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Antichain Simplices

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Abstract

Associated with each lattice simplex Δ is a poset encoding the additive structure of lattice points in the fundamental parallelepiped for Δ. When this poset is an antichain, we say Δ is antichain. For each partition λ of n, we define a lattice simplex Δ_λ having one unimodular facet, and we investigate their associated posets. We give a number-theoretic characterization of the relations in these posets, as well as a simplified characterization in the case where each part of λ is relatively prime to $n-1$. We use these characterizations to experimentally study Δ_λ for all partitions of n with $n \leq 73$. Further, we experimentally study the prevalence of the antichain property among simplices with a restricted type of Hermite normal form, suggesting that the antichain property is common among simplices with this restriction. Finally, we explain how this work relates to Poincaré series for the semigroup algebra associated with Δ, and we prove that this series is rational when Δ is antichain.

1 Introduction

Given a lattice simplex Δ, the structure of the lattice points in the fundamental parallelepiped of the cone over Δ reflects a wealth of arithmetic and combinatorial properties of Δ. In this work, we study a partial order on these lattice points that encodes the additive relations among these points. Thus, our main results are primarily arithmetic in nature, and will hopefully be of interest to those working in areas where fundamental parallelepipeds play a role, e.g., Ehrhart theory, partition identities, coding theory, optimization, etc. Our motivation for this investigation comes from questions regarding rationality of Poincaré series for infinite graded resolutions of graded algebras, where it is of particular interest when the partial order associated with Δ has no relations. Thus, after developing our main results using number-theoretic techniques, we explain their algebraic implications.

More precisely, in Section 2 we define the fundamental parallelepiped poset $P(\Delta)$ associated with Δ, where we say Δ is *antichain* if $P(\Delta) \setminus \{0\}$ has no relations. For each partition λ of n, we define a lattice simplex Δ_λ having one unimodular facet, and we investigate the posets for these simplices in depth. In Theorem 15 we give a number-theoretic characterization of the relations in $P(\Delta_\lambda)$, and in Corollary 16 we give a simplified characterization in the case where each part of λ is relatively prime to $n-1$. In Section 3 we use these characterizations to generate empirical data, experimentally studying those Δ_λ for all partitions of n with $n \leq 73$. These experiments reveal that a substantial fraction of those λ satisfying the relatively prime condition appear to have Δ_λ that are antichain. Further, we experimentally study the prevalence of the antichain property among simplices with a restricted type of Hermite normal form, suggesting that the antichain property is common among simplices with this restriction. Finally, in Section 4 we explain the algebraic implications of our work to Poincaré series of semigroup algebras associated with Δ. Specifically, we prove that if Δ is antichain, then the associated Poincaré series is rational.

2 Fundamental parallelepiped posets and their structure

2.1 Lattice simplices and associated posets

Details regarding polytopes, cones, Hilbert bases, etc. as discussed in the following can be found in standard references [3, 13]. For a collection $\mathcal{V} = \{v_0, \dots, v_m\}$ of points in $\mathbb{R}^d$, we let $\mathrm{conv}(\mathcal{V})$ denote the convex hull of $\mathcal{V}$. In the case that $m = d$ and the set $\mathcal{V}^\circ := \{(v_1 - v_0), \dots, (v_d - v_0)\}$ is a vector space basis of $\mathbb{R}^d$, then we call $\Delta := \mathrm{conv}(\mathcal{V})$ a d-simplex. We call the v_i's the *vertices* of Δ, and if each v_i is an integer point, i.e., lies in $\mathbb{Z}^d$, we call Δ a *lattice simplex*. We define the *conical hull* of $\mathcal{V}$ to be the set

$$\mathrm{cone}(\mathcal{V}) := \left\{ \sum_{i=0}^{m} \gamma_i v_i \text{ such that } 0 \leq \gamma_i \right\} \subset \mathbb{R}^d.$$

Notice that the conical hull is unbounded, as in particular it contains the rays $\mathbb{R}_{\geq 0} \cdot v_i$ for $0 \leq i \leq m$.

We are particularly interested in conical hulls of the following kind. Let $\mathcal{V} = \{v_0, \dots, v_d\}$ with Δ a lattice d-simplex. Then the *cone over* Δ is

$$\mathrm{cone}\{(1, v_0), \dots, (1, v_d)\} \subset \mathbb{R}^{d+1},$$

and is denoted $\mathrm{cone}(\Delta)$. We next recall the fundamental parallelepiped, a distinguished subset of $\mathrm{cone}(\Delta)$.

Definition 1. For a lattice d-simplex Δ with vertices v_0 through v_d, the *fundamental parallelepiped* Π_Δ is the set

$$\Pi_\Delta := \left\{ \sum_{i=0}^{d} \gamma_i (1, v_i) \text{ such that } 0 \leq \gamma_i < 1 \right\} \subset \operatorname{cone}(\Delta).$$

Interest in the fundamental parallelepiped Π_Δ arises mainly from the following well-known fact: every lattice point in $\operatorname{cone}(\Delta)$ can be written uniquely as a non-negative integer combination of the $(1, v_i)$'s and a lattice point in Π_Δ. To see this, note that because any element z of $\operatorname{cone}(\Delta) \cap \mathbb{Z}^{d+1}$ lies in $\operatorname{cone}(\Delta)$, it is a non-negative linear combination of the $(1, v_i)$'s, i.e., there exist non-negative real coefficients g_i such that

$$z = \sum_{i=0}^{d} g_i (1, v_i) = \left(\sum_{i=0}^{d} \lfloor g_i \rfloor (1, v_i) \right) + \left(\sum_{i=0}^{d} \{g_i\} (1, v_i) \right)$$

where $\{g_i\}$ means the fractional part of g_i. By setting γ_i equal to $\{g_i\}$, we see that any point z may be written as a non-negative integral combination of the $(1, v_i)$'s and an integer point in $\Pi_\Delta \cap \mathbb{Z}^{d+1}$. In particular, it is well-known that the set $\operatorname{cone}(\Delta) \cap \mathbb{Z}^{d+1}$ has a unique finite minimal additive generating set.

Definition 2. The unique minimal additive generating set of $\operatorname{cone}(\Delta) \cap \mathbb{Z}^{d+1}$ is called the *Hilbert basis* $\mathcal{H}$ of $\operatorname{cone}(\Delta)$. It consists of the $(1, v_i)$'s and some lattice points h_1 through h_m in Π_Δ such that

$$\operatorname{cone}(\Delta) \cap \mathbb{Z}^{d+1} = \left\{ \left(\sum_{i=0}^{d} r_i (1, v_i) \right) + \left(\sum_{j=1}^{m} s_i \, h_i \right) \text{ such that } r_i \,,\, s_j \in \mathbb{Z}_{\geq 0} \right\}.$$

The Hilbert basis consists of the cone generators $(1, v_i)$ together with the additively minimal elements h_j of $\Pi_\Delta \cap \mathbb{Z}^{d+1}$. If the matrix whose columns are given by $(1, v_i)$ has determinant $\pm \boldsymbol{v}$, we say that the simplex Δ has *normalized volume* $\boldsymbol{v}$. Since $\boldsymbol{v}$ is precisely the index of the sub-lattice generated by $(1, v_0)$ through $(1, v_d)$, we see that the normalized volume is equal to the number of lattice points in Π_Δ. If the normalized volume of Δ is equal to one, then we call Δ a *unimodular* simplex.

The set of lattice points $\mathbb{Z}^{d+1} \cap \Pi_\Delta$ can be equipped with the following partial order, inherited from a well-known partial order on the lattice points in $\mathbb{Z}^{d+1} \cap \operatorname{cone}(\Delta)$.

Definition 3. The set $\mathbb{Z}^{d+1} \cap \Pi_\Delta$ is partially ordered by letting $\sigma \prec \mu$ if and only if $\mu - \sigma$ is an element of $\mathbb{Z}^{d+1} \cap \Pi_\Delta$. We call this poset the *fundamental parallelepiped poset* $\mathrm{P}(\Delta)$.

