi_{-\infty,1}(D)$, while the matrix D grows to the south east region.

Although the bi-infinite triangle $\phi_{-\infty,1}(D)$ is not a Riordan array, this can be approximated by a Riordan array $\phi_{-m,1}(D)$ when m is large. For practical calculations, one can thus work with a Riordan array $\phi_{-m,1}(D)$ after choosing m large enough to fit the purpose.

Indeed, the matching between the columns of both bi-infinite triangles can also be seen from the large m approximation of $\phi_{-\infty,1}(D)$. We have

$$\phi_{-m,1}(D)_{k,n} = [t^{k-m-1}]d(t)h(t)^{n-m-1}, \quad \forall k \geq n \geq 1.$$

Let $k' = k-m-1$ and $n' = n-m-1$. Then, when m is large, say m approaches infinity, both k' and n' run through all integers in $\mathbb{Z}$. Hence, the array $\phi_{-\infty,1}(D)$ has the same columns as the recursive matrix $\chi(d(t), h(t))$. However, since the image of a Riordan array under an automorphism is a Riordan array, the expansion of the columns of $\phi_{-m,1}(D)$ never involves formal Laurent series, only formal power series whose coefficient sequences coincide with the coefficient sequences of the formal Laurent series in the corresponding recursive matrices. Note that in this case, there is a phase shift of $m+1$ units between the index of a column in $\phi_{-m,1}(D)$ and the index of the same column in $\chi(d(t), h(t))$. A recursive matrix $\chi(D)$ can thus be studied through a Riordan array $\phi_{-m,1}(D)$ after choosing an appropriate value for m.

Furthermore, the matrix $\mathfrak{L}_D$ (read from right to left) mirrors a well-defined Riordan array.

Definition 19. Let D be any Riordan array. Consider the matrix sequence $(\phi_{-m,1}(D))_{m\geq 1}$. We define the lower triangular matrix $\mathfrak{l}_D$ as one whose mth column is the first column of the matrix $\phi_{-m,1}(D)$ for each $m \geq 1$, with m rows of zeros added to the top of the column. Hence, the entries of $\mathfrak{l}_D$ read as follows:

$$(\mathfrak{l}_D)_{k,n} = [t^{k-2n}]d(t)h(t)^{-n}, \quad k \geq n \geq 1.$$

Thus, by definition, $\mathfrak{l}_D$ is the following Riordan array:

$$\mathfrak{l}_D = \left(\frac{t}{h(t)}d(t), \frac{t^2}{h(t)}\right). \tag{9}$$

Note that a similar idea to that of defining the matrices $\phi_{-\infty,1}(D)$ and $\mathfrak{l}_D$ from a sequence of images of the Riordan array D under some automorphisms, was recently given by the authors for the construction of some square matrices related to Riordan arrays [19].

Remark 20. Let D be any Riordan array. The bi-infinite triangle $\phi_{-\infty,1}(D)$ corresponds to the following pair of Riordan arrays:

$$(\mathfrak{l}_D, D) = \left(\left(\frac{t}{h(t)}d(t), \frac{t^2}{h(t)}\right), \big(d(t), h(t)\big)\right),$$

in the sense of yielding exactly the same columns.

Now, recall briefly that for $m \in \mathbb{Z}$, the $[m]$-complementary array $\chi(D)^{[m]}$ of a recursive matrix $\chi(D)$, is defined by $\chi(D)^{[m]}_{k,n} = \chi(D)_{m-n,m-k}$ [17]. Next, for fixed $m \geq 1$, we introduce the following lower triangular array with m columns $\mathfrak{L}_D^{(m)}$:

$$(\mathfrak{L}_D^{(m)})_{k,n} = [t^{k-m-1}]d(t)h(t)^{n-m-1}, \quad 1 \leq n \leq k, m.$$

That is,

$$(\mathfrak{L}_D^{(m)})_{k,n} = \phi_{-\infty,1}(D)_{k-m,n-m} = \chi(D)^{[-m]}_{-n+1,-k+1}.$$

We also remark that the $[m]$-complementary array of a Riordan array D reads as

$$D^{[m]} = \left(\left(\frac{t}{\bar{h}(t)} \right)^{m+1} d(\bar{h}(t))\bar{h}'(t), \bar{h}(t) \right)$$

yields $\varphi_{-(m+1),-1,1}(D^{-1})$, where Bacher's automorphism $\varphi_{-(m+1),-1,1}$ is defined by formula (5).

