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A Graph-Theoretic Model for a Generalized Fibonacci Gem

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Abstract

We extend a charming Fibonacci pleasantry to Fibonacci, Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials; and then confirm the resulting polynomial delights using graph-theoretic tools.

1 Introduction

Generalized Fibonacci polynomials $z_n(x)$ are defined by the recurrence $z_n(x) = a(x)z_{n-1}(x) + b(x)z_{n-2}(x)$, where x is an arbitrary complex variable; $a(x), b(x), z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 2$.

Let $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [2, 3, 12, 13].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [7, 10].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [5, 6]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

The polynomials $f_n(x), l_n(x), J_n(x)$, and $j_n(x)$ can also be defined explicitly using *Binet-like* formulas:

$$f_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}; \qquad l_n(x) = \alpha^n + \beta^n;$$
$$J_n(x) = \frac{u^n - v^n}{u - v}; \qquad j_n(x) = u^n + v^n,$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are the solutions of the equation $t^2 - xt - 1$, and $u = u(x)$; and $v = v(x)$ are those of $t^2 - t - x = 0$. Notice that $\alpha - \beta = \sqrt{x^2 + 4}$ and $u - v = \sqrt{4x + 1}$.

In the interest of brevity, clarity, and convenience, we *omit* the argument in the functional notation, when there is *no* ambiguity; so z_n means $z_n(x)$. In addition, we let $g_n = f_n$ or l_n; $b_n = p_n$ or q_n; and $c_n = J_n(x)$ or $j_n(x)$; and correspondingly, $G_n = F_n$ or L_n; $B_n = P_n$ or Q_n; and $C_n = J_n$ or j_n.

2 Q-matrix and digraph

Gibonacci polynomials f_n and l_n can be studied using the Q-matrix

$$Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},$$

where $Q = Q(x) = (q_{ij})_{2\times 2}$ [11, 14]. It then follows by induction that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$.

The Q-matrix has a graph-theoretic appeal. It can be interpreted as the *weighted adjacency matrix* of a *weighted digraph* D_1 with vertices v_1 and v_2 [11, 14]; see Figure 1. Notice that a *weight* is assigned to each edge.

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

We can employ the weighted adjacency matrix to compute the weight of a walk of length n from any vertex v_i to any vertex v_j, as the following theorem shows [9, 11].

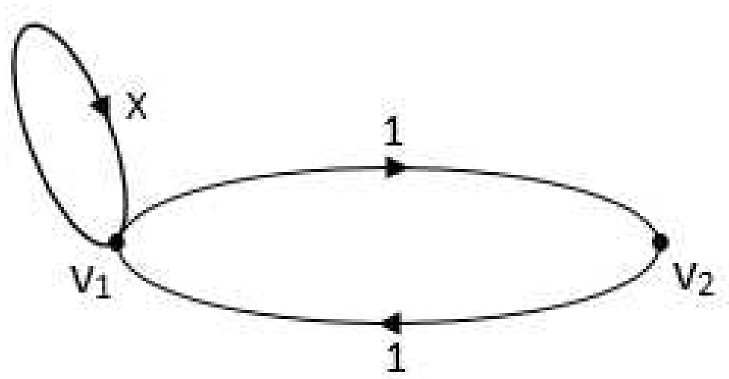

Figure 1: Weighted digraph D_1

Theorem 1. *Let A be the weighted adjacency matrix of a weighted and connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the ijth entry of the matrix A^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.*

This theorem implies the following result.

Corollary 2. *The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i, j \leq 2$.*

It follows by this corollary that the sum of the weights of all closed walks of length n originating in the digraph model is f_{n+1}, and that of walks of length n originating at v_2 is f_{n-1}. So the sum of the weights of all closed walks of length n is $f_{n+1} + f_{n-1} = l_n$.

3 A Gibonacci delight

In 1963, H. W. Gould established a charming identity for Fibonacci squares [8, 13]:

$$F_{n+3}^2 = 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2. \tag{1}$$

It has a simple, but delightful geometric interpretation [13].