Observe that the zero element of $\mathbb{Z}^{d+1} \cap \Pi_\Delta$ is below every other element of $\mathrm{P}(\Delta)$, and that the minimal elements of $\mathrm{P}(\Delta) \smallsetminus \{0\}$ are precisely the elements $h_1, \ldots, h_m$ of the Hilbert basis of $\operatorname{cone}(\Delta)$. Our interest is in the case where the Hilbert basis contains all the elements of $\mathrm{P}(\Delta) \smallsetminus \{0\}$, leading to the following definition.

Definition 4. If Δ is a simplex such that $P(\Delta) \setminus \{0\}$ is an antichain, we call Δ an *antichain simplex* and say Δ is *antichain*.

Example 5. Recall that an empty simplex is one whose only lattice points are its vertices. It is known that any 2-dimensional empty simplex is unimodularly equivalent to the convex hull of the origin and the standard basis vectors; this is easily verified to be antichain. Let Δ be a 3-dimensional empty simplex, for which a classification of such simplices was given by G. K. White [20]. Since Δ is empty, no lattice points of Π_Δ have 0-th coordinate equal to one, and thus the only possible 0-th coordinates of lattice points in Π_Δ are 2 or 3. However, sums of pairs of such lattice points have 0-th coordinate equal to 4, 5, or 6, and hence $\mathrm{P}(\Delta)$ has no relations. Thus, every 3-dimensional empty simplex is antichain.

When attempting to determine whether or not Δ is antichain, the first problem encountered is to enumerate the elements of $\Pi_\Delta \cap \mathbb{Z}^{d+1}$. As an initial attempt in this direction, let the matrix A have columns given by $\{(1, v_i)\}_{0 \le i \le d}$, where the v_i's are the vertices of Δ. Recall that the normalized volume $\boldsymbol{v}$, the number of elements of Π_Δ, may be computed by $\boldsymbol{v} = |\det A|$. Recall also that the set Π_Δ is the image of $[0,1)^{d+1}$ under the linear transformation A, so that the preimage of a lattice point of Π_Δ must be a rational point of $[0,1)^{d+1}$ with denominator $\boldsymbol{v}$. We may therefore compute the set of points in $\Pi_\Delta \cap \mathbb{Z}^{d+1}$ by considering each element of the form

$$\left\{ A \cdot \left(\frac{b_0}{\boldsymbol{v}}, \cdots, \frac{b_d}{\boldsymbol{v}} \right)^T \text{ such that } 0 \le b_i < \boldsymbol{v} \right\},$$

throwing out the ones which are not integer points. Unfortunately, this test set grows as $\boldsymbol{v}^{d+1}$, and there is no easy way to describe the lattice points among them.

The software `Normaliz` [7] gives a more efficient implementation based on the fact that (possibly after a lattice translation) the matrix A has a representation $A = UH$ where U is a unimodular matrix and H is in Hermite normal form. Bruns et al. [8] show that, for $\{c_{i,i}\}_{0 \le i \le d}$ given by the diagonal entries of the matrix H, lattice points in

$$[0, c_{0,0}) \times \cdots \times [0, c_{d,d})$$

are representatives of the quotient classes (in $\mathbb{Z}^{d+1}$ modulo the $(1, v_i)$'s) of the elements of $\Pi_\Delta \cap \mathbb{Z}^{d+1}$. It is then sufficient to consider the image under A of the elements $(A^{-1} \cdot x) \bmod \mathbb{Z}^{d+1}$ for $x \in [0, c_{0,0}) \times \cdots \times [0, c_{d,d})$. This modular arithmetic is implemented in a computer easily enough, but introduces number theory to any analysis of the poset $P(\Delta)$.

2.2 Lattice simplices with a unimodular facet and their posets

In this work, we study a restricted class of simplices in order to avoid both of the methods described above for determining $\Pi_\Delta \cap \mathbb{Z}^{d+1}$.

Definition 6. We say that a lattice d-simplex has a *unimodular facet* if there exists a permutation π in $\mathfrak{S}_{d+1}$ such that $\mathrm{conv}(\{v_{\pi_1}, \ldots, v_{\pi_d}\})$ is a unimodular lattice $(d-1)$-simplex.

One way to represent these simplices is through their Hermite normal form, which is of the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{d-1} \\ 0 & 0 & 0 & \cdots & 0 & n \end{bmatrix}$$

where for $i = 1, \dots, d-1$ we have $0 \le a_i < n$. Simplices with Hermite normal form having only a single non-trivial column, such as the ones given above, were previously considered by Hibi, Higashitani, and Li [11, Section 3] in the context of Ehrhart theory.

Alternatively, if Δ has a unimodular facet, then we may define a lattice preserving transformation taking Δ to $\mathrm{conv}(e_1, \dots, e_d, z)$ where the e_i are the standard basis vectors of $\mathbb{R}^d$ and z is a lattice point in $\mathbb{Z}^d$. This is the general approach we will take in this paper, and we will focus our analysis on those Δ having z with strictly positive entries. Our goal in this section is to find a description of the relations in $P(\Delta)$ in terms of the coordinates of the point z for such simplices.

Note that in the following definition, we can assume that λ is a partition of n, as permuting the entries of λ corresponds to a unimodular transformation of the Δ_λ that is defined. This provides us with a unique lattice simplex for each integer partition, which is our motivation for this restriction.

Definition 7. Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be a lattice point in $\mathbb{N}^d$ such that $\sum_{i=1}^d \lambda_i = n$. We define $\Delta_\lambda := \mathrm{conv}(e_1, \dots, e_d, \lambda) \subset \mathbb{R}^d$ and use the shortened notation $\Pi_\lambda := \Pi_{\Delta_\lambda}$ and $P(\lambda) := P(\Delta_\lambda)$.

Remark 8. The simplices Δ_λ are defined in a similar manner to the simplices $\Delta_{(1,q)}$ that have recently been studied by multiple authors [4, 5, 6, 9, 14, 16, 17]. However, these are not the same families of simplices. Specifically, the matrix giving the Hermite normal form of Δ_λ (after translating Δ_λ by $-e_1$) is

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \lambda_2 \\ 0 & 0 & 1 & \cdots & 0 & \lambda_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda_d \\ 0 & 0 & 0 & \cdots & 0 & -1 + \sum_i \lambda_i \end{bmatrix}.$$

Note that $\sum_{i=2}^d \lambda_i \le -1 + \sum_{i=1}^d \lambda_i$.

Setting $Q = 1 + \sum_i q_i$, the Hermite normal form for $\Delta_{(1,q)}$ is

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & Q - q_2 \\ 0 & 0 & 1 & \cdots & 0 & Q - q_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & Q - q_d \\ 0 & 0 & 0 & \cdots & 0 & Q \end{bmatrix}.$$

Note that for $d \geq 3$, we have $\sum_{i=2}^{d} Q - q_i > Q$. Thus, these are distinct classes of simplices.

The following is a straightforward determinant calculation that also follows from the Hermite normal form given in Remark 8.

Proposition 9. *The number of lattice points in Π_λ, which is equal to the normalized volume of Δ_λ, is $\sum_{i=1}^{d} \lambda_i - 1 = n - 1$.*

We can now describe the integer points in Π_λ using only the entries of λ.

Proposition 10. *For each integer b with $0 \leq b < n-1$, there is a unique lattice point $p(b)$ in Π_λ given by*

$$p(b) = \left(\left(\sum_{i=1}^{d} \left\lceil \frac{b\lambda_i}{n-1} \right\rceil \right) - b\, ,\ \left\lceil \frac{b\lambda_1}{n-1} \right\rceil\ ,\ \ldots\ ,\ \left\lceil \frac{b\lambda_d}{n-1} \right\rceil \right) . \tag{1}$$

Every integer point in Π_λ arises in this manner, and thus we identify the integer b with the lattice point $p(b)$.