Additionally, the first $l \geq 1$ rows of the matrix $\mathfrak{L}_D^{(m)}$ yield a finite matrix, say, $\mathfrak{L}_D^{(l,m)}$. Moreover, consider the submatrix, say $D^{(l',m')}$, of the base array D consisting of the first $l' \geq 1$ rows and $m' \geq 1$ columns of D. In this context, for fixed $q \geq 0$ we consider the following finite $(2q+1) \times (2q+1)$ and $(2q+2) \times (2q+2)$ matrices:

$$A_q = \begin{bmatrix} \mathfrak{L}_D^{(2q+1,q)} & \mathfrak{R}_D^{(q+1,q+1)} \end{bmatrix}, \quad \text{where} \quad \mathfrak{R}_D^{(q+1,q+1)} = \begin{bmatrix} \updownarrow q \text{ rows of zeros} \\ D^{(q+1,q+1)} \end{bmatrix},$$

$$B_q = \begin{bmatrix} \mathfrak{L}_D^{(2q+2,q)} & \mathfrak{R}_D^{(q+2,q+2)} \end{bmatrix}, \quad \text{where} \quad \mathfrak{R}_D^{(q+2,q+2)} = \begin{bmatrix} \updownarrow q \text{ rows of zeros} \\ D^{(q+2,q+2)} \end{bmatrix}.$$

Note that the west side matrices do not occur in A_0 and B_0. We observe that $(A_q)_{q\geq 0}$ and $(B_q)_{q\geq 0}$ are sequences of finite matrices such that $A_q = \gamma_q(D)$, $B_q = \delta_q(D)$, where $(\gamma_q)_{q\geq 0}$ and $(\delta_q)_{q\geq 0}$ are sequences of projections introduced by Luzón, Merlini, Morón, Prieto-Martinez, and Sprugnoli [18].

Example 21. Consider Pascal's array $P = (p(t), tp(t))$, where $p(t) = 1/(1-t)$, sequence A007318:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & \cdots \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{10}$$

Hence, $P_{k,n} = \binom{k-1}{n-1}$, $n \geq k \geq 1$. Let $m \geq 1$. We now examine the matrix $\phi_{-m,1}(P) = ((1-t)^{m-1}, t/(1-t))$ for large m. For instance, we have for $m = 10$:

$$\phi_{-10,1}(P) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 36 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -84 & 28 & -7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 126 & -56 & 21 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -126 & 70 & -35 & 15 & -5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 84 & -56 & 35 & -20 & 10 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -36 & 28 & -21 & 15 & -10 & 6 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 9 & -8 & 7 & -6 & 5 & -4 & 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We observe that the matrix $\mathfrak{L}_P^{(10)}$ is zero below the equator. Note that is not true in general for the matrix $\mathfrak{L}_D^{(m)}$, with $m \geq 1$ associated with an arbitrary Riordan array D. Indeed, this happens for all $m \geq 1$ with the Pascal array by chance, for only the first two coefficients of $p^{-1}(t)$ are different from zero.

When m is large, in the north west region of the matrix $\phi_{-m,1}(P)$, the transpose of the reflection of the matrix $\mathfrak{L}_P$ (read from right to left) over the equator yields the matrix $P^{-1} = (1/(1+t), t/(1+t))$, the inverse of Pascal's array. The columns of $\mathfrak{L}_P$ (read from right to left) also coincide with the sequence A110555 when seen as an upper right triangle. Besides, the lower triangular matrix $\mathfrak{l}_P$ of formula (9) is

$$\mathfrak{l}_P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & -4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 6 & -5 & 1 & \cdots \\ 0 & 0 & 0 & 0 & -4 & 10 & -6 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -10 & 15 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 5 & -20 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -1 & 15 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, $(\iota_P)_{k,n} = (-1)^{k-n}\binom{n-1}{k-n}$. The diagonals of ι_P yield the columns in the north east region of the bi-infinite matrix of Barnabei [6, p. 1132]. Note that a similar correspondence also holds for the matrix of p. 1139 of the same paper.