The next theorem extends identity (1) to gibonacci polynomials g_n.

Theorem 3.

$$g_{n+3}^2 = f_3 g_{n+2}^2 + f_3 g_{n+1}^2 - g_n^2. \tag{2}$$

Proof. Using the gibonacci recurrence, we have

$$\begin{aligned} g_{n+3}^2 + g_n^2 &= (xg_{n+2} + g_{n+1})^2 + (g_{n+2} - xg_{n+1})^2 \\ &= (x^2+1)g_{n+2}^2 + (x^2+1)g_{n+1}^2. \end{aligned}$$

This yields the desired identity. (It also follows by Binet's formulas.) $\square$

4 Graph-theoretic models

With these tools at our finger tips, we can give graph-theoretic interpretations of the gibonacci results in Theorem 3. The essence of our technique hinges on Corollary 2, and the “weighted”version of *Fubini's principle* [1, 13]: *Counting the number of elements in a set in two different ways yields the same result.*

We begin our discourse with $g_n = f_n$.

4.1 Interpretation with $g_n = f_n$

It follows by Corollary 2 that the sum of the weights of closed walks of length $n+2$ originating at v_1 is f_{n+3}. The sum S of the weights of ordered pairs (v, w) of such closed walks is the product of the sum of the weights of such walks v and w. Consequently, $S = f_{n+3}^2$.

We will now compute the sum S in a different way.

Proof. Case 1. Suppose v and w begin with a loop at v_1. The sum of the weights of pairs (v, w) of such closed walks of length $n + 2$ is $(xf_{n+2})(xf_{n+2}) = x^2 f_{n+2}^2$.

Case 2. Suppose v begins with a loop at v_1, but w does *not*. The sum of the weights of pairs of such closed walks is $(xf_{n+2})(1 \cdot 1 \cdot f_n) = xf_{n+2}f_n$.

Case 3. On the other hand, suppose v does *not* begin with a loop, but w does. The sum of the weights of pairs of such closed walks is $(1 \cdot 1 \cdot f_n)(xf_{n+2}) = xf_{n+2}f_n$.

Case 4. Finally, suppose neither v nor w begins with a loop. The contribution of pairs of such walks toward the sum S is $(1 \cdot f_{n+1})(1 \cdot f_{n+1}) = f_{n+1}^2$.

Combining the four cases, we also get

$$\begin{aligned} S &= x^2 f_{n+2}^2 + f_{n+1}^2 + 2xf_{n+2}f_n \\ &= (x^2 + 1)f_{n+2}^2 + (x^2 + 1)f_{n+1}^2 - f_n^2, \end{aligned}$$

as in the proof of Theorem 3.

Equating the cumulative sums yields the desired result. □

As a byproduct, this discourse then gives a graph-theoretic proof of the Pell identity

$$p_{n+3}^2 = p_3 p_{n+2}^2 + p_3 p_{n+1}^2 - p_n^2.$$

Next we investigate the graph-theoretic interpretation of identity (2) with $g_n = l_n$.

4.2 Interpretation with $g_n = l_n$

Proof. Let A denote the set of closed walks of length $n+3$ originating at v_1, and B that of length $n+3$ originating at v_2. Let $C = A \cup B$, where $A \cap B = \emptyset$. The sum of the weights of all closed walks in C equals $f_{n+4} + f_{n+2} = l_{n+3}$. Consequently, the sum S of the weights of ordered pairs $(v, w) \in C \times C$ is given by $S = l_{n+3}^2$.

To compute this sum in a different way, first we make an interesting observation. By Theorem 3, we have

$$\begin{aligned} x^2 f_{n+3}^2 + 4f_{n+2}^2 + 4x f_{n+3} f_{n+2} &= (x f_{n+3} + 2f_{n+2})^2 \\ &= (f_{n+4} + f_{n+2})^2 \\ &= l_{n+3}^2 \\ &= f_3 l_{n+2}^2 + f_3 l_{n+1}^2 - l_n^2. \end{aligned} \tag{3}$$

Consequently, it suffices to establish graph-theoretically the equivalent identity

$$x^2 f_{n+3}^2 + 4f_{n+2}^2 + 4x f_{n+3} f_{n+2} = l_{n+3}^2. \tag{4}$$

We will accomplish this using four cases for an arbitrary element $(v, w) \in C \times C$.