Proof. For an element $\sum_{i=1}^{d} \gamma_i(1, e_i) + \gamma_{d+1}(1, \lambda) \in \Pi_\lambda \cap \mathbb{Z}^{d+1}$, we have

$$\left(\left(\sum_{i=1}^{d+1} \gamma_i \right)\ ,\ (\gamma_1 + \gamma_{d+1}\lambda_1)\ ,\ \ldots\ ,\ (\gamma_d + \gamma_{d+1}\lambda_d) \right) \in \mathbb{Z}^{d+1} .$$

Because of the condition that each γ_i is strictly less than one, for each i we have

$$\gamma_i = \lceil \gamma_{d+1}\lambda_i \rceil - \gamma_{d+1}\lambda_i \, ,$$

thus

$$\begin{aligned} &\left(\gamma_{d+1} + \sum_{i=1}^{d} \left(\lceil \gamma_{d+1}\lambda_i \rceil - \gamma_{d+1}\lambda_i \right), \lceil \gamma_{d+1}\lambda_1 \rceil, \ldots, \lceil \gamma_{d+1}\lambda_d \rceil \right) \\ = &\left(\gamma_{d+1} \left(1 - \sum_{i=1}^{d} \lambda_i \right) + \sum_{i=1}^{d} \lceil \gamma_{d+1}\lambda_i \rceil, \lceil \gamma_{d+1}\lambda_1 \rceil, \ldots, \lceil \gamma_{d+1}\lambda_d \rceil \right) . \end{aligned}$$

Observe that the first coordinate of this vector is an integer, hence

$$\gamma_{d+1} \left(1 - \sum_{i=1}^{d} \lambda_i \right) = \gamma_{d+1}(1-n) \in \mathbb{Z} \, .$$

It follows that γ_{d+1} is a rational number of the form $b/(n-1)$, and every lattice point arises in this manner and is of the form

$$\left(\left(\sum_{i=1}^{d} \left\lceil \frac{b\lambda_i}{n-1} \right\rceil \right) - b\, ,\ \left\lceil \frac{b\lambda_1}{n-1} \right\rceil\ ,\ \ldots\ ,\ \left\lceil \frac{b\lambda_d}{n-1} \right\rceil \right) .$$

Since there are $n-1$ lattice points in Π_λ by Proposition 9, there must be one unique lattice point for each $0 \leq b < n-1$. ∎

Using the notation from (1), for $0 \le b < n-1$ we have that the zeroth coordinate of $p(b)$ is

$$p(b)_0 := \left(\sum_{i=1}^{d} \left\lceil \frac{b\lambda_i}{n-1} \right\rceil \right) - b\,.$$

Recall that we freely identify the integer b with the lattice point $p(b)$. The following lemma provides a connection between the parameterization of the integer points in Π_λ and the order in $P(\lambda)$.

Lemma 11. *For $i, j \in P(\lambda)$ with $i \neq j$, we have $i \prec j$ if and only if $i < j$ and $p(i)+p(j-i) = p(j)$.*

Proof. For the forward direction, if $i \prec j$, then by Proposition 10 there exists a point $p(\ell) \in P(\lambda)$ such that $p(i) + p(\ell) = p(j)$. Note that $\ell > 0$ since $p(0) = 0$. It follows that for all $1 \le t \le d$, we have

$$\left\lceil \frac{i\lambda_t}{n-1} \right\rceil + \left\lceil \frac{\ell\lambda_t}{n-1} \right\rceil = \left\lceil \frac{j\lambda_t}{n-1} \right\rceil .$$

Given this, we have that $p(i)_1 + p(\ell)_1 = p(j)_1$ reduces to $i + \ell = j$, forcing $\ell = j - i > 0$, as desired.

For the reverse direction, if $i < j$ and $p(i) + p(j-i) = p(j)$, then we have $i \prec j$ by definition. □

We now give two propositions demonstrating how Lemma 11 can be used in practice.

Proposition 12. *If $0 \neq i \prec j$ in $P(\lambda)$, then also $j - i \prec j$ in $P(\lambda)$.*

Proof. By Lemma 11, we have $0 \neq i \prec j$ if and only if $0 \neq i < j$ and $p(i) + p(j-i) = p(j)$ if and only if $0 \neq j - i < j$ and $p(i) + p(j-i) = p(j)$ if and only if $0 \neq j - i \prec j$. □

Proposition 13. *Let $\lambda = (n-2, 2)$. Then $P(n-2, 2)$ is equal to the following poset on the elements $\{1, 2, \ldots, n-2\}$: The minimal elements of $P(n-2, 2)$ are $\{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ and the maximal elements are $\{\lfloor \frac{n-1}{2} \rfloor + 1, \ldots, n-2\}$. The cover relations are that the maximal element $\lfloor \frac{n-1}{2} \rfloor + j$ covers $\{j, j+1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.*

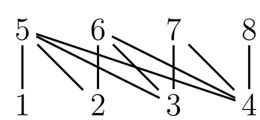

Figure 1: The poset $P(6, 2)$.

Proof. By Lemma 11, we see that $i \prec j$ if and only if $i < j$ and the following hold:

$$\left\lceil \frac{2i}{n-1} \right\rceil + \left\lceil \frac{2(j-i)}{n-1} \right\rceil = \left\lceil \frac{2j}{n-1} \right\rceil \tag{2}$$

$$\left\lceil \frac{i(n-2)}{n-1} \right\rceil + \left\lceil \frac{(j-i)(n-2)}{n-1} \right\rceil = \left\lceil \frac{j(n-2)}{n-1} \right\rceil \tag{3}$$

It is straightforward to verify that these equations hold for the values claimed in the proposition statement.

To show that no other pairs $i < j$ lead to relations $i \prec j$, suppose that $1 \le i < j \le \left\lfloor \frac{n-1}{2} \right\rfloor$. Then in (2), we obtain $1 + 1 = 1$, which is false. Similarly, if $\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \le i < j \le n-2$, then in (2) we obtain $2 + 2 = 2$, which is again false. $\square$

2.3 Characterizing the relations in $P(\lambda)$

While Lemma 11 is a reasonable first tool, as Propositions 12 and 13 illustrate, in general it is difficult to compute these relations directly. Thus, we need to create a more sophisticated mechanism through which to study $P(\lambda)$. In this section, we establish in Theorem 15 a number-theoretic characterization of the relations in $P(\lambda)$. Further, Corollary 16 provides a particularly simple characterization in the case where each part of λ is relatively prime to $n-1$.

For $0 \le i < n-1$, define the non-negative integers $r_{t,i}$ and $0 \le s_{t,i} < n-1$ by

$$i\lambda_t = r_{t,i}(n-1) + s_{t,i}\,. \tag{4}$$

Lemma 14. *We have $i \prec j$ in $P(\lambda)$ if and only if $i < j$ and for every $t \in \{1, \ldots, d\}$ we have*

$$\frac{s_{t,i} + s_{t,j-i} - s_{t,j}}{n-1} = \left\lceil \frac{s_{t,i}}{n-1} \right\rceil + \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil - \left\lceil \frac{s_{t,j}}{n-1} \right\rceil \tag{5}$$

Proof. After adding and subtracting (4) for the values i, $j-i$, and j, we obtain

$$r_{t,i} + r_{t,j-i} - r_{t,j} = \frac{-s_{t,i} - s_{t,j-i} + s_{t,j}}{n-1}\,. \tag{6}$$

By dividing both sides of (4) by $n-1$ and taking the ceiling of both sides, we see that

$$\left\lceil \frac{\ell\lambda_t}{n-1} \right\rceil = r_{t,\ell} + \left\lceil \frac{s_{t,\ell}}{n-1} \right\rceil\,. \tag{7}$$

Adding (7) with itself for ℓ equal to i and $j-i$, then subtracting the equation with $\ell = j$, and further applying (6), we obtain

$$
\begin{aligned}
&\left\lceil \frac{i\lambda_t}{n-1} \right\rceil + \left\lceil \frac{(j-i)\lambda_t}{n-1} \right\rceil - \left\lceil \frac{j\lambda_t}{n-1} \right\rceil \\
&= r_{t,i} + r_{t,j-i} - r_{t,j} + \left\lceil \frac{s_{t,i}}{n-1} \right\rceil + \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil - \left\lceil \frac{s_{t,j}}{n-1} \right\rceil \\
&= \frac{-s_{t,i} - s_{t,j-i} + s_{t,j}}{n-1} + \left\lceil \frac{s_{t,i}}{n-1} \right\rceil + \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil - \left\lceil \frac{s_{t,j}}{n-1} \right\rceil .
\end{aligned}
$$

Recall that $i \prec j$ in $P(\lambda)$ if and only if $p(i) + p(j-i) = p(j)$ if and only if for all t, we have that

$$\left\lceil \frac{i\lambda_t}{n-1} \right\rceil + \left\lceil \frac{(j-i)\lambda_t}{n-1} \right\rceil - \left\lceil \frac{j\lambda_t}{n-1} \right\rceil = 0\,,$$

which by our computation above holds if and only if

$$\frac{s_{t,i} + s_{t,j-i} - s_{t,j}}{n-1} = \left\lceil \frac{s_{t,i}}{n-1} \right\rceil + \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil - \left\lceil \frac{s_{t,j}}{n-1} \right\rceil .$$

□

Theorem 15. *Let λ be a partition of n. We have $i \prec j$ in $P(\lambda)$ if and only if $i < j$ and for each $t \in \{1, \dots, d\}$, one of the following holds:*

1. $s_{t,i} > s_{t,j} > 0$,
2. $s_{t,i} = 0$ *and* $s_{t,j} = s_{t,j-i}$, *or*
3. $s_{t,j} = s_{t,i} > 0$ *and* $s_{j-i} = 0$.

