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Diagonal Sums in the Pascal Pyramid, II: Applications

Hacène Belbachir and Abdelghani Mehdaoui
USTHB
Faculty of Mathematics
RECITS Laboratory
BP 32
El Alia 16111
Bab Ezzouar
Algiers
Algeria
hacenebelbachir@gmail.com
mehabdelghani@gmail.com

László Szalay
University of Sopron
Institute of Mathematics
Bajcsy-Zsilinszky utca 4
H-9400 Sopron
Hungary
szalay.laszlo@uni-sopron.hu

Abstract

In a previous paper, we derived a theorem which describes, in the language of linear recurrences, the sum of diagonal elements lying along a finite ray of a certain type crossing the 3-dimensional Pascal pyramid. The corresponding generating function was also determined. In this paper, we apply these results to prove several (more precisely 24) recurrence relations previously conjectured, or not given in the *On-Line Encyclopedia of Integer Sequences*. Moreover, we provide many (exactly 75) new summatory identities linked to sequences listed in the Encyclopedia.

1 Introduction

Let $(F_n)_{n=0}^{\infty}$ denote the Fibonacci sequence given by the initial values $F_0 = 0$ and $F_1 = 1$, and by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Probably Binet [4] was the first to publish the identity

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots, \quad (n \geq 0)$$

about the sum of elements of ascending diagonals, which motivated the paper [2] to investigate diagonals of arbitrary direction similarly generating finite sums. It turned out that such sums can also be described by suitable linear recurrences. In this paper, we recall the two main theorems and some corollaries of [3] which study the analogous question in the 3D Pascal pyramid (in short: PP3D), and we apply the results to many sequences appearing in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [4].

It is known that trinomial coefficients $\binom{n}{i,\, j,\, k} = \frac{n!}{i! \cdot j! \cdot k!}$ (where i, j, k are non-negative integers and $i + j + k = n$) appear in the expansion of $(x + y + z)^n$ as

$$(x + y + z)^n = \sum_{i+j+k=n} \binom{n}{i,\ j,\ k} x^i\ y^j\ z^k \tag{1}$$

(see, for example, Feinberg [5] or Anatriello and Vincenzi [1]).

We adapted the trinomial coefficients to the 3D coordinate-system — more precisely, to the vertices of the lattice $\mathbb{Z}^3$. The location of $\binom{n}{i,\, j,\, k}$ is the point $(i, j, -n)$ (recall that $n = i + j + k$).

Take two arbitrary vertices $P_0(x_0, y_0, z_0) \in \mathbb{Z}^3$ and $P_1(x_1, y_1, z_1) \in \mathbb{Z}^3$ of the pyramid (that is, such that the conditions $x_i \geq 0$, $y_i \geq 0$, $z_i \leq 0$, $x_i + y_i + z_i \leq 0$ are fulfilled) such that if $r = z_1 - z_0$ holds, then

$$\alpha_1 = x_1 - x_0 > 0, \qquad \alpha_2 = y_1 - y_0 > 0, \qquad \alpha_1 + \alpha_2 + r > 0.$$

We also require that $x_0 + y_0 + z_0 < 0$, excluding the case when the vector $\overrightarrow{v} = \overrightarrow{P_0P_1}$ is on the plane $x + y + z = 0$.

Along the straight line determined by the points P_0 and P_1 we translate $-\overrightarrow{v}$ from P_0 as many times as possible: let $x_0 = s_1\alpha_1 + \ell_1$ with $0 \leq \ell_1 < \alpha_1$, and let $x_1 = s_2\alpha_2 + \ell_2$ with $0 \leq \ell_2 < \alpha_2$. Put $s = \min\{s_1, s_2\}$, and then set

$$\theta_1 = x_0 - s\alpha_1, \qquad \theta_2 = y_0 - s\alpha_2.$$

The point we obtain, say P, has coordinates $(\theta_1, \theta_2, z_0 - sr)$. Now we raise the point P parallel to the z-axis until it reaches the plane $x + y + z = 0$. Then we consider the uppermost ray defined by the quintuple $(\alpha_1, \alpha_2, r, \theta_1, \theta_2)$. Note that, seemingly, the conditions $\alpha_1 > 0$ and $\alpha_2 > 0$ together would not cover all the possible directions in PP3D. Nevertheless, here they do because of the symmetry of rotation around the vertical axis (before fitting PP3D to the coordinate system).

