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Probabilistic Proofs of Some Beta-Function Identities

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Abstract

Using a probabilistic approach, we derive some interesting identities involving beta functions. These results generalize certain well-known combinatorial identities involving binomial coefficients and gamma functions.

1 Introduction

There are several interesting combinatorial identities involving binomial coefficients, gamma functions, and hypergeometric functions (see, for example, Riordan [9], Bagdasaryan [1], Vellaisamy [15], and the references therein). One of these is the following famous identity that involves the convolution of the central binomial coefficients:

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n. \tag{1}$$

In recent years, researchers have provided several proofs of (1). A proof that uses generating functions can be found in Stanley [12]. Combinatorial proofs can also be found, for example, in Sved [13], De Angelis [3], and Mikić [6]. A related and an interesting identity for the alternating convolution of central binomial coefficients is

$$\sum_{k=0}^{n}(-1)^k \binom{2k}{k}\binom{2n-2k}{n-k} = \begin{cases} 2^n\binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

Nagy [7], Spivey [11], and Mikić [6] discussed the combinatorial proofs of the above identity. Recently, there has been considerable interest in finding simple probabilistic proofs for combinatorial identities (see Vellaisamy and Zeleke [14] and the references therein). Pathak [8] gave a probabilistic proof of the identity in (2). Chang and Xu [2] extended the result in (1) and presented a probabilistic proof of the identity

$$\sum_{\substack{k_j \geq 0,\ 1\leq j\leq m; \\ \sum_{j=1}^{m} k_j = n}} \binom{2k_1}{k_1}\binom{2k_2}{k_2}\cdots\binom{2k_m}{k_m} = \frac{4^n}{n!}\frac{\Gamma(n+\frac{m}{2})}{\Gamma(\frac{m}{2})}, \tag{3}$$

where $k_1, \ldots, k_m$ are nonnegative integers, and m and n are positive integers. Mikić [6] discussed a combinatorial proof of (3) based on the method of recurrence relations and telescoping. Duarte and Guedes de Oliveira [4] discussed a generalization of the result in (3) and proved the following identity (see their Theorem 2), using combinatorial arguments,

$$\sum_{\sum_{j=1}^{m} k_j = n} \binom{2k_1+l_1}{k_1}\binom{2k_2+l_2}{k_2}\cdots\binom{2k_m+l_m}{k_m} = 4^n\binom{n+\frac{m}{2}-1}{n}, \tag{4}$$

where $l_1, \ldots, l_m$ are reals such that $l_1 + \cdots + l_m = 0$.

Our goal in this paper is to generalize the combinatorial identities in (2) and (3), using a simple probabilistic approach. Indeed, we derive certain identities involving beta functions. Our method uses the Dirichlet-multinomial distribution and also the moments of the difference of two gamma random variables.

2 Identities involving beta functions

Let the random variable T follow the beta distribution with parameters $a, b > 0$ and with probability density

$$f(t) = \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1},\ t > 0,$$

where $B(a,b)$ is the beta function. Note that the beta function $B(x,y)$ is related to the gamma function by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, x > 0$. The beta function is symmetric (that is, $B(x,y) = B(y,x)$) and satisfies the basic identity

$$B(x,y) = B(x,y+1) + B(x+1,y), \text{ for } x,y > 0. \tag{5}$$

It is easy to see that the derivative of the beta function is

$$\frac{\partial}{\partial y}B(x,y) = B(x,y)\left(\psi(y) - \psi(x+y)\right), \tag{6}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

We start with a result that relates binomial coefficients and beta functions on one side to a simple rational expression on the other side.

Theorem 1. *For $s > 0$ and an integer $n \geq 0$,*

$$\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^j\binom{n}{j}\frac{B(j+1,s)}{s+i} = \frac{1}{(s+n)^2}.$$

Proof. Let the random variable Y follow the beta distribution with parameters 1 and $n+s$. Using the density of Y, we get

$$\begin{aligned}
&\int_0^\infty (1-y)^{n+s-1}dy = B(1,n+s) &&= \frac{1}{n+s};\\
\Longrightarrow &\int_0^\infty \left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}y^j\right)(1-y)^{s-1}dy &&= \frac{1}{n+s};\\
\Longrightarrow &\sum_{j=0}^{n}(-1)^j\binom{n}{j}\int_0^\infty y^j(1-y)^{s-1}dy &&= \frac{1}{n+s};\\
\Longrightarrow &\sum_{j=0}^{n}(-1)^j\binom{n}{j}B(j+1,s) &&= \frac{1}{n+s}.
\end{aligned} \tag{7}$$