Proof. Forward implication: Suppose that $i \prec j$ in $P(\lambda)$, and thus by Lemma 14 the s-values satisfy (5). We consider five cases:

- $s_{t,i} = 0$
- $s_{t,i} > s_{t,j} = 0$
- $s_{t,i} = s_{t,j} > 0$
- $s_{t,i} > s_{t,j} > 0$
- $s_{t,j} > s_{t,i} > 0$

Case 1: $s_{t,i} = 0$. If $s_{t,i} = 0$, then by (5) we have that

$$\frac{s_{t,j-i} - s_{t,j}}{n-1} = \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil - \left\lceil \frac{s_{t,j}}{n-1} \right\rceil .$$

Thus $\dfrac{s_{t,j-i} - s_{t,j}}{n-1}$ is equal to an integer, and the fact that $0 \le s_{t,\ell} < n-1$ implies that $s_{t,j-i} - s_{t,j} = 0$. Thus, we must have $s_{t,j-i} = s_{t,j}$. This establishes the second condition in the theorem statement.

Case 2: $s_{t,i} > s_{t,j} = 0$. In this case, (5) implies

$$\frac{s_{t,i} + s_{t,j-i}}{n-1} = \left\lceil \frac{s_{t,i}}{n-1} \right\rceil + \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil .$$

Thus $\dfrac{s_{t,i} + s_{t,j-i}}{n-1}$ is an integer, and again since $0 \le s_{t,\ell} < n-1$ and $0 < s_{t,i} < n-1$ we have that Thus, it is impossible to have $s_{t,i} > s_{t,j} = 0$.

Case 3: $s_{t,i} = s_{t,j} > 0$. In this case, (5) implies

$$\frac{s_{t,j-i}}{n-1} = \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil .$$

This forces $s_{t,j-i} = 0$, resulting in the third condition in the theorem statement.

Case 4: $s_{t,i} > s_{t,j} > 0$. In this case, (5) implies $\dfrac{s_{t,i} + s_{t,j-i} - s_{t,j}}{n-1}$ is equal to an integer, and the fact that every $0 \le s_{t,\ell} < n-1$ implies this integer is 0 or 1. Since $n-1 > s_{t,i} - s_{t,j} > 0$, we must have $\dfrac{s_{t,i} + s_{t,j-i} - s_{t,j}}{n-1} = 1$, and also the right-hand side of (5) is equal to 1. Thus, the first condition in the theorem statement is possible if $i \prec j$.

Case 5: $s_{t,j} > s_{t,i} > 0$. Following the same logic as in the previous case, we must have $\dfrac{s_{t,i} + s_{t,j-i} - s_{t,j}}{n-1} = 0$ and thus $s_{t,j-i} \neq 0$. But then the right-hand side of (5) is equal to 0 while the right-hand side is equal to 1, a contradiction.

Reverse implication: We verify that each of the three conditions listed in the theorem statement imply that (5) is valid.

First, by equation (6) we have $\dfrac{s_{t,i} + s_{t,j-i} - s_{t,j}}{n-1} \in \mathbb{Z}$. Combining $n-1 > s_{t,i} > s_{t,j} > 0$ and the general bounds $0 \le s_{t,\ell} < n-1$ for all ℓ, it follows that $\dfrac{s_{t,i} + s_{t,j-i} - s_{t,j}}{n-1} = 1$. Thus, $s_{t,i} + s_{t,j-i} - s_{t,j} = n-1$. Since $n-1 > s_{t,i} - s_{t,j} > 0$, we have $s_{t,j-i} = n-1-(s_{t,i} - s_{t,j}) > 0$, and thus

$$\left\lceil \frac{s_{t,i}}{n-1} \right\rceil + \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil - \left\lceil \frac{s_{t,j-i}}{n-1} \right\rceil = 1 .$$

We conclude that equation (5) holds.

Second, if $s_{t,i} = 0$ and $s_{t,j} = s_{t,j-i}$, then it is immediate that (5) holds.

Finally, if $s_{t,j} = s_{t,i} > 0$ and $s_{j-i} = 0$, then again it is immediate that (5) holds. $\square$

The following corollary illustrates a special case of Theorem 15 that we will focus on in the remainder of this paper.

Corollary 16. *Let λ be a partition of n where each coordinate is coprime to $n-1$, i.e.,* $\gcd(n-1, \lambda_t) = 1$. *Then $i \prec j$ in $P(\lambda)$ if and only if $s_{t,i} > s_{t,j} > 0$ for every t.*

Proof. If $\gcd(n-1, \lambda_t) = 1$, then $s_{t,i} \neq 0$ for all i. Thus, the second and third conditions in Theorem 15 do not apply. $\square$

Remark 17. Ehrhart-theoretic properties of simplices Δ_λ that satisfy the relatively prime condition in Corollary 16 have been previously studied by Hibi, Higashitani, and Li [11, Section 3].

We can use Corollary 16 to prove the following structural result regarding $P(\lambda)$ in the case where each part of λ is coprime to $n-1$.

Theorem 18. *Let λ be a partition of n such that each λ_t is coprime to $n-1$. Then $P(\lambda)$ is self-dual.*

Proof. We claim that $\phi : x \to n-1-x$ for $x \in [n-2]$ is an order-reversing poset isomorphism. It is clear that ϕ is a bijection. To see that ϕ is order-reversing, observe that by Corollary 16, we have that $i \prec j$ if and only if

$$s_{t,i} > s_{t,j} \text{ for all } t\,. \tag{8}$$

Due to the fact that $\gcd(n-1, \lambda_t) = 1$, we have that $s_{t,i} + s_{t,n-i-i} = n-1$ for all i and t, and thus (8) holds if and only if

$$s_{t,n-1-j} > s_{t,n-1-i} \text{ for all } t\,. \tag{9}$$

This final condition holds if and only if $n-1-j \prec n-1-i$, as desired. $\square$

2.4 Partitions with one distinct part

As an example of how these results can be applied, we consider the case where $\lambda = (x, x, \ldots, x)$ has v occurrences of x. In this case, it is immediate that x is coprime to $n-1 = vx-1$. The following theorem shows that $P(\lambda)$ has a direct interpretation as a subposet of $\mathbb{Z}^2$.

Theorem 19. *For $\lambda = (x, x, \ldots, x)$ with v occurrences of x, we have that $P(\lambda)$ is isomorphic to the poset with elements*

$$\{(r,p) : 0 \leq r < x, 0 \leq p < v\} \setminus \{(0,0), (x-1, v-1)\}$$

and order relation $(r,p) \prec (r',p')$ if both $p > p'$ and $r' > r$.