Example 22. We compute the array $\iota_T(x, y)$ of formula (9), where $T(x, y)$ is a Pascal-like array of polynomials in the variables x and y, introduced by Barry [7]:

$$\begin{aligned} T(x,y) &= \left(\frac{1}{1-yt}, \frac{yt(1+xt)}{1-yt}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ y & y & 0 & 0 & 0 & \cdots \\ y^2 & xy+2y^2 & y^2 & 0 & 0 & \cdots \\ y^3 & 2xy^2+3y^3 & 2xy^2+3y^3 & y^3 & 0 & \cdots \\ y^4 & 3xy^3+4y^4 & x^2y^2+6xy^3+6y^4 & 3xy^3+4y^4 & y^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

The general expression for the entries is:

$$T_{k,n}(x,y) = \sum_{j=0}^{n-1} \binom{n-1}{j}\binom{k-1-j}{n-1} x^j y^{k-1-j}, \quad k, n \geq 1.$$

Then, the matrix ι_T is

$$\begin{aligned} \iota_T(x,y) &= \left(\frac{1}{y(1+xt)}, \frac{t(1-yt)}{y(1+xt)}\right) \\ &= \begin{bmatrix} \frac{1}{y} & 0 & 0 & 0 & 0 & \cdots \\ -\frac{x}{y} & \frac{1}{y^2} & 0 & 0 & 0 & \cdots \\ \frac{x^2}{y} & -\frac{2x+y}{y^2} & \frac{1}{y^3} & 0 & 0 & \cdots \\ -\frac{x^3}{y} & \frac{3x^2+2xy}{y^2} & -\frac{3x+2y}{y^3} & \frac{1}{y^4} & 0 & \cdots \\ \frac{x^4}{y} & -\frac{4x^3+3x^2y}{y^2} & \frac{6x^2+6xy+y^2}{y^3} & -\frac{4x+3y}{y^4} & \frac{1}{y^5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

Hence, the complete central coefficients sequence introduced by Barry [7] of the matrix $T(x, y)$ is the sequence:

$$T_{k,\lceil\frac{k}{2}\rceil}(x,y) = \sum_{j=0}^{\lceil\frac{k}{2}\rceil-1} \binom{\lceil\frac{k}{2}\rceil-1}{j}\binom{k-1-j}{\lceil\frac{k}{2}\rceil-1} x^j y^{k-1-j}, \quad k, n \geq 1.$$

Note that Barry considers $\lfloor k/2 \rfloor$ instead of $\lceil k/2 \rceil$ for in [7] $k, n \geq 0$, while here $k, n \geq 1$. The general term of the Riordan array $\iota_T(x, y)$ is given by:

$$(\iota_T)_{k,n}(x,y) = \frac{(-1)^{k+1}}{y^n} \sum_{j=0}^{n-1} \binom{n-1}{j}\binom{k-1-j-1}{n-1} x^{k-j-n} y^j, \quad k, n \geq 1.$$

Therefore, the complete central coefficients sequence of Barry of the matrix $\mathfrak{l}_T(x,y)$ is the sequence:

$$(\mathfrak{l}_T)_{k,\lceil\frac{k}{2}\rceil}(x,y) = \frac{(-1)^{k+1}}{y^{\lceil\frac{k}{2}\rceil}} \sum_{j=0}^{\lceil\frac{k}{2}\rceil-1} \binom{\lceil\frac{k}{2}\rceil-1}{j}\binom{k-1-j-1}{\lceil\frac{k}{2}\rceil-1} x^{k-j-\lceil\frac{k}{2}\rceil} y^j, \quad k,n \geq 1.$$