Differentiate both sides of (7) with respect to $s > 0$ to get

$$\begin{aligned}
\frac{-1}{(s+n)^2} &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\partial}{\partial s}B(j+1,s)\\
&= \sum_{j=0}^{n}(-1)^j\binom{n}{j}B(j+1,s)\left(\psi(s) - \psi(j+1+s)\right),
\end{aligned} \tag{8}$$

using (6). Further, it is known that the digamma function $\psi(x)$ satisfies

$$\psi(x+1) - \psi(x) = \frac{1}{x}. \tag{9}$$

Using (9) iteratively leads to

$$\psi(x+j+1) - \psi(x) = \sum_{i=0}^{j} \frac{1}{x+i}, \tag{10}$$

for a nonnegative integer j. The result follows by putting (10) into (8). $\square$

Remark 2. The identity in (7) itself is an interesting identity. When $n = 2$, it reduces to

$$B(1,s) - 2B(2,s) + B(3,s) = \frac{1}{s+2},$$

which can also be verified using the basic identity in (5).

Next we extend the identity in (2). Let X be a gamma random variable with parameter $p > 0$, denoted by $X \sim G(p)$. The density of X is given by

$$f(x|p) = \frac{1}{\Gamma(p)} e^{-x} x^{p-1},\ x > 0,\ p > 0.$$

Then, it follows (see Rohatgi and Saleh [10]) that

$$E(X^n) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-x} x^{p+n-1} dx = \frac{\Gamma(p+n)}{\Gamma(p)}.$$

Theorem 3. *Let $p > 0$ and n be a positive integer. Then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} B(p+k, p+n-k) = \begin{cases} \dfrac{n!\Gamma(p)\Gamma(p+\frac{n}{2})}{\Gamma(\frac{n}{2}+1)\Gamma(2p+n)}, & \textit{if } n \textit{ is even;} \\ 0, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Consider the random variable $X = X_1 - X_2$, where X_1 and X_2 are independent gamma random variables with the same parameter $p > 0$, that is, $X_i \sim G(p)$, for $i = 1, 2$. Since X_1 and X_2 are independent and identically distributed, we have $X \overset{d}{=} -X$ (that is, X and $-X$ have the same distributions on $\mathbb{R}$). This implies the density of X is symmetric about zero. Hence, $E(X^n) = 0$ if n is an odd integer.

Next we compute the even moments of X. Finding the moments of X using the probability density function is tedious. This is because the density of X is very complicated and it involves Whittaker's W-function (see Mathai [5]). Therefore, we use the moment generating function (MGF) approach to find the moments of X.

It is known (see Rohatgi and Saleh [10]) that the MGF of X_1 is

$$M_{X_1}(t) = E(e^{tX_1}) = (1-t)^{-p}.$$

Hence, the MGF of X is

$$M_X(t) = M_{X_1}(t)M_{X_2}(-t) = (1-t)^{-p}(1+t)^{-p} = (1-t^2)^{-p},$$

which exists for $|t| < 1$. Using the result

$$(1-q)^{-p} = \sum_{n=0}^{\infty} \frac{\Gamma(n+p)q^n}{\Gamma(n+1)\Gamma(p)}, \text{ for } p > 0 \text{ and } |q| < 1,$$

we have

$$M_X(t) = (1-t^2)^{-p} = \sum_{n=0}^{\infty} \frac{\Gamma(n+p)t^{2n}}{\Gamma(n+1)\Gamma(p)}. \tag{11}$$

Hence, for $n \geq 1$, we have from (11)

$$E(X^{2n}) = M_X^{(2n)}(t)|_{t=0} = \frac{\Gamma(n+p)(2n)!}{\Gamma(n+1)\Gamma(p)},$$

where $f^{(k)}$ denotes the k-th derivative of f. Thus, we have shown that

$$E(X^n) = \begin{cases} \dfrac{n!\Gamma(\frac{n}{2}+p)}{\Gamma(\frac{n}{2}+1)\Gamma(p)}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{12}$$

Next, we compute the moments of X, using the binomial theorem. Note that

$$\begin{aligned} E(X^n) =& E(X_1 - X_2)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E(X_1^k)E(X_2^{n-k}) \\ =& \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{\Gamma(p+k)}{\Gamma(p)}\right) \left(\frac{\Gamma(p+n-k)}{\Gamma(p)}\right). \end{aligned} \tag{13}$$

Equating (12) and (13), we get

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \Gamma(p+k)\Gamma(p+n-k) = \begin{cases} \dfrac{n!\Gamma(\frac{n}{2}+p)\Gamma(p)}{\Gamma(\frac{n}{2}+1)}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \tag{14}$$

which is an interesting identity involving gamma functions and binomial coefficients. Dividing both sides of (14) by $\Gamma(2p+n)$, the result follows. □

We will show now that the identity in (2) follows as a special case.

