Journal of Integer Sequences, Vol. 22 (2019),
Article 19.6.8

Approximating Sums of Consecutive Integral Roots

Pongpol Ruankong and Kantaphon Kuhapatanakul
Department of Mathematics
Faculty of Science
Kasetsart University
Bangkok 10900
Thailand
fscippru@ku.ac.th
fscikpkk@ku.ac.th

Abstract

We present an alternative proof of Saltzman and Yuan's result on the sums of consecutive integral roots. We use the AM-GM-HM inequality to prove the main result. Moreover, the lower bound for which the result holds is greatly improved.

1 Introduction

For each real number x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. The sums of consecutive integral roots have been studied by many mathematicians (see [1]-[5]). For instance, the following identities hold for every positive integer n:

(1) $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$

(2) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rfloor = \lfloor \sqrt{9n+8} \rfloor$

(3) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} \rfloor = \lfloor \sqrt{16n+20} \rfloor$

(4) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} \rfloor = \lfloor \sqrt{25n+49} \rfloor$

(5) $\lfloor \sqrt[3]{n} + \sqrt[3]{n+1} \rfloor = \lfloor \sqrt[3]{8n+3} \rfloor$

(6) $\lfloor \sqrt[3]{n} + \sqrt[3]{n+1} + \sqrt[3]{n+2} \rfloor = \lfloor \sqrt[3]{27n+26} \rfloor$.

On the contrary, Zhan [5] showed that for any real number c, there is a positive integer n such that

$$\left\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} + \sqrt{n+5} \right\rfloor \neq \left\lfloor \sqrt{36n+c} \right\rfloor .$$

Saltzman and Yuan [3] presented the following similar formula for $n \geq 4$.

$$\left\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} + \sqrt{n+5} \right\rfloor = \left\lfloor \sqrt{36n+89} \right\rfloor ,$$

and gave the general formula: for all integers $p, m \geq 2$,

$$\left\lfloor \sqrt[p]{n} + \sqrt[p]{n+1} + \sqrt[p]{n+2} + \cdots + \sqrt[p]{n+m-1} \right\rfloor = \left\lfloor \sqrt[p]{m^p n + \frac{m^p(m-1)}{2} - 1} \right\rfloor \qquad (1)$$

holds for all positive integers

$$n > \frac{m^p(m-1)(2m-1)(p-1)}{12p}. \qquad (2)$$

They utilized properties of concave functions to approximate the sums of consecutive integral roots.

In this paper, we give an alternative proof of (1) using the AM-GM-HM inequality and obtain a new lower bound on n, which is approximately half of (2).

2 Results

In the sequel, let m and p be two positive integers greater than 1, let $M = \frac{(p-1)m^p(m^2-1)}{12p}$, and let $i_1, i_2, \ldots, i_p \in \{0, 1, 2, \ldots, m-1\}$. Denoted by A_p, G_p and H_p the arithmetic mean, the geometric mean, and the harmonic mean of the positive integers $n+i_1, n+i_2, \ldots, n+i_p$, respectively, i.e.,

$$A_p = \frac{1}{p}\sum_{j=1}^{p}(n+i_j),$$

$$G_p = \sqrt[p]{(n+i_1)\ldots(n+i_p)},$$

$$H_p = \frac{p}{\sum_{j=1}^{p}\frac{1}{n+i_j}}.$$

By the well-known AM-GM-HM inequality,

$$H_p \le G_p \le A_p$$

where equality holds if and only if $n+i_1 = n+i_2 = \cdots = n+i_p$. To prove the main theorem, we first present two technical lemmas.