Thus the direction $(\alpha_1, \alpha_2, r, \theta_1, \theta_2)$ defines diagonal rays in the Pascal pyramid. Such a ray contains the elements

$$A_k = \binom{n - rk}{\theta_1 + \alpha_1 k,\ \theta_2 + \alpha_2 k,\ \eta_k} x^{\theta_1+\alpha_1 k} y^{\theta_2+\alpha_2 k} z^{\eta_k}, \tag{2}$$

where $k = 0, \ldots, \lfloor \frac{n-\theta_1-\theta_2}{\alpha_1+\alpha_2+r} \rfloor$ and $\eta_k = n - \theta_1 - \theta_2 - (\alpha_1 + \alpha_2 + r)k$. Here x, y and z are nonzero real parameters. What most interests us is the sum

$$T_n^{(\alpha_1,\alpha_2,r,\theta_1,\theta_2)} = \sum_{k=0}^{\lfloor \frac{n-\theta_1-\theta_2}{\alpha_1+\alpha_2+r} \rfloor} A_k,$$

but computing it is a hard problem in general.

Belbachir, Mehdaoui, and Szalay [3] considered only

$$T_n^{(r)} = \sum_{k=0}^{\lfloor \frac{n}{r+2} \rfloor} \binom{n - rk}{k,\ k,\ n-(r+2)k} x^k y^k z^{n-(2+r)k}$$

corresponding to the particular case $(\alpha_1, \alpha_2, r, \theta_1, \theta_2) = (1, 1, r, 0, 0)$. They also assumed $r \geq -1$ to ensure the finiteness of the rays. Here we mention that Shannon [7] found a connection between such sums and ternary recurrences. In fact, our approach is more general.

In the next section, we summarize the main result of [3] and its consequences.

2 Previous results

Assume $t = xy$. For notational convenience, set $T_n = T_n^{(r)}$.

Theorem 1. *The terms of the sequence (T_n) given by*

$$T_n = \sum_{k=0}^{\lfloor n/(r+2)\rfloor} \binom{n-rk}{k,\ k,\ n-(r+2)k} t^k z^{n-(r+2)k} \tag{3}$$

satisfy the linear recurrence relation

$$nT_n = (2n-1)zT_{n-1} - (n-1)z^2T_{n-2} + 2(2n-(r+2))tT_{n-r-2}. \tag{4}$$

Observe that the last subscript which appears in the right-hand side of (4) is $n-r-2$. If $r=-1$ or $r=0$, then T_{n-r-2} can be joined to T_{n-1} or T_{n-2}, respectively. This is the Morgan-Voyce phenomenon, and under such conditions we obtain the following corollaries.

Corollary 2. *If $r=0$, then the recurrence relation (4) simplifies to*

$$nT_n = (2n-1)zT_{n-1} + (n-1)(4t-z^2)T_{n-2}. \tag{5}$$

Corollary 3. *Assume $r=-1$. Then (4) provides*

$$nT_n = (2n-1)(2t+z)T_{n-1} - (n-1)z^2T_{n-2}. \tag{6}$$

The comparison of the two formulae (5) and (6) makes it possible to establish equalities between two sums belonging to the same recurrence. This idea leads to the following result.

Corollary 4. *We have the identity*

$$\sum_{k}^{\lfloor n/2\rfloor} \binom{n}{k,\ k,\ n-2k} (t_1\tau)^k (\tau+t_1)^{n-2k} = \sum_{k}^{n} \binom{n+k}{k,\ k,\ n-k} t_1^k (\tau-t_1)^{n-k}, \tag{7}$$

where $\tau \neq 0$ and $\tau \neq \pm t_1$.

Finally, we were able to determine the generating function of sequence (3) [3].

Theorem 5. *The generating function of variable u of the sequence given by (3) is*

$$g(u) = \frac{1}{\sqrt{(zu-1)^2 - 4tu^{r+2}}}. \tag{8}$$

Now we list several examples of the type of Corollary 4 we found in the OEIS [7]. In the sequel, keeping the notation of [7] we use (a_n) for the sequences (instead of (T_n)).

3 List of new identities, and a combinatorial interpretation of Theorem 1

3.1 Supplement of OEIS

We use the generating function for the identification of sequences since a sequence in the OEIS is usually defined by its generating function or the generating function is often given as a feature of the sequence. In other words, the generating function is the connection between the sequence, its recurrence relation, and the corresponding sum(s).

We split our results into four blocks. The first one contains the sequences for which the appropriate recurrence relations were given in the OEIS with the note "conjecture". The application of Theorem 5 (identification) and Theorem 1 provide the proofs automatically. So we showed

Theorem 6. *Each sequence of the first column of Table 1 satisfies the linear recurrence rule located in the same row of the second column.*

Table 2 gives the sequences for which the recurrence relations were not given in the OEIS.