Corollary 4. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2n-2k}{n-k}=\begin{cases}2^n\binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Let $p=\frac{1}{2}$ in (14) and it suffices to consider the case when n is even. Then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(n-k+\frac{1}{2}\right)=\frac{n!\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}.$$

That is,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)\left(\frac{\Gamma(n-k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)=\frac{n!\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2}+1)}. \tag{15}$$

Note that,

$$\begin{aligned}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}&=\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\\&=\frac{(2n-1)\cdot(2n-3)\cdots 3\cdot 1}{2^n}\\&=\frac{(2n)!}{n!4^n}.\end{aligned} \tag{16}$$

Using (16) in (15), we get

$$\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}\frac{(2k)!}{4^k k!}\frac{(2n-2k)!}{4^{(n-k)}(n-k)!}=\frac{n!n!}{4^{\frac{n}{2}}(\frac{n}{2})!(\frac{n}{2})!}.$$

That is,

$$\sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2n-2k}{n-k}=\frac{n!4^n}{(\frac{n}{2})!(\frac{n}{2})!4^{\frac{n}{2}}}=2^n\binom{n}{\frac{n}{2}},$$

which proves the result. ∎

Finally, we discuss an extension of the identity given in (3). Let $p_i>0$ for $1\le i\le m$. Let

$$B(p_1,\ldots,p_m)=\frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_m)}{\Gamma(p_1+\cdots+p_m)}$$

denote the beta function of m variables, and $\binom{n}{k_1,\ldots,k_m}=\frac{n!}{k_1!\cdots k_m!}$ denote the multinomial coefficient.

Let $X = (X_1, \ldots, X_m)$ be a discrete nonnegative random vector and $Y = (Y_1, \ldots, Y_m)$ be a continuous positive random vector such that $\sum_1^m X_i = n$ and $\sum_1^m Y_i = 1$. Let $(X|Y) \sim \mathrm{MN}(n; Y_1, \ldots, Y_m)$, the multinomial distribution, with

$$P\left(X_1 = k_1, \ldots, X_m = k_m | Y_1 = y_1, \ldots, Y_m = y_m\right) = \binom{n}{k_1, \ldots, k_m} y_1^{k_1} \ldots y_m^{k_m}$$

and $Y \sim \mathrm{Dir}(p_1, \ldots, p_m)$, the Dirichlet distribution, with density

$$f(y_1, \ldots, y_m) = \frac{1}{B(p_1, \ldots, p_m)} y_1^{p_1 - 1} \cdots y_m^{p_m - 1}.$$

Then the marginal distribution of X follows the Dirichlet-multinomial distribution with

$$P\left(X_1 = k_1, \ldots, X_m = k_m\right) = \frac{1}{B(p_1, \ldots, p_m)} \binom{n}{k_1, \ldots, k_m} B(k_1 + p_1, \ldots, k_m + p_m),$$

where $k_1 + \cdots + k_m = n$.

The next result follows trivially as the sum of the above probabilities is unity.

Theorem 5. *Let $p_1, \ldots, p_m > 0$. Then for any nonnegative integer n,*

$$\sum_{\substack{k_j \geq 0,\ 1 \leq j \leq m; \\ \sum_{j=1}^m k_j = n}} \binom{n}{k_1, \ldots, k_m} B(k_1 + p_1, \ldots, k_m + p_m) = B(p_1, \ldots, p_m). \tag{17}$$

It is interesting to note that the identity in (3) follows as a special case.

Corollary 6. *When $p_1 = p_2 = \cdots = p_m = \frac{1}{2}$, the identity in (17) reduces to*

$$\sum_{\substack{k_j \geq 0,\ 1 \leq j \leq m; \\ \sum_{j=1}^m k_j = n}} \binom{2k_1}{k_1} \binom{2k_2}{k_2} \cdots \binom{2k_m}{k_m} = \frac{4^n}{n!} \frac{\Gamma(n + \frac{m}{2})}{\Gamma(\frac{m}{2})}, \tag{18}$$

for all integers $m, n \geq 1$.