Proof. For $1 \leq i \leq vx-2$, write

$$i = r_i v + p_i$$

where $0 \le r_i < x$ and $0 \le p_i < v$, but we do not have simultaneously $r_i = x - 1$ and $p_i = v - 1$. Then

$$\begin{aligned}
s_i &= ix - \left\lfloor \frac{ix}{xv-1} \right\rfloor (xv-1) \\
&= x(r_iv + p_i) - \left\lfloor \frac{(r_iv+p_i)x}{xv-1} \right\rfloor (xv-1) \\
&= xr_iv + xp_i - \left\lfloor \frac{xr_iv - r_i + r_i + p_ix}{xv-1} \right\rfloor (xv-1) \\
&= xr_iv + xp_i - \left(r_i + \left\lfloor \frac{r_i + p_ix}{xv-1} \right\rfloor \right) (xv-1) \\
&= r_i + xp_i - \left\lfloor \frac{r_i + p_ix}{xv-1} \right\rfloor (xv-1) \\
&= r_i + xp_i,
\end{aligned}$$

where the final equality is a result of the bounds on r_i and p_i forcing the floor function to be zero. Thus, if $i = r_iv + p_i$ and $j = r_jv + p_j$, then we have $i \prec j$ in $P(\lambda)$ if and only if $i < j$ and $s_i > s_j$, which happens if and only if the following two conditions simultaneously occur:

- $p_i > p_j$ or $p_i = p_j$ with $r_i > r_j$
- $r_j > r_i$ or $r_j = r_i$ with $p_j > p_i$

The only way for both conditions to simultaneously occur is to have $p_i > p_j$ and $r_j > r_i$, and thus our proof is complete. $\square$

The following corollary follows immediately.

Corollary 20. *The posets for $\lambda = (x, x, \ldots, x)$ where x occurs v times and $\lambda' = (v, v, \ldots, v)$ where v occurs x times are isomorphic.*

Corollary 20 is interesting because the two lattice simplices corresponding to λ and λ' are in different dimensions. As an aside, we remark that the order on the lattice points within a rectangular grid given in Theorem 19 corresponds to the reflexive closure of the direct product of two strict total orders.

Example 21. Figures 2 and 3 show the Hasse diagrams of the posets $P(4, 4, 4, 4, 4, 4)$ and $P(6, 6, 6, 6)$, respectively, embedded in $\mathbb{Z}^2$ as described in Theorem 19. This illustrates the isomorphism obtained by switching the roles of x and v.

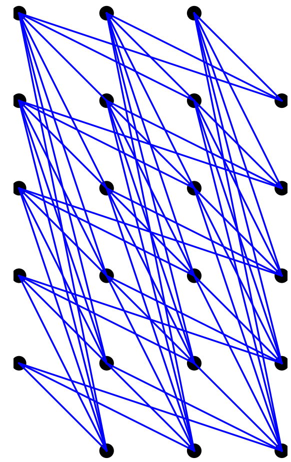

Figure 2: $P(4,4,4,4,4,4)$.

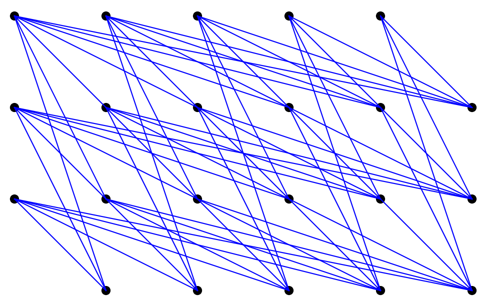

Figure 3: $P(6,6,6,6)$.

3 Experimental results

3.1 Exhaustive search of Δ_λ over all partitions of n

The results in Section 2, particularly Theorem 15 and Corollary 16, provide explicit tools for studying the relations in $P(\lambda)$. Also, Corollary 16 and Theorem 18 demonstrate that the condition that the parts of λ be relatively prime to $\sum_i \lambda_i - 1$ imposes additional structure on $P(\lambda)$, leading us to the following definition.

Definition 22. Given a partition λ of n, if $\gcd(\lambda_i, n-1) = 1$ for all i, we say λ *satisfies the relatively prime condition*. Let $\mathrm{Part}(n)$ denote the set of partitions of n, and let $\mathrm{part}(n) := |\mathrm{Part}(n)|$. Let $\mathrm{Relprime}(n)$ denote the set of partitions of n that satisfy the relatively prime condition, and set $\mathrm{relprime}(n) := |\mathrm{Relprime}(n)|$. Finally, let $\mathrm{Rpac}(n)$ denote the subset of $\mathrm{Relprime}(n)$ for which Δ_λ is an antichain simplex, and set $\mathrm{rpac}(n) := |\mathrm{Rpac}(n)|$.

Using `SageMath` [18] via `CoCalc.com` [12], we computed $\mathrm{part}(n)$, $\mathrm{relprime}(n)$, and $\mathrm{rpac}(n)$ for all $1 \le n \le 73$; the results are given in Table 1. Figure 4 plots $\mathrm{relprime}(n)/\mathrm{part}(n)$ for these values, and Figure 5 plots the ratio rpac(n)/relprime(n).

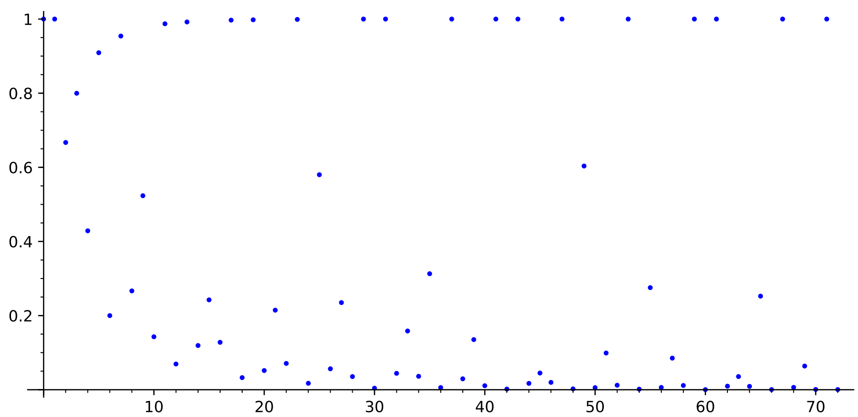

Figure 4: The ratio relprime(n)/part(n) for $1 \leq n \leq 73$.

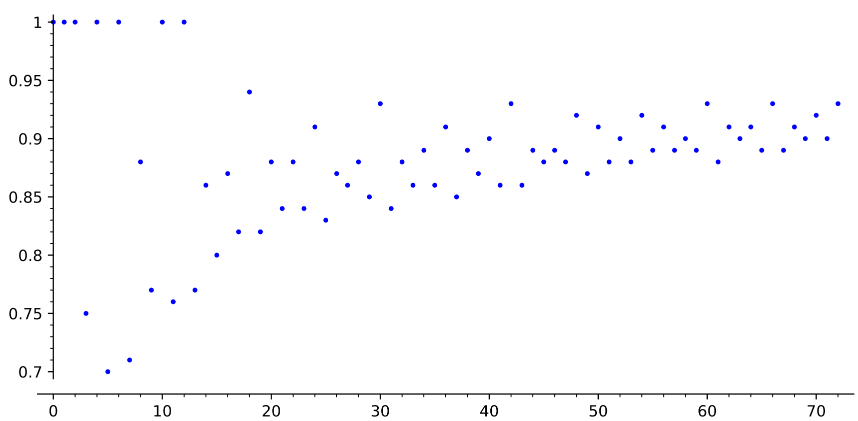

Figure 5: The ratio rpac(n)/relprime(n) for $1 \leq n \leq 73$.

What follows are some observations regarding this experimental data.

1. Figure 5 shows that regardless of the total value of relprime(n), rpac(n)/relprime(n) is generally above 0.8 and as n grows it is clustering between 0.85 and 0.95. Thus, these experiments suggest that when λ satisfies the relatively prime condition, it is likely that Δ_λ is antichain.

2. Figure 4 shows that when $n-1$ is not prime or the square of a prime, the ratio relprime(n)/part(n) appears to be small, and thus our consideration of the relatively prime condition does not broadly apply to partitions in this case. However, it is immediate that when $n-1$ is prime, every partition of n except for $1+(n-1)$ satisfies

the relatively prime condition, and thus $\text{rpac}(n)/\text{relprime}(n) = \text{rpac}(n)/(\text{part}(n) - 1)$. Thus, it appears that one likely source of antichain simplices are those Δ_λ for which $n - 1$ is prime.