Case 1. Suppose $v, w \in A$. Suppose both v and w begin with a loop. The sum of the weights of pairs (v, w) of such closed walks is $(x f_{n+3})(x f_{n+3}) = x^2 f_{n+3}^2$. If v begins with a loop at v_1 and w does *not*, then $v \in A$ and $w \in B$. The sum of the weights of all such pairs (v, w) of closed walks equals $(x \cdot f_{n+3})(1 \cdot 1 \cdot f_{n+2}) = x f_{n+3} f_{n+2}$. Suppose v does *not* begin with a loop, but w does. Then $v \in B$ and $w \in A$. The sum of the weights of all such pairs (v, w) of closed walks equals $(1 \cdot 1 \cdot f_{n+2})(x \cdot f_{n+3}) = x f_{n+3} f_{n+2}$. Suppose neither v nor w begins with a loop. The total contribution by the corresponding pairs (v, w) is $(1 \cdot 1 \cdot f_{n+2})(1 \cdot 1 \cdot f_{n+2}) = f_{n+2}^2$.

Thus, when $v, w \in A$, the sum of the weights of such closed walks of length $n+3$ is given by

$$S_1 = x^2 f_{n+3}^2 + 2x f_{n+3} f_{n+2} + f_{n+2}^2.$$

Case 2. Suppose $v \in A$ and $w \in B$. If v begins with a loop, then the sum of the weights of products of such closed walks of length $n+3$ is $(x f_{n+3})(f_{n+2}) = x f_{n+3} f_{n+2}$. On the other hand, suppose v does not begin with a loop. The corresponding sum is $(1 \cdot 1 \cdot f_{n+2})(f_{n+2}) = f_{n+2}^2$. Consequently, the total contribution from this case is

$$S_2 = x f_{n+3} f_{n+2} + f_{n+2}^2.$$

Case 3. Suppose $v \notin A$, but $w \in B$. Then $v \in B$. If w begins with a loop, the resulting contribution is $(f_{n+2})(x f_{n+3}) = x f_{n+3} f_{n+2}$. If w does not begin with a loop, then the corresponding contribution is $(f_{n+2})(1 \cdot 1 \cdot f_{n+2}) = f_{n+2}^2$. Consequently, the total contribution from Case 3 toward the cumulative sum is

$$S_3 = x f_{n+3} f_{n+2} + f_{n+2}^2.$$

Case 4. Suppose $v, w \in B$. Clearly, the resulting contribution from this case toward S is

$$S_4 = (f_{n+2})(f_{n+2}) = f_{n+2}^2.$$

Collecting all contributions from the four cases and using identities (2) and (3), we get

$$\begin{aligned} S &= S_1 + S_2 + S_3 + S_4 \\ &= x^2 f_{n+3}^2 + 4f_{n+2}^2 + 4x f_{n+3} f_{n+2} \\ &= l_{n+3}^2, \end{aligned}$$

as desired. □

An Alternate Proof.

Proof. Alternatively, by focusing on the closed walks at v_1 alone, we can establish identity (3). To see this, let C denote the set of closed walks of length $n+3$ at v_1, and D that of length $n+1$ at v_1. Let $E = C \cup D$, where $C \cap D = \emptyset$. The sum of the weights of the walks in E is $f_{n+4} + f_{n+2} = l_{n+3}$. Consequently, the sum S of the weights of elements in $E \times E$ is $S = l_{n+3}^2$.

We will now compute S in a different way. (In the interest of brevity, we highlight the key steps only.) To this end, let (v, w) be an arbitrary element in $E \times E$.