Example 23. We compute the Riordan array $\mathfrak{l}_{FC}$, where

$$\begin{aligned} FC &= \left(\frac{1}{1-t-t^2}, tc(t)\right) \\ &= \left(\frac{1}{1-t-t^2}, \frac{1-\sqrt{1-4t}}{2}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 5 & 3 & 1 & 0 & 0 & \cdots \\ 5 & 12 & 9 & 4 & 1 & 0 & \cdots \\ 8 & 31 & 26 & 14 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{aligned}$$

is the Fibonacci-Catalan triangle whose first column yields the Fibonacci numbers A000045 (without the first element which is zero). Moreover, the second column and the row sum sequence number both yield the convolution of the Fibonacci and the Catalan numbers A090826. The sequence numbers for the first three descending diagonals are A000012, A000027, and A000096. The next three descending diagonals yield:

$$\begin{aligned} FC_{3+n,n} &= \frac{n^3+9n^2+20n-12}{3!}, \quad n \geq 1, \\ FC_{4+n,n} &= \frac{n^4+18n^3+107n^2+162n-168}{4!}, \quad n \geq 1, \\ FC_{5+n,n} &= \frac{n^5+30n^4+335n^3+1530n^2+1824n-2760}{5!}, \quad n \geq 1. \end{aligned}$$

The array $\mathfrak{l}_{FC}$ of formula (9) is

$$\begin{aligned} \mathfrak{l}_{FC} &= \left(\frac{2t}{(1-t-t^2)(1-\sqrt{1-4t})}, \frac{2t^2}{1-\sqrt{1-4t}}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & 0 & \cdots \\ -2 & -1 & -2 & 1 & 0 & 0 & \cdots \\ -7 & -4 & -1 & -3 & 1 & 0 & \cdots \\ -23 & -10 & -4 & 0 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

The coefficients of the generating function $2t/(1-\sqrt{1-4t})$ yield the sequence A115140. Then, the first column of $\mathfrak{l}_{FC}$ is the convolution of A115140 and A000045 (after removing the first element of the latter). The first descending diagonal is of course A000012. The second is A001478. The third is A000096 from the fourth element on. We now give recursion formulas for the fourth and fifth descending diagonals from the sixth and seventh element on, respectively. The following holds:

$$\begin{aligned}
(\mathfrak{l}_{FC})_{3+n,n} &= -\frac{n^3-12n^2+41n-18}{3!}, \quad n \geq 6,\\
(\mathfrak{l}_{FC})_{4+n,n} &= \frac{n^4-22n^3+167n^2-434n+120}{4!}, \quad n \geq 7.
\end{aligned}$$

Alternatively, we have the following recursion formulas:

$$\begin{aligned}
(\mathfrak{l}_{FC})_{9,6} &= -2, \quad (\mathfrak{l}_{FC})_{10,7} = -4,\\
(\mathfrak{l}_{FC})_{3+n,n} &= -\big((\mathfrak{l}_{FC})_{2+n,n-2} - 2(\mathfrak{l}_{FC})_{3+n,n-1} + n - 5\big), \quad n \geq 8,\\
(\mathfrak{l}_{FC})_{11,7} &= 5, \quad (\mathfrak{l}_{FC})_{12,8} = 7, \quad (\mathfrak{l}_{FC})_{13,9} = 11,\\
(\mathfrak{l}_{FC})_{4+n,n} &= 3(\mathfrak{l}_{FC})_{4+n,n-1} - 3(\mathfrak{l}_{FC})_{3+n,n-2} + (\mathfrak{l}_{FC})_{2+n,n-3} + n - 7, \quad n \geq 10.
\end{aligned}$$

5 Acknowledgments

We would like to thank the anonymous referee for his valuable suggestions. This research was made within the activities of the Group for Linear, Algebraic and Combinatorial Structures of the Center for Functional Analysis, Linear Structures and Applications (University of Lisbon, Portugal). The first and the second authors were supported by the research fellowships SFRH/BPD/100567/2014 and SFRH/BPD/111317/2015, respectively, provided by the Portuguese Foundation for Science and Technology (FCT).

References

[1] M. Aigner, Catalan and other numbers: a recurrent theme, in *Algebraic Combinatorics and Computer Science: A Tribute to Gian-Carlo Rota*, Springer, 2001, pp. 347–390.

[2] M. Aigner, Enumeration via ballot numbers, *Discrete Math.* **308** (2008), 2544–2563.

[3] J. Agapito, Â. Mestre, P. Petrullo, and M. M. Torres, On one-parameter Catalan arrays, *J. Integer Sequences* **18** (2015), Article 15.5.1.

