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On Certain Sums Divisible by the Central Binomial Coefficient

Jovan Mikić
J.U. SŠC "Jovan Cvijić"
74480 Modriča
Republic of Srpska
jnmikic@gmail.com

Abstract

We prove that some sums, which arise as generalizations of known binomial coefficient identities, are divisible by the central binomial coefficient. A new method is used. In particular, we show that an alternating sum concerning the product of a power of a binomial coefficient with two Catalan numbers is always divisible by the central binomial coefficient.

1 Introduction

Let n be a non-negative integer and let m be a positive integer. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the nth Catalan number.

In this paper, we consider the following four binomial sums:

$$P(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{2k}{k}\binom{2n-2k}{n-k}, \tag{1}$$

$$Q(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{2k}{k}\binom{2n-2k}{n-k}(n-k), \tag{2}$$

$$R(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m C_k \binom{2n-2k}{n-k}, \tag{3}$$

$$T(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m C_k C_{n-k}. \tag{4}$$

For $m = 1$, all four sums (1), (2), (3), and (4) reduce to interesting combinatorial identities. For example, it is well-known that [9, Example 3.6.2, p. 45]

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\binom{4n-2k}{2n-k}=\binom{2n}{n}^2. \tag{5}$$

By Eq. (5), it follows that $P(2n,1)=\binom{2n}{n}^2$.

Recently, the following two binomial coefficient identities involving the Catalan numbers were discovered [8]:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}C_k\binom{2n-2k}{n-k}=\binom{n}{\lfloor\frac{n}{2}\rfloor}^2, \tag{6}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_k\,C_{2n-k}=C_n\binom{2n}{n}. \tag{7}$$

In this paper we present, among other results, two generalizations of Eqns. (6) and (7). See Remark 23. By Eqns. (6) and (7), it follows that $R(n,1)=\binom{n}{\lfloor\frac{n}{2}\rfloor}^2$ and $T(2n,1)=C_n\binom{2n}{n}$. Our main results are as follows:

Theorem 1. *The sum $P(2n,m)$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and for all positive integers m.*

Theorem 2. *The sum $Q(2n-1,m)$ is divisible by $2(2n-1)\binom{2(n-1)}{n-1}$ for all positive integers n and m.*

Theorem 3. *The sum $R(2n,m)$ is divisible by $(n+1)\binom{2n}{n}$ for all non-negative integers n and for all positive integers m.*

Corollary 4. *The sum $T(2n,m)$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and for all positive integers m.*

Theorem 5. *The sum $R(2n-1,m)$ is divisible by $\binom{2n-1}{n}$ for all positive integers n and m.*

For proving our main results, we use a method we call the "method of D sums".

Definition 6. Let n, j, and t be non-negative integers such that $j\le\lfloor\frac{n}{2}\rfloor$, and let m be a positive integer. Let $S(n,m)=\sum_{k=0}^{n}\binom{n}{k}^m F(n,k)$, where $F(n,k)$ is an integer-valued function that depends only on n and k. Then the D sums for $S(n,m)$ are

$$D_S(n,j,t)=\sum_{l=0}^{n-2j}\binom{n-j}{l}\binom{n-j}{j+l}\binom{n}{j+l}^t F(n,j+l). \tag{8}$$

First, note that all four sums (1), (2), (3), and (4) are instances of $S(n,m)$. For $m \geq 2$, by Eq. (8), it follows that

$$S(n,m) = D_S(n,0,m-2). \tag{9}$$

Furthermore, D sums satisfy the following two recurrence relations [6, Thm. 2, Thm. 3, p. 2]:

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n}{j+u}\binom{n-j}{u} D_S(n,j+u,t), \tag{10}$$

$$D_S(n,j,0) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{j+u}\binom{n-2j-u}{u} \sum_{v=0}^{n-2j-2u} \binom{n-2j-2u}{v} F(n,j+u+v). \tag{11}$$

The idea is to calculate the D sums for (1), (2), (3), and (4) by using Relations 10 and 11. We show that their D sums have interesting divisibility properties. For example, we show that $D_P(2n,j,0)$ is divisible by $\binom{2n}{n}$ for all non-negative integers j and n such that $j \leq n$. Surprisingly, this result is sufficient to prove Theorem 1 for $m \geq 2$. Namely, then by Relation 10 and induction, it can be shown that all $D_P(2n,j,t)$ are divisible by $\binom{2n}{n}$. By Relation 9, it follows that $P(2n,m)$ is divisible by $\binom{2n}{n}$ for all $m \geq 2$. See [6, Section 5].

To calculate the sum $D_P(2n,j,0)$, we derive the following binomial coefficient identity:

$$\sum_{k=t}^{2n-t} (-1)^k \binom{2n-2t}{k-t}\binom{2k}{k}\binom{4n-2k}{2n-k} = (-1)^t \frac{\binom{2n}{n}\binom{2t}{t}\binom{2(n-t)}{n-t}}{\binom{2n-t}{t}}, \tag{12}$$

where n and t are non-negative integers such that $t \leq n$. For $t = 0$, Eq. (12) becomes Eq. (5). Therefore, we can see Eq. (12) as a generalization of Eq. (5).

2 Motivation

The first application of D sums [6, Section 8] was for proving Calkin's result [1, Thm. 1]. In 1998, Calkin proved that the alternating binomial sum $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m.