3. In Figure 5, the values of $\text{rpac}(n)/\text{relprime}(n)$ for $n \geq 13$ that lie on the upper hull of the data plot arise from n in $\{13, 19, 31, 43, 61, 67, 73\}$. These are all prime numbers, and an OEIS [19] search finds that these values arise in three known sequences, including the sequence A040047 of those primes p such that $x^3 = 6$ has no solution mod p.

4. Again in Figure 5, the values of $\text{rpac}(n)/\text{relprime}(n)$ for $70 \geq n \geq 20$ that lie on the lower hull of the data plot arise from n in $\{20, 26, 32, 38, 44, 50, 62, 68\}$. These values are all of the form $6k + 2$, though whether due to coincidence or mathematics it is not clear.

5. When $n - 1$ is a superabundant A004394 or highly composite A002182 number, one might expect to see particularly low numbers of relatively prime antichain simplices, which is supported by the data given in Table 1.

At this time, the authors do not have an explanation for why the ratio $\text{rpac}(n)/\text{relprime}(n)$ appears to be clustering as n grows, leading to the following problems.

Problem 23. Determine if there is a limiting value to which the sequence $\text{rpac}(n)/\text{relprime}(n)$ converges as n increases. Alternatively, determine if there are any connections between a liminf or limsup value for $\text{rpac}(n)/\text{relprime}(n)$ and the values of n corresponding to subsequences achieving those values, as hinted at in the observations above.

3.2 Random sampling of simplices with one non-trivial column in Hermite normal form

It is worthwhile to compare the results for our restricted Δ_λ simplices to arbitrary simplices with Hermite normal form given by

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{d-1} \\ 0 & 0 & 0 & \cdots & 0 & n \end{bmatrix}$$

where for $i = 1, \ldots, d - 1$ we have $0 \leq a_i < n$. We will call a simplex of this form a *one-column (n,d) Hermite normal form simplex.* Let $\text{OCH}(n, d)$ denote the family of one-column (n, d) Hermite normal form simplices, and let

$$\text{OCH}^+(n, d) := \{A \in \text{OCH}(n, d) : 1 \leq a_i < n \text{ for all } i\}.$$

n	rpac(n)	relprime(n)	part(n)
1	1	1	1
2	2	2	2
3	2	2	3
4	3	4	5
5	3	3	7
6	7	10	11
7	3	3	15
8	15	21	22
9	7	8	30
10	17	22	42
11	8	8	56
12	58	76	77
13	7	7	101
14	103	134	135
15	18	21	176
16	45	56	231
17	33	38	297
18	316	384	385
19	15	16	490
20	513	626	627
21	36	41	792
22	180	215	1002
23	78	89	1255
24	1317	1574	1575
25	31	34	1958
26	1169	1414	2436
27	148	170	3010
28	750	874	3718
29	143	162	4565
30	4779	5603	5604
31	26	28	6842
32	7050	8348	8349
33	392	448	10143
34	1675	1951	12310
35	478	539	14883
36	4850	5625	17977
37	115	126	21637
38	22109	26014	26015
39	816	918	31185
40	4410	5047	37338

n	rpac(n)	relprime(n)	part(n)
41	433	481	44583
42	45819	53173	53174
43	104	112	63261
44	64731	75174	75175
45	1362	1522	89134
46	4192	4747	105558
47	2202	2468	124754
48	129242	147272	147273
49	365	399	173525
50	106948	123165	204226
51	1233	1362	239943
52	24641	27874	281589
53	3597	3986	329931
54	339300	386154	386155
55	623	679	451276
56	128590	145176	526823
57	3426	3781	614154
58	54230	60927	715220
59	8575	9496	831820
60	864231	966466	966467
61	302	324	1121505
62	1146930	1300155	1300156
63	13151	14458	1505499
64	55541	61850	1741630
65	16496	18200	2012558
66	522255	586074	2323520
67	1012	1091	2679689
68	2761384	3087734	3087735
69	20580	22503	3554345
70	234794	261034	4087968
71	3040	3287	4697205
72	4875893	5392782	5392783
73	2715	2931	6185689

Table 1: Experimental data

Thus, $\mathrm{OCH}^+(n,d)$ contains those simplices in $\mathrm{OCH}(n,d)$ that are not obviously arising as lattice pyramids over simplices of smaller dimension.

There are $(n-1)^{d-1}$ simplices in $\mathrm{OCH}^+(n,d)$. For 67 random choices (uniform without replacement) of $(n,d) \in \{3,\ldots,20\} \times \{3,\ldots,20\}$, we selected n^3 random samples from $\mathrm{OCH}^+(n,d)$ and computed the resulting fraction $f(n,d)$ of antichain simplices in this sample. We plotted the points $(n/d, f(n,d))$ in Figure 6. It is particularly interesting that when d is large relative to n, the percentage of antichain simplices among those sampled appears to be close to 1, leading to the following problem.

Problem 24. Fix $n \geq 2$. Is it true that the fraction of antichain simplices in $\mathrm{OCH}^+(n,d)$ goes to 1 as $d \to \infty$? Alternatively, let $ac_n(d)$ denote the fraction of $\displaystyle\bigcup_{j=3}^{d} \mathrm{OCH}^+(n,j)$ that are antichain simplices; what is the liminf of $ac_n(d)$ as $d \to \infty$?

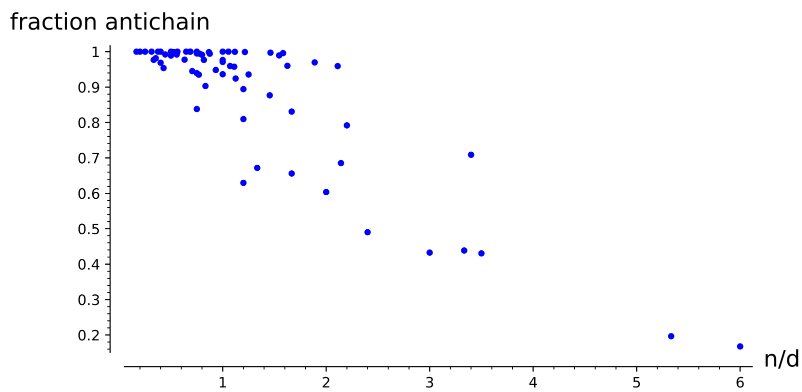

Figure 6: For 67 random choices (uniform without replacement) of $(n,d) \in \{3,\ldots,20\} \times \{3,\ldots,20\}$, we plot the point $(n/d, f(n,d))$ where $f(n,d)$ is the fraction of antichain simplices among n^3 random samples from $\mathrm{OCH}^+(n,d)$.

4 Algebraic implications

In this section, we discuss the algebraic implications of our analysis of the fundamental parallelepiped poset. Our main result is Theorem 32 showing that the Poincaré series for the semigroup algebra associated with an antichain simplex is rational. Unlike previous work of the authors [4] establishing rationality of Poincaré series for lattice simplex semigroup algebras, our proof technique in this work involves the bar resolution.

4.1 A review of resolutions and Poincaré series

For all background regarding graded resolutions of algebras, see the book by Peeva [15]. Recall that the semigroup $(\Lambda, +)$ associated with a d-simplex Δ is the intersection $\Lambda := \mathrm{cone}(\Delta) \cap \mathbb{Z}^{d+1}$ with $+$ given by the usual coordinate-wise addition on $\mathbb{Z}^{d+1}$. The *semigroup algebra* $K[\Lambda]$ associated with a semigroup $\Lambda \subset \mathbb{Z}^{d+1}$ is the K-vector space with basis $\{e_\alpha\}_{\alpha\in\Lambda}$ equipped with the product $e_\alpha \cdot e_\beta = e_{\alpha+\beta}$. For K a field, a K-algebra R is called *graded* with respect to $\mathbb{Z}^n$ if it can be written as a direct sum

$$R = \bigoplus_{\alpha\in\mathbb{Z}^n} R_\alpha,$$

where for $x \in R_\alpha$ and $y \in R_\beta$, we have that $x \cdot y \in R_{\alpha+\beta}$. It is immediate that $K[\Lambda]$ is a $\mathbb{Z}^{d+1}$-graded K-algebra. It is common to "coarsen" the grading of $K[\Lambda]$ by considering it to be a $\mathbb{Z}$-graded algebra with grading given by the zeroth coordinate of its $\mathbb{Z}^{d+1}$-grading.