Suppose $v, w \in C$. Then the sum of the weights of the pairs (v, w) of such closed walks is given by

$$S_1 = x^2 f_{n+3}^2 + f_{n+2}^2 + 2x f_{n+3} f_{n+2}.$$

On the other hand, let $v \in C$ and $w \in D$. The total contribution from such pairs (v, w) is

$$\begin{aligned} S_2 &= x^2 f_{n+3}^2 f_{n+1} + x f_{n+3} f_n + x f_{n+2} f_{n+1} + f_{n+2} f_n \\ &= f_{n+2}^2 + x f_{n+3} f_{n+2}. \end{aligned}$$

When $v, w \in D$, the total contribution from the corresponding pairs is

$$\begin{aligned} S_3 &= x^2 f_{n+1}^2 + 2x f_{n+1} f_n + f_n^2 \\ &= f_{n+2}^2. \end{aligned}$$

Finally, let $v \in D$ and $w \in C$. The corresponding contribution is

$$\begin{aligned} S_4 &= x^2 f_{n+3} f_{n+1} + x f_{n+3} f_n + x f_{n+2} f_{n+1} + f_{n+2} f_n \\ &= f_{n+2}^2 + x f_{n+3} f_{n+2}. \end{aligned}$$

Thus the cumulative sum S of the weights of all pairs $(v, w) \in E \times E$ is also given by

$$\begin{aligned} S_1 + S_2 + S_3 + S_4 &= x^2 f_{n+3}^2 + 4f_{n+2}^2 + 4x f_{n+3} f_{n+2} \\ &= l_{n+3}^2, \end{aligned}$$

as expected. □

Since [4, 14]

$$g_{n+1}^2 + g_n^2 = \begin{cases} f_{2n+1}, & \text{if } g_n = f_n; \\ (x^2+4)f_{2n+1}, & \text{if } g_n = l_n; \end{cases}$$

these models also give a graph-theoretic interpretation of the identity [2, 4, 14]

$$\begin{aligned} g_{n+3}^2 + g_n^2 &= (x^2+1)(g_{n+2}^2 + g_{n+1}^2) \\ &= \begin{cases} (x^2+1)f_{2n+3}, & \text{if } g_n = f_n; \\ (x^2+1)(x^2+4)f_{2n+3}, & \text{if } g_n = l_n. \end{cases} \end{aligned}$$

We now add that using the bijection algorithm in [11], we can translate the graph-theoretic models into tiling models with squares and dominoes, where weight(square) = x; weight(domino) = 1; and the weight of a tiling is the product of the weights of tiles in the tiling.

Replacing x with $2x$ in this discourse yields a graph-theoretic proof of the Pell-Lucas identity

$$\begin{aligned} q_{n+3}^2 &= 4x^2p_{n+3}^2 + 4p_{n+2}^2 + 8xp_{n+3}p_{n+2} \\ &= p_3q_{n+2}^2 + p_3q_{n+1}^2 - q_n^2. \end{aligned}$$

Finally, it follows from identity (4) that

$$F_{n+3}^2 + 4F_{n+2}^2 + 4F_{n+3}F_{n+2} = L_{n+3}^2.$$

Consequently, an $L_{n+3} \times L_{n+3}$ floor can be tessallated with nine tiles: one $F_{n+3} \times F_{n+3}$ tile; four $F_{n+2} \times F_{n+2}$ tiles; and four $F_{n+3} \times F_{n+2}$ tiles, where $n \geq 0$.

5 Jacobsthal implications

Using the gibonacci-Jacobsthal relationships $J_n(x) = x^{(n-1)/2}f_n(u)$ and $j_n(x) = x^{n/2}l_n(u)$ [12], we can easily find the Jacobsthal counterparts of identities (2) and (3), where $u = 1/\sqrt{x}$:

$$\begin{aligned} c_{n+3}^2 &= J_3(x)c_{n+2}^2 + xJ_3(x)c_{n+1}^2 - x^3c_n^2; \qquad (5) \\ j_{n+1}^2(x) &= J_{n+1}^2(x) + 4x^2J_n^2(x) + 4xJ_{n+1}(x)J_n(x), \end{aligned}$$

respectively. (We have *omitted* the basic algebra for brevity and convenience.)