Theorem 7. *The sequences given in Table 2 satisfy the corresponding recurrence relations.*

The next theorem, thanks to Theorem 1 and Corollary 4, collects at least two new trinomial sums for some sequences. Note that the parameter r of the direction $(\alpha_1, \alpha_2, r) = (1, 1, r)$ appears, for example, in the upper index of the trinomial coefficient in (3). Hence one can easily conclude the value of r from the corresponding sum.

Theorem 8. *Columns 2–4 of Table 3 provide new sum identities for the sequences of the first column.*

Finally, we found a few new identities for certain sequences when only one sum exists or we could add only one new sum to the detected one(s).

Theorem 9. *The sums in Table 4 give new descriptions for the corresponding sequences.*

3.2 Powers of integers

Corollary 10. *Formula (3) simplifies to*

$$T_n = \sum_{k=0}^{n} \binom{n+k}{k,\ k,\ n-k} (-1)^{n-k} = 1 \tag{9}$$

if $r = -1$, $t = 1$, and $z = -1$.

OEIS code	Recurrence relation conjectured in the OEIS
A084768	$na_n = 7(2n-1)a_{n-1} - (n-1)a_{n-2}$
A084769	$na_n = 9(2n-1)a_{n-1} - (n-1)a_{n-2}$
A084771	$na_n = 5(2n-1)a_{n-1} - 9(n-1)a_{n-2}$
A098270	$na_n = 10(2n-1)a_{n-1} - 4(n-1)a_{n-2}$
A098333	$na_n = (2n-1)a_{n-1} - 13(n-1)a_{n-2}$
A098334	$na_n = (2n-1)a_{n-1} - 17(n-1)a_{n-2}$
A098336	$na_n = 2(2n-1)a_{n-1} - 12(n-1)a_{n-2}$
A098337	$na_n = 2(2n-1)a_{n-1} - 20(n-1)a_{n-2}$
A098338	$na_n = 3(2n-1)a_{n-1} - 13(n-1)a_{n-2}$
A098340	$na_n = 3(2n-1)a_{n-1} - 21(n-1)a_{n-2}$
A098341	$na_n = 3(2n-1)a_{n-1} - 25(n-1)a_{n-2}$
A098411	$na_n = 8(2n-1)a_{n-1} - 48(n-1)a_{n-2}$
A098439	$na_n = (2n-1)a_{n-1} + 47(n-1)a_{n-2}$
A098456	$na_n = 2(2n-1)a_{n-1} + 64(n-1)a_{n-2}$
A098479	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2} + 2(2n-3)a_{n-3}$
A098480	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2} + 4(2n-3)a_{n-3}$
A106186	$na_n = 2(2n-1)a_{n-1} - 4(n-1)a_{n-2} + 8(2n-3)a_{n-3}$
A115864	$na_n = 8(2n-1)a_{n-1} - 16(n-1)a_{n-2}$
A116092	$na_n = -4(2n-1)a_{n-1} - 64(n-1)a_{n-2}$
A116093	$na_n = -2(2n-1)a_{n-1} - 12(n-1)a_{n-2}$
A122868	$na_n = 3(2n-1)a_{n-1} + 3(n-1)a_{n-2}$

Table 1: Conjectured recurrence relations we proved.

OEIS code	Recurrence relation we discovered
A113179	$na_n = 2(2n-1)a_{n-1} - 4(n-1)a_{n-2} + 4(2n-3)a_{n-3}$
A248168	$na_n = 7(2n-1)a_{n-1} - 33(n-1)a_{n-2}$
A258723	$na_n = 6(2n-1)a_{n-1} - 48(n-1)a_{n-2}$

Table 2: Recurrence relation discovered.

OEIS code	Sum 1	Sum 2	Sum 3
A012000	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} (-3)^k\, 2^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-1)^k\, 4^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^k\, (-4)^{n-k}$
A084771	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 4^k\, 5^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k\, (-3)^{n-k}$
A084772	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 5^k\, 6^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^k\, (-4)^{n-k}$
A084773	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 8^k\, 6^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k 2^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k\, (-2)^{n-k}$
A084774	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 10^k\, 7^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k 3^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^k\, (-3)^{n-k}$
A098269	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 15^k\, 8^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^k 2^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^k\, (-2)^{n-k}$
A098270	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 24^k\, 10^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k 2^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 6^k\, (-2)^{n-k}$
A098341	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} (-4)^k\, 3^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-1)^k 5^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k\, (-5)^{n-k}$
A098659	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 6^k\, 7^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 6^k\, (-5)^{n-k}$
A115864	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 12^k\, 8^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k 4^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 6^k\, (-4)^{n-k}$
A115865	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 27^k\, 12^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^k 6^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 9^k\, (-6)^{n-k}$
A116091	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} (-3)^k\, (-2)^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-3)^k 4^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-4)^{n-k}$
A116092	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} (-12)^k\, (-4)^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-6)^k 8^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k\, (-8)^{n-k}$
A069835	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 3^k\, 4^{n-2k}$	already known	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^k\, (-2)^{n-k}$
A084768	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 12^k\, 7^{n-2k}$	already known	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k\, (-1)^{n-k}$
A084769	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 20^k\, 9^{n-2k}$	already known	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^k\, (-1)^{n-k}$
A098332	already known	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-1)^k 3^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k (-3)^{n-k}$