Proof. Putting $p_1 = p_2 = \cdots = p_m = \frac{1}{2}$ in (17), we obtain

$$\sum_{\substack{k_j \geq 0,\ 1 \leq j \leq m; \\ \sum_{j=1}^m k_j = n}} \binom{n}{k_1, \ldots, k_m} B\left(\frac{1}{2} + k_1, \ldots, \frac{1}{2} + k_m\right) = B\left(\frac{1}{2}, \ldots, \frac{1}{2}\right).$$

This implies,

$$\sum_{\substack{k_j \geq 0,\ 1 \leq j \leq m; \\ \sum_{j=1}^m k_j = n}} \binom{n}{k_1, \ldots, k_m} \frac{\Gamma\left(\frac{1}{2} + k_1\right) \cdots \Gamma\left(\frac{1}{2} + k_m\right)}{\Gamma\left(n + \frac{m}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right) \cdots \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)},$$

or, equivalently,

$$\sum_{\substack{k_j\geq 0,\ 1\leq j\leq m;\\ \sum_{j=1}^m k_j=n}} \binom{n}{k_1,\ldots,k_m} \frac{\Gamma\left(\frac{1}{2}+k_1\right)\cdots\Gamma\left(\frac{1}{2}+k_m\right)}{\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma\left(n+\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}.$$

Using (16), we get

$$\sum_{\substack{k_j\geq 0,\ 1\leq j\leq m;\\ \sum_{j=1}^m k_j=n}} \binom{n}{k_1,\ldots,k_m} \frac{(2k_1)!\cdots(2k_m)!}{4^{k_1}(k_1)!\cdots 4^{k_m}(k_m)!} = \frac{\Gamma\left(n+\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)},$$

which is equivalent to the identity in (18). ☐

Remark 7. Obviously, when $m=2$, the identity in (17) reduces to

$$\sum_{\substack{k_j\geq 0,\ 1\leq j\leq 2;\\ k_1+k_2=n}} \binom{n}{k_1,k_2} B(p_1+k_1,p_2+k_2)$$

$$=\sum_{k=0}^{n}\binom{n}{k}B(p_1+k,p_2+n-k)=B(p_1,p_2).$$

Also, in view of Corollary 6, when $p_1=p_2=\frac{1}{2}$, the above equation reduces to (1).

Remark 8. Let m be even so that $m=2l$ for some positive integer l. Then the right hand side of (18) is

$$\frac{4^n}{n!}\frac{\Gamma(n+l)}{\Gamma(l)} = 4^n\binom{n+l-1}{n} = 4^n\binom{n+\frac{m}{2}-1}{n}.$$

Similarly, when m is odd, say $m=2l+1$,

$$\begin{aligned}
\frac{4^n}{n!}\frac{\Gamma(n+\frac{m}{2})}{\Gamma(\frac{m}{2})} &= \frac{4^n}{n!}\frac{\Gamma(n+l+\frac{1}{2})}{\Gamma(l+\frac{1}{2})} = \frac{4^n}{n!}\left(\frac{\Gamma(n+l+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(l+\frac{1}{2})}\right)\\
&= \binom{2n+2l}{2n}\left(\frac{(2n)!}{n!\,n!}\right)\left(\frac{l!\,n!}{(n+l)!}\right) \text{ (using (16))}\\
&= \frac{\binom{2n+2l}{2n}\binom{2n}{n}}{\binom{n+l}{n}}\\
&= \frac{\binom{2n+m-1}{2n}\binom{2n}{n}}{\binom{n+\frac{m-1}{2}}{n}},
\end{aligned}$$

since $2l = m - 1$. Thus, we have, from (18),

$$\sum_{\substack{k_j \geq 0,\ 1 \leq j \leq m; \\ \sum_{j=1}^m k_j = n}} \binom{2k_1}{k_1}\binom{2k_2}{k_2}\cdots\binom{2k_m}{k_m} = \begin{cases} 4^n \dbinom{n + \frac{m}{2} - 1}{n}, & \text{if } m \text{ is even;} \\ \dfrac{\binom{2n+m-1}{2n}\binom{2n}{n}}{\binom{n+\frac{m-1}{2}}{n}}, & \text{if } m \text{ is odd,} \end{cases}$$

which is equation (3) of Mikić [6]. Indeed, Mikić [6] provided a combinatorial proof of the above result based on recurrence relations.

Corollary 9. *Let $k_1, \ldots, k_m$ be nonnegative integers and $l_1, \ldots, l_m$ be integers such that $0 \leq k_i + l_i \leq n$ and $\sum_{i=1}^m l_i = 0$. Then*

$$\sum_{\sum_{j=1}^m k_j = n} \binom{2k_1 + 2l_1}{k_1 + l_1}\binom{2k_2 + 2l_2}{k_2 + l_2}\cdots\binom{2k_m + 2l_m}{k_m + l_m} = \frac{4^n}{n!}\frac{\Gamma(n + \frac{m}{2})}{\Gamma(\frac{m}{2})},$$

for all integers $m, n \geq 1$.

The above corollary, which follows from Corollary 6, is similar to (4). It is not clear if the identity in (4) can be obtained through probabilistic considerations.

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