Lemma 1. *For every positive integer n,*

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} (A_p - H_p) < \frac{M}{n^{p-1}} \left(n + \frac{m-1}{2}\right)^{p-2}.$$

Proof. Observe that

$$\begin{aligned} A_p - H_p &= \frac{1}{p}\sum_{j=1}^{p}(n+i_j) - \frac{p}{\sum_{j=1}^{p} \frac{1}{n+i_j}} \\ &= \frac{\left(\left(n + \frac{1}{p}\sum_{j=1}^{p} i_j\right)\left(\frac{1}{p}\sum_{k=1}^{p} \frac{\prod_{j=1}^{p}(n+i_j)}{n+i_k}\right) - \prod_{j=1}^{p}(n+i_j)\right)}{\frac{1}{p}\sum_{k=1}^{p} \frac{\prod_{j=1}^{p}(n+i_j)}{n+i_k}}. \end{aligned}$$

Hence

$$\frac{A_p - H_p}{p} \sum_{k=1}^{p} \frac{\prod_{j=1}^{p}(n+i_j)}{n+i_k} = \left(n + \frac{1}{p}\sum_{j=1}^{p} i_j\right)\left(\frac{1}{p}\sum_{k=1}^{p} \frac{\prod_{j=1}^{p}(n+i_j)}{n+i_k}\right) - \prod_{j=1}^{p}(n+i_j).$$

Next, we determine the coefficients of

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\left(n + \frac{1}{p}\sum_{j=1}^{p} i_j\right)\left(\frac{1}{p}\sum_{k=1}^{p} \frac{\prod_{j=1}^{p}(n+i_j)}{n+i_k}\right) - \prod_{j=1}^{p}(n+i_j)\right) \tag{3}$$

as a polynomial in variable n.

It is easy to see that the coefficients of n^p and n^{p-1} in (3) are zero. Moreover, we claim that, for each $t = 2, 3, \ldots, p$, the coefficient of n^{p-t} in (3) is equal to $M\binom{p-2}{t-2}\left(\frac{m-1}{2}\right)^{t-2}$ implying that the polynomial is $M\left(n + \frac{m-1}{2}\right)^{p-2}$.

The coefficient of n^{p-t} in (3), for $t = 2, 3, \ldots, p$, is equal to the sums of the following:

(i) $$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\frac{1}{p^2}\sum_{j=1}^{p} i_j \left(\sum_{S} \sum_{\substack{j_1<\cdots<j_{t-1} \\ j_1,\ldots,j_{t-1}\in S}} i_{j_1}\cdots i_{j_{t-1}}\right)\right)$$

(ii) $$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\frac{1}{p}\sum_{S} \sum_{\substack{j_1<\cdots<j_t \\ j_1,\ldots,j_t\in S}} i_{j_1}\cdots i_{j_t}\right)$$

$$\text{(iii)} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(- \sum_{\substack{j_1<\cdots<j_t \\ j_1,\ldots,j_t\in\mathbb{N}_p}} i_{j_1}\cdots i_{j_t} \right)$$

where $\sum_S$ represents the sum over all $(p-1)$-subsets S of $\mathbb{N}_p = \{1, 2, \ldots, p\}$. (Note that for the case $t = p$, (ii) is an empty sum which is conventionally zero. This coincides with the fact that (ii) does not occur in the case $t = p$.)

Distributing $\sum_{j=1}^{p} i_j$ inside the double sum, (i) becomes

$$\frac{1}{p^2} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\sum_S \sum_{\substack{j_1<\cdots<j_{t-1} \\ j_1,\ldots,j_{t-1}\in S}} \left(\sum_{r=1}^{t-1} i_{j_r}(i_{j_1}\cdots i_{j_{t-1}}) + \sum_{r\notin\{1,\ldots,t-1\}} i_{j_r}(i_{j_1}\cdots i_{j_{t-1}}) \right) \right).$$

Since the double sum consists of exactly $p\binom{p-1}{t-1}$ terms, using the symmetry of the i_j and combining the like terms, (i) amounts to

$$\frac{1}{p}\binom{p-1}{t-1} \left((t-1) \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} i_1^2 \cdots i_{t-1} + (p-t+1) \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} i_1 \cdots i_t \right). \tag{4}$$