Table 3: Sums discovered.

OEIS code	Sum	OEIS code	Sum
A001850	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k\,(-1)^{n-k}$	A006134	$\sum_k^{\lfloor n/3\rfloor} \binom{n-k}{k,\,k,\,n-3k} 3^{n-3k}$
A006442	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 6^k\,5^{n-2k}$	A026375	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 3^{n-2k}$
A059304	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 4^k\,4^{n-2k}$	A080609	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 2^k\,4^{n-2k}$
A081671	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 4^{n-2k}$	A084605	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 4^k$
A084770	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 5^k\,2^{n-2k}$	A098339	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} (-2)^k 3^{n-2k}$
A098409	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 5^{n-2k}$	A098410	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 6^{n-2k}$
A098411	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 4^k 8^{n-2k}$	A098430	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 16^k 8^{n-2k}$
A098443	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 5^k 4^{n-2k}$	A098444	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 5^k 3^{n-2k}$
A098453	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 4^k 2^{n-2k}$	A098455	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 10^k 2^{n-2k}$
A098456	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 17^k 2^{n-2k}$	A098658	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 9^k 6^{n-2k}$
A106186	$\sum_k^{\lfloor n/3\rfloor} \binom{n-k}{k,\,k,\,n-3k} 4^k 2^{n-3k}$	A106258	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 6^k 4^{n-2k}$
A106259	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 12^k 6^{n-2k}$	A106260	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 20^k 8^{n-2k}$
A106261	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 30^k 10^{n-2k}$	A122868	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 3^k 3^{n-2k}$
A248168	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} 4^k 7^{n-2k}$	A258723	$\sum_k^{\lfloor n/2\rfloor} \binom{n}{k,\,k,\,n-2k} (-3)^k 6^{n-2k}$

Table 4: Sums discovered.

Proof. Obviously, (9) satisfies the recurrence rule (6). Hence for $n \geq 2$ we have

$$nT_n = (2\,n-1)\,T_{n-1} + (n-1)\,T_{n-2},$$

which is equivalent to

$$n\,\underbrace{(T_n - T_{n-1})}_{U_n} = (n-1)\,\underbrace{(T_{n-1} - T_{n-2})}_{U_{n-1}},$$

and then to

$$\underbrace{n\,U_n}_{V_n} = \underbrace{(n-1)\,U_{n-1}}_{V_{n-1}}.$$

Consequently, $V_n = V_1$ holds for $n \geq 2$. The initial values $T_0 = T_1 = 1$ imply $V_1 = 0$. Thus $nU_n = 0$ gives $U_n = 0$, and then $T_n = T_{n-1}$ follows, which together with the initial values proves $T_n = 1$ for all $n \geq 0$. □

Corollary 11. *Assume that $r = -1$ and $z = -t$ hold in (3). Then we have $T_n = t^n$.*

Proof. Combine (3), Corollary 10, and the conditions here to get

$$T_n = \sum_{k=0}^{n} \binom{n+k}{k,\ k,\ n-k} t^k(-t)^{n-k} = t^n.$$

□

Replacing $r = -1$ and $t = -z$ in (9), together with Corollary 10 immediately leads to

Corollary 12.

$$T_n = \sum_{k=0}^{n} \binom{n+k}{k,\ k,\ n-k} (-z)^k z^{n-k} = (-z)^n.$$

3.3 A combinatorial interpretation

A combinatorial interpretation of sequence (3) corresponding to the direction $(r, 1, 1)$ follows easily from the sum of (3).

Theorem 13. *The sequence T_n in (3) gives the sum of products of weighs associated with the walk in the square lattice from the point $(0, 0)$ to (n, n) when the steps allowed are in $\{(2+r, 0), (0, 2+r), (1, 1)\}$. The first and second steps of the walk have weight $\sqrt{t}$, while the last one has z.*

Example 14. The triple $(r, t, z) = (-1, 3, 2)$ yields a Morgan-Voyce direction, and provides the sequence A098269. It can be given by

$$T_n = \sum_{k=0}^{n} \binom{n+k}{k,\ k,\ n-k} 3^k\, 2^{n-k},$$

the first few elements are: $1, 8, 94, 1232, 16966, 240368, 3468844, \ldots$.