[4] J. Agapito, Â. Mestre, P. Petrullo, and M. M. Torres, Combinatorics of a generalized Narayana identity, *Linear Algebra Appl.* **503** (2016), 56–82.

[5] R. Bacher, Sur le groupe d'interpolation, preprint, 2006. Available at https://arxiv.org/abs/math/0609736.

[6] M. Barnabei, Polynomial sequences of integral type and recursive matrices, *Comput. Math. Appl.* **41** (2001), 1125–1141.

[7] P. Barry, The central coefficients of a family of Pascal-like triangles and colored lattice paths , *J. Integer Sequences* **22** (2019), Article 19.1.3.

[8] G.-S. Cheon and S.-T. Jin, Structural properties of Riordan matrices and extending the matrices, *Linear Algebra Appl.* **435** (2011), 2019–2032.

[9] T. X. He, Parametric Catalan numbers and Catalan triangles, *Linear Algebra Appl.* **438** (2013), 1467–1484.

[10] T.-X. He, Matrix characterizations of Riordan arrays, *Linear Algebra Appl.* **465** (2015), 15–42.

[11] T.-X. He and L. W. Shapiro, Row sums and alternating sums of Riordan arrays, *Linear Algebra Appl.* **507** (2016), 77–95.

[12] E. Jabotinsky, Sur la représentation de la composition de fonctions par un produit de matrices. Application à l'itération de e^z et de $e^z - 1$, *C. R. Acad. Sci. Paris* **224** (1947), 323–324.

[13] E. Jabotinsky, Sur les fonctions inverses, *C. R. Acad. Sci. Paris* **229** (1949), 508–509.

[14] E. Jabotinsky, Analytic iteration, *Trans. Amer. Math. Soc.* **108** (1963), 457–477.

[15] C. Jean-Louis and A. Nkwanta, Some algebraic structure of the Riordan group, *Linear Algebra Appl.* **438** (2013), 2018–2035.

[16] D. L. Johnson, The group of formal power series under substitution, *J. Aust. Math. Soc. Ser. A* **45** (1988), 296–302.

[17] A. Luzón, D. Merlini, M. A. Morón, and R. Sprugnoli, Complementary Riordan arrays, *Discrete Appl. Math.* **172** (2014), 75–87.

[18] A. Luzón, D. Merlini, M. A. Morón, L. F. Prieto-Martínez, and R. Sprugnoli, Some inverse limit approaches to the Riordan group, *Linear Algebra Appl.* **491** (2016), 239–262.

[19] Â. Mestre and J. Agapito, Square matrices generated by sequences of Riordan arrays, *J. Integer Sequences* **22** (2019), Article 19.8.4.

[20] B. Muckenhoupt, Automorphisms of formal power series under substitution, *Trans. Amer. Math. Soc.* **99** (1961), 373–383.

[21] C. Radoux, Addition formulas for polynomials built on classical combinatorial sequences, *J. Comput. Appl. Math.* **115** (2000), 471–477.

[22] D. G. Rogers, Pascal triangles, Catalan numbers, and renewal arrays, *Discrete Math.* **22** (1978), 301–310.

[23] L. W. Shapiro, A Catalan triangle, *Discrete Math.* **14** (1976), 83–90.

[24] L. W. Shapiro, Bijections and the Riordan group, *Theoret. Comput. Sci.* **307** (2003), 403–413.

[25] L. W. Shapiro, S. Getu, W. Woan, and L. Woodson, The Riordan group, *Discrete Appl. Math.* **34** (1991), 229–239.

[26] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`, 2018.

[27] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994), 267–290.

[28] S.-L. Yang, Some inverse relations determined by Catalan matrices, *Int. J. Comb.* (2013), Art. ID 528584.

2010 *Mathematics Subject Classification*: Primary 05B20; Secondary 18A30, 20E18, 20H20.
Keywords: Riordan group, Riordan subgroup, automorphism, isomorphism, bi-infinite triangle.

(Concerned with sequences A000012, A000027, A000045, A000096, A000108, A001478, A005586, A007318, A014137, A014138, A090826, A110555, and A115140.)

Received July 10 2019; revised versions received December 23 2019; December 24 2019. Published in *Journal of Integer Sequences*, December 26 2019.

Return to Journal of Integer Sequences home page.