Similarly, (ii) amounts to

$$\binom{p-1}{t} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} i_1 \cdots i_t, \tag{5}$$

and (iii) amounts to

$$-\binom{p}{t} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} i_1 \cdots i_t. \tag{6}$$

Note that

$$\frac{p-t+1}{p}\binom{p-1}{t-1} + \binom{p-1}{t} - \binom{p}{t} = -\frac{t-1}{p}\binom{p-1}{t-1}$$

for $t = 2, \ldots, p$. Hence adding (4), (5) and (6) yields

$$\begin{aligned}
&\frac{t-1}{p}\binom{p-1}{t-1}\left(\sum_{i_1=0}^{m-1}\cdots\sum_{i_p=0}^{m-1} i_1^2\cdots i_{t-1} - \sum_{i_1=0}^{m-1}\cdots\sum_{i_p=0}^{m-1} i_1\cdots i_t\right) \\
&\qquad = \frac{p-1}{p}\binom{p-2}{t-2}\left(m^{p-t+1}\sum_{i=0}^{m-1} i^2\left(\sum_{i=0}^{m-1} i\right)^{t-2} - m^{p-t}\left(\sum_{i=0}^{m-1} i\right)^{t}\right) \\
&\qquad = \frac{p-1}{p}\binom{p-2}{t-2} m^{p-t}\left(\sum_{i=0}^{m-1} i\right)^{t-2}\left(m\sum_{i=0}^{m-1} i^2 - \left(\sum_{i=0}^{m-1} i\right)^2\right) \\
&\qquad = M\binom{p-2}{t-2}\left(\frac{m-1}{2}\right)^{t-2},
\end{aligned}$$

which is the coefficient of n^{p-t} in $M\left(n+\frac{m-1}{2}\right)^{p-2}$. Therefore,

$$\begin{aligned}
\sum_{i_1=0}^{m-1}\cdots\sum_{i_p=0}^{m-1}(A_p - H_p) &< \frac{1}{n^{p-1}}\sum_{i_1=0}^{m-1}\cdots\sum_{i_p=0}^{m-1}\left(\frac{A_p-H_p}{p}\sum_{k=1}^{p}\frac{\prod_{j=1}^{p}(n+i_j)}{n+i_k}\right) \\
&< \frac{M}{n^{p-1}}\left(n+\frac{m-1}{2}\right)^{p-2}.
\end{aligned}$$

□

Lemma 2. *For every positive integer* $n \geq M + \frac{(p-2)(m-1)}{2}$,

$$n^{p-1} \geq M\left(n+\frac{m-1}{2}\right)^{p-2}.$$

Proof. For $n \geq M + \frac{(p-2)(m-1)}{2}$,

$$\begin{aligned}
n^{p-1} &\geq Mn^{p-2} + \frac{(p-2)(m-1)}{2}n^{p-2} \\
&\geq Mn^{p-2} + \frac{(p-2)(m-1)}{2}Mn^{p-3} + \left(\frac{(p-2)(m-1)}{2}\right)^2 n^{p-3} \\
&\ \ \vdots \\
&\geq M\sum_{j=0}^{p-3}\left(\frac{(p-2)(m-1)}{2}\right)^j n^{p-2-j} + \left(\frac{(p-2)(m-1)}{2}\right)^{p-2} n \\
&\geq M\sum_{j=0}^{p-2}\left(\frac{(p-2)(m-1)}{2}\right)^j n^{p-2-j}.
\end{aligned}$$

Since $(p-2)^j \geq \binom{p-2}{j}$ for all $0 \leq j \leq p-2$,

$$n^{p-1} \geq M \sum_{j=0}^{p-2} \binom{p-2}{j} \left(\frac{m-1}{2}\right)^j n^{p-2-j} = M \left(n + \frac{m-1}{2}\right)^{p-2}$$

as desired. $\square$

Now we are ready to state and prove the main result.