Consequently,

$$\begin{aligned} C_{n+3}^2 &= 3C_{n+2}^2 + 6C_{n+1}^2 - 8C_n^2; \\ j_{n+1}^2 &= J_{n+1}^2 + 16J_n^2 + 8J_{n+1}J_n. \qquad (6) \end{aligned}$$

Identity (6) implies that a $j_{n+1} \times j_{n+1}$ floor can be tiled with 25 tiles: one $J_{n+1} \times J_{n+1}$ tile; sixteen $J_n \times J_n$ tiles; and eight $J_{n+1} \times J_n$ tiles, where $n \geq 1$.

5.1 A Jacobsthal digraph

Next we confirm independently identity (5) using graph-theoretic tools. To this end, we first present a weighted digraph D_2; see Figure 2. Its weighted adjacency matrix is

$$M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}.$$

Then

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_n(x) & xJ_{n-1}(x) \end{bmatrix},$$

where $n \geq 1$; we can confirm this using induction.

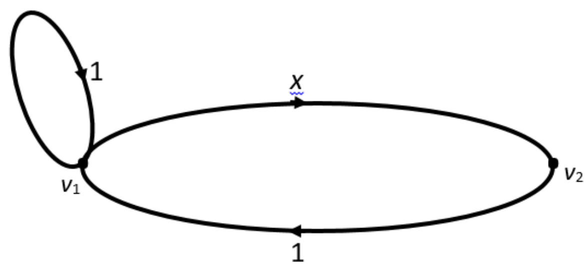

Figure 2: Weighted digraph D_2

It then follows that the sum of the weights of closed walks of length n originating at v_1 is $J_{n+1}(x)$, and that of those originating at v_2 is $xJ_{n-1}(x)$. Consequently, the sum of all closed walks of length in the digraph D_2 is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$. These facts play a central role in the graph-theoretic proof.

With these tools at our finger tips, we now present the proof of each part.

Proof.

Part 1. To establish part 1, we let A be the set of closed walks of length $n+2$ starting at v_1. The sum of the weights of all such closed walks is $J_{n+3}(x)$; so the sum S of the weights of all ordered pairs $(v, w) \in A \times A$ is $J^2_{n+3}(x)$.

We will now compute S in a different way. Again, let (v, w) be an arbitrary element of $A \times A$. Suppose both v and w begin with a loop; the sum of the weights of such pairs (v, w) is $[1 \cdot J_{n+2}(x)][1 \cdot J_{n+2}(x)] = J^2_{n+2}(x)$. If v begins with a loop and w does *not*, the corresponding sum is $[1 \cdot J_{n+2}(x)][x \cdot 1 \cdot J_{n+1}(x)] = xJ_{n+2}(x)J_{n+1}(x)$. Suppose v does *not* begin with a loop, but w does; then also the resulting sum is $[x \cdot 1 \cdot J_{n+1}(x)][1 \cdot J_{n+2}(x)] = xJ_{n+2}(x)J_{n+1}(x)$.

Finally, if both v and w do *not* begin with a loop, the contribution from such pairs equals $[x \cdot 1 \cdot J_{n+1}(x)][x \cdot 1 \cdot J_{n+1}(x)] = x^2 J_{n+1}^2(x)$.

Thus the cumulative contribution of pairs (v, w) all closed walks of length $n+2$ starting at v_1 is given by

$$\begin{aligned}
S &= J_{n+2}^2(x) + 2xJ_{n+2}(x)J_{n+1}(x) + x^2 J_{n+1}^2(x) \\
&= J_{n+2}^2(x) + xJ_{n+2}(x)[J_{n+2}(x) - xJ_n(x)] + xJ_{n+1}(x)[J_{n+1}(x) + xJ_n(x)] + x^2 J_{n+1}^2(x) \\
&= (x+1)J_{n+2}^2(x) + x(x+1)J_{n+1}^2(x) - x^2 J_n(x)[J_{n+2}(x) - J_{n+1}(x)] \\
&= (x+1)J_{n+2}^2(x) + x(x+1)J_{n+1}^2(x) - x^3 J_n(x).
\end{aligned}$$

□

Combining the two values of S yields identity (5) when $c_n = J_n(x)$.