In this context, the seemingly arbitrary definition of the cone over a simplex Δ is shown to be natural and helpful by the following observation. For a point $x = (x_0, x_1, \dots, x_d)$ in $\mathbb{R}^{d+1}$, we define the *height* of x to be $\mathrm{height}(x) = x_0$. Letting X_n denote the collection of points $x \in \mathbb{R}^{d+1}$ with height equal to n, we have the set equality

$$X_n \cap \mathrm{cone}(\Delta) \ = \ \{(n, n\cdot x) \in \mathbb{R}^{d+1} \text{ such that } x \in \Delta\}.$$

Observe that the set $\mathbb{Z}^{d+1} \cap X_n \cap \mathrm{cone}(\Delta)$ is in bijection with the set of lattice points of $n\Delta$ (by dropping the zeroth coordinate). Thus, the coarsened grading of $K[\Lambda]$ corresponds to the height function in the cone.

We need to consider complexes of vector spaces in order to define free resolutions of K-algebras. Given a collection of vector spaces $\{F_i\}_{i\in\mathbb{Z}_{\geq 0}}$, together with linear maps ∂_i from F_i to F_{i-1}, we call the sequence

$$F: \quad F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_i} F_i \xleftarrow{\partial_{i+i}} F_{i+1} \xleftarrow{\partial_{i+2}} \cdots$$

a *complex* of vector spaces if the image of ∂_{i+1} is contained in the kernel of ∂_i for all $i \geq 1$. The ith homology of the complex F is the quotient vector space $H_i(F) := \ker \partial_i / \mathrm{im}\partial_{i+1}$. Let M be a finitely generated graded module over R, F_i be a free R-module and ∂_i be a graded R-module homomorphism such that the image of ∂_{i+1} is equal to the kernel of ∂_i for all $i \geq 1$. Then the complex F is a *free resolution* of M over R if $M \cong F_0/\mathrm{im}\partial_1$. Because it is graded, we may split the free resolution F into a direct sum of K-vector space complexes by writing each F_i as a direct sum $\bigoplus_{\alpha\in\mathbb{Z}^n} F_{i,\alpha}$.

For (F, ∂) a complex of free R-modules, we can define a tensor complex $(M \otimes F, \mathrm{Id} \otimes \partial)$. If F is a graded free resolution of M, the Betti number $\beta^R_{i,\alpha}(M)$ of a graded R-module M is the vector space dimension of the ith homology of the graded component of $K \otimes F$ of degree α. This leads to our primary object of interest.

Definition 25. The Poincaré series $P_R^M(z;\boldsymbol{t})$ is the ordinary generating function for the Betti numbers of the R-module M, i.e.,

$$P_R^M(z;\boldsymbol{t}) \;=\; \sum_{\alpha\in\mathbb{Z}^n}\sum_{i\geq 0}\beta_{i,\alpha}^R(M)z^i\boldsymbol{t}^\alpha.$$

In the case that R is a polynomial ring in n variables, the Hilbert Syzygy Theorem says that the Poincaré series $P_R^M(z;\boldsymbol{t})$ is a polynomial for any finitely generated R-module M. However, when R is not a polynomial ring, the growth of the Betti numbers is not so simple.

4.2 Rational Poincaré series

We call a $\mathbb{Z}^n$-graded algebra R *connected* if $R_0 \cong K$ (as in the case of a semigroup ring $K[\Lambda]$ associated with a lattice simplex Δ). By a slight abuse of notation, we write

$$\mathfrak{m} \;:=\; \bigoplus_{\alpha\in\Lambda\setminus 0} R_\alpha \text{ and } K \;\cong\; R/\mathfrak{m}$$

as R-modules. It has been shown [10] that if the Poincaré series for the ground field K as an R-module is rational for all R, then the Poincaré series is rational for any finitely generated module, leading to the question of Serre-Kaplansky:

Question 26. Is the Poincaré series of the ground field K over R rational for all K-algebras R?

This question was answered in the negative by Anick [1], and much subsequent work has focused on determining the properties of R that lead to rationality or irrationality. Our interest is in the rationality of the Poincaré series for $K[\Lambda]$, which leads us to define a related algebra as follows.

Because $K[\Lambda]$ is finitely generated (by its Hilbert basis $\mathcal{H}$ given in Definition 2) it has a presentation

$$0 \to \ker\varphi \to K[V_0,\ldots,V_d,x_1,\ldots,x_m] \xrightarrow{\varphi} K[\Lambda] \to 0, \tag{10}$$

where the map φ is defined by the image of variables: the image of V_i is the vector space basis element $e_{(1,v_i)}$ associated with the Hilbert basis element $(1,v_i)$ in Λ, and the image of x_i is e_{h_i} where the h_i are the remaining elements of the Hilbert basis. This defines a surjective degree map $\deg(\cdot)$ from the set of monomials of $K[V_1,\ldots,V_{d+1},x_1,\ldots,x_m]$ onto Λ by

$$\deg\left(\prod V_i^{s_i}\cdot\prod x_j^{r_j}\right) = \sum s_i(1,v_i) + \sum r_j h_j\,.$$

Extending $\deg(\cdot)$ K-linearly, we see that $\ker\varphi$ is the toric ideal I generated by all binomials

$$\mathbf{V}^{u_V}\mathbf{x}^{u_x} - \mathbf{V}^{w_V}\mathbf{x}^{w_x}$$

such that $\deg\left(\mathbf{V}^{u_V}\mathbf{x}^{u_x}\right) = \deg(\mathbf{V}^{w_V}\mathbf{x}^{w_x})$.

Definition 27. The *Fundamental Parallelepiped Algebra* $\mathrm{FPA}(\Delta)$ associated with the simplex Δ may be constructed in two ways; firstly as the quotient

$$K[V_0, \ldots, V_d, x_1, \ldots, x_m] \,/\, \ker \varphi + \big(V_0, \ldots, V_d\big),$$

and secondly as the algebra with K-vector space basis

$$\big\{e_\sigma \text{ such that } \sigma \in \mathbb{Z}^{d+1} \cap \Pi_\Delta\big\}$$

and with multiplication given by

$$e_\sigma \cdot e_\mu = \begin{cases} e_{\sigma+\mu}, & \text{if } \sigma + \mu \in \mathbb{Z}^{d+1} \cap \Pi_\Delta; \\ 0, & \text{otherwise.} \end{cases}$$

One inspiration for defining this algebra is the fact that, due to an argument presented earlier, every element of Λ may be written uniquely as a non-negative sum of points $(1, v_i)$ and a single point in Π_Δ. Because the generators $e_{(1,v_i)}$ form a linear system of parameters for $K[\Lambda]$, we have the following result [2, Prop. 3.3.5].

Theorem 28. *For the $\mathbb{Z}$-graded algebra $K[\Lambda]$, we have the following equality:*

$$\begin{aligned} P^K_{K[\Lambda]}(z; \boldsymbol{t}) &= \prod_i \big(1 + z\boldsymbol{t}^{(1,v_i)}\big) \cdot P^K_{\mathrm{FPA}(\Delta)}(z; \boldsymbol{t}) \\ &= P^K_{K[V_0,\ldots,V_d]}(z; \boldsymbol{t}) \cdot P^K_{\mathrm{FPA}(\Delta)}(z; \boldsymbol{t}). \end{aligned}$$

4.3 Bar resolutions and antichain simplices

We will use the Bar resolution of K, with K as a module over a graded K-algebra, which is a standard construction. We will define the Bar resolution in the special case of a fundamental parallelepiped algebra. In the following, we use the bar symbol $|$ to mean a tensor over K, and reserve the tensor symbol $\otimes$ to mean a tensor over the ring under consideration.