Theorem 3. *For every positive integer $n \geq M + \frac{(p-2)(m-1)}{2}$,*

$$\lfloor \sqrt[p]{n} + \sqrt[p]{n+1} + \cdots + \sqrt[p]{n+m-1} \rfloor = \left\lfloor \sqrt[p]{m^p n + \frac{m^p(m-1)}{2} - 1} \right\rfloor.$$

Proof. Let

$$S = \sqrt[p]{n} + \sqrt[p]{n+1} + \sqrt[p]{n+2} + \cdots + \sqrt[p]{n+m-1}.$$

One obtains

$$S^p = \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \sqrt[p]{(n+i_1)\cdots(n+i_p)} = \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} G_p.$$

By the AM-GM-HM inequality,

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} A_p - \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} (A_p - H_p) < S^p < \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} A_p.$$

Observe that

$$\begin{aligned} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} A_p &= \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} n + \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\frac{1}{p} \sum_{j=1}^{p} i_j\right) \\ &= m^p n + \frac{m^p(m-1)}{2}. \end{aligned}$$

Moreover, by Lemma 1 and Lemma 2, we have

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} (A_p - H_p) < \frac{M}{n^{p-1}} \left(n + \frac{m-1}{2}\right)^{p-2} \leq 1.$$

Thus

$$m^p n + \frac{m^p(m-1)}{2} - 1 < S^p < m^p n + \frac{m^p(m-1)}{2}$$

holds for all positive integers $n \geq M + \frac{(p-2)(m-1)}{2}$. Equivalently,

$$\lfloor \sqrt[p]{n} + \sqrt[p]{n+1} + \cdots + \sqrt[p]{n+m-1} \rfloor = \left\lfloor \sqrt[p]{m^p n + \frac{m^p(m-1)}{2} - 1} \right\rfloor$$

holds for all positive integers $n \geq M + \frac{(p-2)(m-1)}{2}$. $\square$

The table below compares the lower bounds of n between Saltzman and Yuan's work, $N_{SY} = \frac{m^p(m-1)(2m-1)(p-1)}{12p} + 1$, and our work, $N = \frac{(p-1)m^p(m^2-1)}{12p} + \frac{(p-2)(m-1)}{2}$, for small values of p and m.

	$p=3$		$p=4$		$p=5$		$p=6$	
m	N_{SY}	N	N_{SY}	N	N_{SY}	N	N_{SY}	N
2	2	2	4	4	7	8	14	16
3	16	13	51	43	163	133	507	409
4	75	55	337	243	1434	1029	5974	4273
5	251	169	1407	942	7501	5006	39063	29050
6	661	423	4456	2840	28513	18152	178201	113410
7	1487	918	11705	7209	87397	53792	637266	392176
8	2587	1796	26881	16135	229377	137637	1911467	1146894
9	5509	3244	55769	32813	535378	314940	5019166	2952466
10	9501	5505	106876	61884	1140001	660014	11875001	6875018

3 Acknowledgments

The authors greatly appreciate the anonymous referee for his/her careful reading and helpful comments.

References

[1] F. D. Hammer, Problem E3010, *Amer. Math. Monthly* **95** (1988), 133–134.

[2] K. Kuhapatanakul and P. Ruankong, Sums of square roots, *Math. Gazette* **101** (2017), 276–280.

[3] P. W. Saltzman and P. Yuan, On sums of consecutive integral roots, *Amer. Math. Monthly* **115** (2008), 254–261.

[4] Z. Wang, *A City of Nice Mathematics* (in Chinese), The Democracy and Construction Press, Beijing, 2000.

[5] X. Zhan, Formulae for sums of consecutive square roots, *Math. Intelligencer* **27** (2005), 4–5.

2010 *Mathematics Subject Classification* : Primary 11A99; Secondary 11Y55.
Keywords: sum of roots.

Received April 30 2019; revised version received September 2 2019. Published in *Journal of Integer Sequences*, September 25 2019.

Return to Journal of Integer Sequences home page.

.