Part 2. To confirm identity (5) when $c_n = j_n(x)$, we focus on the closed walks of lengths $n+3$ and n in the digraph. Let C be the set of closed walks of length $n+3$ starting at v_1, and D the set of those starting at v_2. Clearly, $C \cap D = \emptyset$, so the sum of the weights of the walks in $F = C \cup D$ is $j_{n+3}(x)$. Consequently, the sum S_1 of the weights of the ordered pairs $(v, w) \in F \times F$ is $j_{n+3}^2(x)$.

Now let R denote the set of closed walks of length n originating at v_1, and S that of those originating at v_2. It follows by the preceding argument that the sum S_2 of the weights of the ordered pairs $(v, w) \in G \times G$ is $j_n^2(x)$, where $G = R \cup S$ and $R \cap S = \emptyset$.

Thus

$$S_1 + x^3 S_2 = j_{n+3}^2(x) + x^3 j_n^2(x).$$

We will now compute the sum $S_1 + x^3 S_2$ in a different way. Again, let (v, w) be an arbitrary element of $F \times F$.

Suppose $v, w \in C$. Then the sum of the weights of pairs (v, w) of such closed walks of length $n+3$ originating at v_1 is $[J_{n+4}(x)][J_{n+4}(x)] = J_{n+4}^2(x)$. If $v \in C$ and $w \in D$, then the resulting sum is $[J_{n+4}(x)][xJ_{n+2}(x) = xJ_{n+4}(x)J_{n+2}(x)$. When $v \in D$ and $w \in C$, the corresponding sum is $[xJ_{n+2}(x)][J_{n+4}(x)] = xJ_{n+4}(x)J_{n+2}(x)$. Finally, when $v, w \in D$, the contribution from such pairs (v, w) is $[xJ_{n+2}(x)][xJ_{n+2}(x)] = x^2 J_{n+2}^2(x)$. Thus

$$S_1 = J_{n+4}^2(x) + 2xJ_{n+4}(x)J_{n+2}(x) + x^2 J_{n+2}^2(x).$$

It then follows that

$$S_2 = J_{n+1}^2(x) + 2xJ_{n+1}(x)J_{n-1}(x) + x^2 J_{n-1}^2(x).$$

Consequently, $S_1 + x^3 S_2 = A + B$, where

$$\begin{aligned}
A &= J_{n+4}^2(x) + x^2 J_{n+2}^2(x) + x^3 J_{n+1}^2(x); \\
B &= x^5 J_{n-1}^2(x) + 2xJ_{n+4}(x)J_{n+2}(x) + 2x^4 J_{n+1}(x)J_{n-1}(x).
\end{aligned}$$

Proof. We will now confirm that $S_1 + x^3 S_2 = (x+1)j_{n+2}^2(x) + x(x+1)j_{n+1}^2(x)$. The proof involves a lot of carefully prepared basic algebra; so in the interest of brevity, clarity, and convenience, we present only the major steps; also we *omit* the argument in the functional notation.