The sums of products of weights are related to the lattice paths from $(0, 0)$ to (n, n) using only the steps $(1, 0), (0, 1)$ and $(1, 1)$, with the weights $\sqrt{3}$, $\sqrt{3}$, 2, respectively. In case of $n = 2$ we illustrate all paths from $(0, 0)$ to $(2, 2)$ in Figure 1, which imply

$$\begin{aligned}
T_2 \;=\; & (\sqrt{3}\times\sqrt{3}\times\sqrt{3}\times\sqrt{3}) + (2\times\sqrt{3}\times\sqrt{3}) + (\sqrt{3}\times\sqrt{3}\times\sqrt{3}\times\sqrt{3}) \\
& +(\sqrt{3}\times 2\times\sqrt{3}) + (2\times 2) + (\sqrt{3}\times\sqrt{3}\times\sqrt{3}\times\sqrt{3}) + (\sqrt{3}\times\sqrt{3}\times\sqrt{3}\times\sqrt{3}) \\
& +(2\times\sqrt{3}\times\sqrt{3}) + (\sqrt{3}\times\sqrt{3}\times 2) + (\sqrt{3}\times\sqrt{3}\times\sqrt{3}\times\sqrt{3}) \\
& +(\sqrt{3}\times\sqrt{3}\times\sqrt{3}\times\sqrt{3}) + (\sqrt{3}\times\sqrt{3}\times 2) + (\sqrt{3}\times 2\times\sqrt{3}) \\
& = 94.
\end{aligned}$$

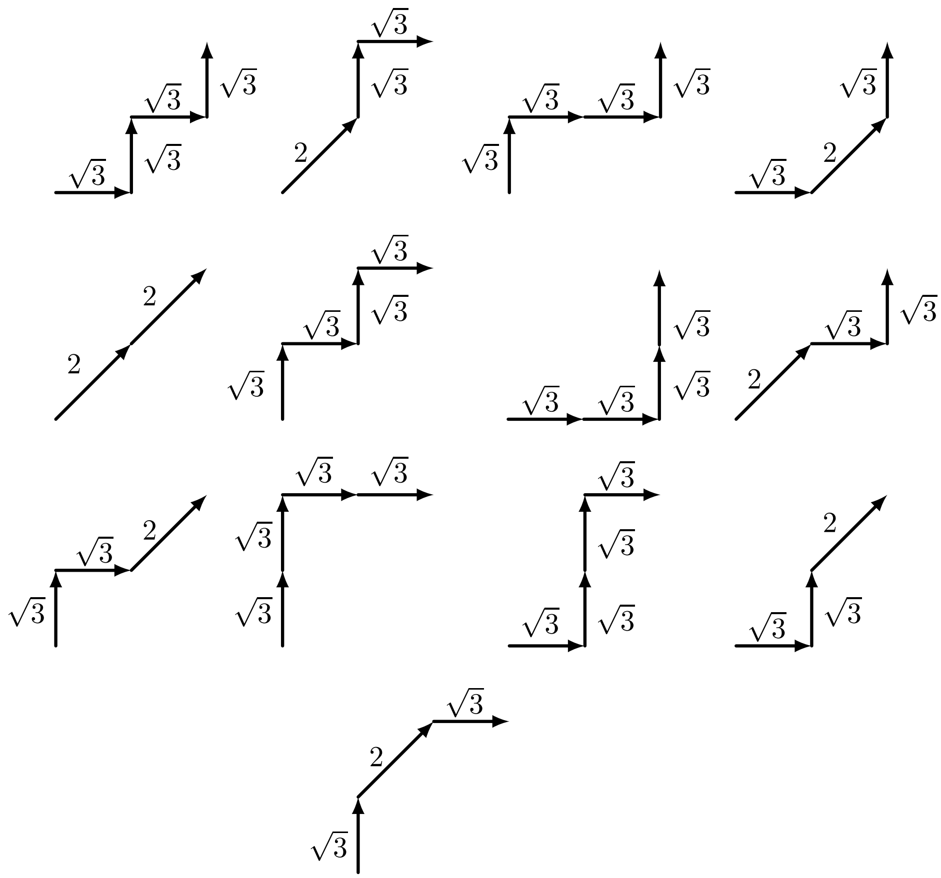

Figure 1: All lattice paths linked to A098269.

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