Definition 29. Consider the $\mathbb{Z}^n$-graded K-algebra $\mathrm{FPA}(\Delta)$. The Bar resolution $\mathbb{B}$ of the module K over $\mathrm{FPA}(\Delta)$ is a sum of graded components, where the component $[\mathbb{B}_i]_\alpha$ has a formal vector space basis given by all $e_{\delta_0} \,|\, e_{\delta_1} \,|\, \cdots \,|\, e_{\delta_i}$ such that

- δ_0 is in Π_Δ,
- δ_j is in $\Pi_\Delta \smallsetminus \{0\}$ for $j \geq 1$, and
- $\sum_{j=0}^{i} \delta_j = \alpha$.

The differential map ∂_i acts by sending $e_{\delta_0} \,|\, \cdots \,|\, e_{\delta_i} \in \mathbb{B}_i$ to the sum

$$\sum_{j=0}^{i-1} (-1)^j \, e_{\delta_0} \,|\, \cdots \,|\, e_{\delta_{j-1}} \,|\, e_{\delta_j} e_{\delta_{j+1}} \,|\, e_{\delta_{j+2}} \,|\, \cdots \,|\, e_{\delta_i}$$

in $\mathbb{B}_{i-1}$.

Example 30. For a unimodular simplex Δ, the FPA(Δ) is one-dimensional as a K-vector space, and has basis e_0. Consequently, $[B_i]_\alpha$ has empty basis (and dimension zero) unless α is equal to zero in Λ and $i = 0$. It follows that the complex B is given by

$$0 \leftarrow K \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

and that

$$\beta_{i,\alpha} = \begin{cases} 1 & \text{if } i = 0 \text{ and } \alpha = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $P^K_{\mathrm{FPA}(\Delta)}(z; \boldsymbol{t}) = 1$. The result is consistent with the fact that $K[\Lambda]$ is a polynomial ring in the case that Δ is unimodular.

Example 31. Consider Δ_λ for $\lambda = (2, 2, 3, 5)$. It can be checked using Theorem 15 that $P(\Delta_\lambda) \setminus \{0\}$ has 10 elements and no relations. Thus, for any pair of non-zero elements $\delta_i, \delta_j \in \Pi_{\Delta_\lambda}$, we have $\delta_i + \delta_j \notin \Pi_{\Delta_\lambda}$, hence $e_{\delta_i} e_{\delta_j} = 0$. Thus, in the Bar resolution for FPA(Δ_λ), the differential maps are uniformly equal to zero.

Recall from the paragraph prior to Definition 25 that in order to compute the Betti number $\beta_{i,\alpha}$, we must compute homology in the tensor complex $B := K \otimes \mathbb{B}$. Because we identify K with the quotient $R_0 \cong R/\mathfrak{m}$ with basis e_0, we see that $[B_i]_\alpha$ is generated as a vector space by the collection $\{e_0 \otimes e_{\delta_0} \mid e_{\delta_1} \mid \cdots \mid e_{\delta_i}\}$. Observe that unless δ_0 is equal to the point $0 \in \Lambda$, the product $e_0 \otimes \delta_0$ is equal to zero, since for σ not equal to zero, $e_0 \cdot e_\sigma$ is equal to zero in the module K, and hence

$$e_0 \otimes e_\sigma \;=\; e_0 \cdot e_\sigma \otimes e_0 \;=\; 0 \otimes e_0 \;=\; 0\,.$$

Consequently, for $i \geq 1$, $[B_i]_\alpha$ has a vector space basis in bijection with the collection of $e_{\delta_1} \mid \cdots \mid e_{\delta_i}$ such that each $\delta_j \in \Pi_\Delta \smallsetminus \{0\}$ and $\sum_{j=1}^{i} \delta_j = \alpha$. We further have that $[B_0]_\alpha$ is the trivial vector space unless α is zero in Λ, and that $[B_0]_0$ is isomorphic to K.

We are now ready to prove our final result.

Theorem 32. *For an antichain simplex Δ, we have*

$$P^K_{\mathrm{FPA}(\Delta)}(z; \boldsymbol{t}) = \left(1 - \sum_{\substack{\sigma \in \mathrm{P}(\Delta) \\ \sigma \neq 0}} z \boldsymbol{t}^\sigma \right)^{-1},$$

and thus the Poincaré series is rational.

Proof. As illustrated by Example 31, for an antichain simplex the differential map is uniformly zero, since $e_{\delta_j} \cdot e_{\delta_{j+1}}$ equals zero for all j. Therefore, $\beta_{i,\alpha}$ is equal to the dimension of $[B_i]_\alpha$. From the description of the basis of $[B_i]_\alpha$ in the Bar resolution, it follows that there is a recurrence

$$\dim_K [B_i]_\alpha \;=\; \sum_{\substack{\sigma \in \mathrm{P}(\Delta) \\ \sigma \neq 0}} \dim_K [B_i]_{\alpha - \sigma}\,,$$

from which our result follows. $\square$

References

[1] David Anick, Construction d'espaces de lacets et d'anneaux locaux à séries de Poincaré-Betti non rationnelles, *C. R. Acad. Sci. Paris Sér. A-B* **290** (1980), A729–A732.

[2] Luchezar L. Avramov, Infinite free resolutions. In *Six Lectures on Commutative Algebra*, Mod. Birkhäuser Class., Birkhäuser Verlag, 2010, pp. 1–118.

[3] Matthias Beck and Sinai Robins, *Computing the Continuous Discretely*, Undergraduate Texts in Mathematics, Springer, 2nd edition, 2015.

[4] Benjamin Braun and Brian Davis, Rationality of Poincaré series for a family of lattice simplices. Preprint, available at https://arxiv.org/abs/1711.04206.

[5] Benjamin Braun, Robert Davis, and Liam Solus, Detecting the integer decomposition property and Ehrhart unimodality in reflexive simplices, *Adv. in Appl. Math.* **100** (2018), 122–142.

[6] Benjamin Braun and Fu Liu, h^*-Polynomials with roots on the unit circle. to appear in *Experimental Mathematics*. Available at https://arxiv.org/abs/1807.00105.

[7] W. Bruns, B. Ichim, T. Römer, R. Sieg, and C. Söger, Normaliz: Algorithms for rational cones and affine monoids, Available at https://www.normaliz.uni-osnabrueck.de.

[8] Winfried Bruns, Bogdan Ichim, and Christof Söger, The power of pyramid decomposition in Normaliz, *J. Symbolic Comput.* **74** (2016), 513–536.

[9] Brian Davis, Predicting the integer decomposition property via machine learning. To appear in *Proceedings of the 2018 Summer Workshop on Lattice Polytopes at Osaka University*. Available at https://arxiv.org/abs/1807.08399.

[10] Tor Holtedahl Gulliksen, Massey operations and the Poincaré series of certain local rings, Preprint series: Pure mathematics, University of Oslo, 1970. Available at https://www.duo.uio.no/bitstream/handle/10852/45414/20150821182559730.pdf.

[11] Takayuki Hibi, Akihiro Higashitani, and Nan Li, Hermite normal forms and δ-vectors, *J. Combin. Theory Ser. A* **119** (2012), 1158–1173.

[12] SageMath Inc., Cocalc collaborative computation online, 2018. https://cocalc.com/.

[13] Ezra Miller and Bernd Sturmfels, *Combinatorial Commutative Algebra*, Vol. 227 of *Graduate Texts in Mathematics*, Springer-Verlag, 2005.

[14] Sam Payne, Ehrhart series and lattice triangulations, *Discrete Comput. Geom.* **40** (2008), 365–376.

[15] Irena Peeva, *Graded Syzygies*, Vol. 14 of *Algebra and Applications*, Springer-Verlag, 2011.

[16] Liam Solus, Local h^*-polynomials of some weighted projective spaces. To appear in *Proceedings of the 2018 Summer Workshop on Lattice Polytopes at Osaka University*. Available at https://arxiv.org/abs/1807.08223.

[17] Liam Solus, Simplices for numeral systems, *Trans. Amer. Math. Soc.* **371** (2019), 2089–2107.

[18] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 8.4)*, 2018. Available at http://www.sagemath.org.

[19] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org, 2019.

[20] G. K. White, Lattice tetrahedra, *Canad. J. Math.* **16** (1964), 389–396.

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