We have

$$\begin{aligned}
A &= (J_{n+3} + xJ_{n+2})^2 + x^2 J_{n+2}^2 + x^3 J_{n+1}^2 \\
&= J_{n+3}^2 + 2x^2 J_{n+2}^2 + 2xJ_{n+2}(J_{n+2} + xJ_{n+1}) + x^3 J_{n+1}^2 \\
&= J_{n+3}^2 + (x^2+x)J_{n+2}^2 + x^3 J_{n+1}^2 + (x^2+x)J_{n+2}^2 + 2x^2 J_{n+2}J_{n+1} \\
&= J_{n+3}^2 + (x^2+x)J_{n+2}^2 + x^3 J_{n+1}^2 + (x^2+x)J_{n+2}^2 + 2x^2 J_{n+1}(J_{n+3} - xJ_{n+1}) \\
&= J_{n+3}^2 + (x^2+x)J_{n+2}^2 + x^3 J_{n+1}^2 + (x^2+x)J_{n+2}^2 + 2x^2 J_{n+3}J_{n+1} - 2x^3 J_{n+1}^2; \\
B &= x^3(J_{n+1} - J_n)^2 + 2xJ_{n+2}(J_{n+3} + xJ_{n+2}) + 2x^3 J_{n+1}(J_{n+1} - J_n) \\
&= x^3 J_{n+1}^2 + x^3 J_n^2 - 2x^3 J_{n+1}J_n + 2xJ_{n+3}J_{n+2} + 2x^2 J_{n+2}^2 + 2x^3 J_{n+1}^2 - 2x^3 J_{n+1}J_n \\
&= x^3 J_{n+1}^2 + x^3 J_n^2 - 2x^3 J_{n+1}J_n + 2xJ_{n+3}(J_{n+1} + xJ_n) + 2x^2 J_{n+2}^2 + 2x^3 J_{n+1}^2 - 2x^3 J_{n+1}J_n \\
&= 2xJ_{n+3}J_{n+1} + x^3 J_n^2 + 2x^3 J_{n+1}J_n + 2x^2 J_{n+2}^2 + 3x^3 J_{n+1}^2 - 4x^3 J_{n+1}J_n \\
&= 2xJ_{n+3}J_{n+1} + x^3 J_n^2 + 2x^2 J_n(J_{n+2} + xJ_{n+1}) + x(J_{n+3} - J_{n+2})^2 + +2x^2 J_{n+2}^2 + 2x^3 J_{n+1}^2 \\
&\quad - 4x^3 J_{n+1}J_n \\
&= 2xJ_{n+3}J_{n+1} + x^3 J_n^2 + 2x^2 J_{n+2}J_n + xJ_{n+3}^2 + 2x^3 J_{n+1}J_n + (2x^2+x)J_{n+2}^2 - 2xJ_{n+3}J_{n+2} \\
&\quad + 2x^3 J_{n+1}^2 - 4x^3 J_{n+1}J_n.
\end{aligned}$$

Then

$$S_1 + x^3 S_2 = C + D + (x^2+x)J_{n+2}^2 - 2x^3 J_{n+1}J_n + (2x^2+x)J_{n+2}^2 - 2xJ_{n+3}J_{n+2},$$

where

$$\begin{aligned}
C &= (x+1)(J_{n+3}^2 + 2xJ_{n+3}J_{n+1}) + x^3 J_{n+1}^2 \\
&= (x+1)(J_{n+3} + xJ_{n+1})^2 - x^2 J_{n+1}^2 \\
&= (x+1)j_{n+2}^2 - x^2 J_{n+1}^2; \\
D &= (x^2+x)J_{n+2}^2 + x^3 J_n^2 + 2x^2 J_{n+2}J_n \\
&= (x^2+x)(J_{n+2} + xJ_n)^2 - 2x^3 J_{n+2}J_n - x^4 J_n^2 \\
&= x(x+1)j_{n+1}^2 - 2x^3 J_{n+2}J_n - x^4 J_n^2.
\end{aligned}$$

Consequently,

$$S_1 + x^3 S_2 = (x+1)j_{n+2}^2 + x(x+1)j_{n+1}^2 + E,$$

where

$$\begin{aligned}
E &= -2x^3 J_{n+1}J_n - 2x^2 J_{n+2}J_{n+1} - 2x^4 J_n^2 + 2x^2 J_{n+2}^2 \\
&= -2x^3 J_n(J_{n+1} + xJ_n) + 2x^2 J_{n+2}(J_{n+2} - J_{n+1}) \\
&= -2x^3 J_{n+2}J_n + 2x^3 J_{n+2}J_n \\
&= 0.
\end{aligned}$$

Thus

$$S_1 + x^3 S_2 = (x+1)j_{n+2}^2 + x(x+1)j_{n+1}^2,$$

as expected. $\square$

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