In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [4, Thm. 1.2, Thm. 1.3, p. 2]. As a special case of [4, Thm. 1.2, p. 2], they gave a direct generalization of Calkin's result [4, Thm. 4.1, p. 8]. Moreover, this generalization implies that the sum D_{S_3} [6, Section 8] is divisible by $\binom{2n}{n}$ and $\binom{2n-j}{n}$ for $t \geq 1$. Take $m = t+2$, $n_1 = n_2 = \cdots = n_{t+1} = n$, and $n_{t+2} = n - j$ in [4, Thm. 4.1, p. 8].

Since the sum in [4, Thm. 4.1, p. 8] is not instance of $S(n,m)$ from our Definition 6, it is clear that D sums cannot prove this direct generalization by Guo, Jouhet, and Zeng. Therefore, method of D sums proves the smallest generalization of Calkin's result.

In this paper, we show how the method of D sums works on harder sums, such as our four sums (1), (2), (3), and (4). Note that our main results are not consequences of [4, Thm. 1.2, Thm. 1.3, Thm. 4.1.].

Let S, F, and D_S be sums according to Definition 6. The main obstacle is to calculate the sum $D_S(n, j, 0)$. In order to make Relation 11 more readable, we introduce the following definition.

Definition 7. Let n and t be non-negative integers such that $t \leq \lfloor \frac{n}{2} \rfloor$. Then $S_t(n)$ denotes

$$\sum_{k=t}^{n-t} \binom{n-2t}{k-t} F(n,k). \tag{13}$$

Obviously, for $t = 0$, it follows that $S_0(n) = S(n, 1)$. Therefore, a sum $S_t(n)$ can be viewed as a generalization of a sum $S(n, 1)$. Furthermore, by substitution $k = u + j + v$, the inner sum of the right-side of (11) becomes

$$\sum_{v=0}^{n-2j-2u} \binom{n-2j-2u}{v} F(n, j+u+v) = S_{j+u}(n). \tag{14}$$

It is readily verified [5, Eq. (1.4), p. 5] that $\binom{n-j}{j+u}\binom{n-2j-u}{u} = \binom{n-j}{u}\binom{n-j-u}{j+u}$. By using this fact and Eq. (14), Relation 11 becomes

$$D_S(n,j,0) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} \binom{n-j-u}{j+u} S_{j+u}(n). \tag{15}$$

From now on, for calculating $D_S(n, j, 0)$ sum, we use Eq. (15) instead of Relation 11.

To find a sum $D_S(n, j, 0)$, we first need to find the appropriate $S_t(n)$. For example, let us consider our first sum $P(2n, m)$. We want to find $D_P(2n, j, 0)$ sum, where j is a non-negative integer such that $j \leq n$. Let t be a non-negative integer such that $t \leq n$. Then what is $P_t(2n)$? By Definitions (6) and (7), $P_t(2n)$ is equal to the left side of Eq. (12). If Eq. (12) holds, then it follows that

$$P_t(2n) = (-1)^t \frac{\binom{2n}{n}\binom{2t}{t}\binom{2(n-t)}{n-t}}{\binom{2n-t}{t}}.$$

Note that formula above is our Lemma 10. Therefore, Lemma 10 is a restatement of Eq. (12). Also we want to find the sums $Q_t(2n-1)$, $R_t(2n)$, $R_t(2n-1)$, and $T_t(2n)$.

This paper consists of two main parts. In the first part, we derive formulas for $P_t(2n)$, $Q_t(2n-1)$, $R_t(2n)$, $R_t(2n-1)$ and $T_t(2n)$ sums. We use recurrences and telescoping. Interestingly, $Q_t(2n)$, $R_t(n)$, and $T_t(n)$ are auxiliary sums for the sum $P_t(2n)$. See [7]. Therefore, if we find the first sum $P_t(2n)$, then the others follow by using recurrence relations. Note that we use telescoping only for the first sum $P_t(2n)$.

In the second part, we apply the method of D sums by using Relations 10, 15, and 9.

The rest of the paper is structured as follows: In Section 3, we begin with some preliminary results for Eqs. (1), (2), (3), and (4). Also, we give some preliminary results for the sums $P_t(n)$, $Q_t(n)$, $R_t(n)$ and $T_t(n)$. In Section 4, we present formulas for the sums $P_t(2n)$, $Q_t(2n-1)$, $R_t(2n)$, $R_t(2n-1)$, and $T_t(2n)$. Also we give recurrence relations between these sums. In Section 5, we prove most of the results from Section 4. In Section 6, we start with the proof of Theorem (1) by using the method of D sums and Lemma 10. Then we prove Theorem 3 and Corollary 4. For brevity and clarity, we omit proofs of Theorems 2 and 5. Namely, the proofs of Theorems 2 and 5 are similar to proofs of Theorems 1 and 3, respectively.

3 Some preliminary results

We start with some preliminary results for Eqs. (1), (2), (3), and (4) sums.

Lemma 8. *Let n be a non-negative integer, and let m be a positive integer. Then we have*

$$P(2n+1,m) = 0, \tag{16}$$

$$Q(2n,m) = nP(2n,m), \tag{17}$$

$$T(2n+1,m) = 0, \tag{18}$$

$$T(2n,m) = \frac{1}{n+1}R(2n,m). \tag{19}$$

Furthermore, we give similar results for the sums $P_t(n)$, $Q_t(n)$, $R_t(n)$ and $T_t(n)$.

Lemma 9. *Let n be a non-negative integer, and let t be a non-negative integer such that $t \le \lfloor \frac{n}{2} \rfloor$. Then we have*

$$P_t(2n+1) = 0, \tag{20}$$

$$Q_t(2n) = nP_t(2n), \tag{21}$$

$$T_t(2n+1) = 0, \tag{22}$$

$$T_t(2n) = \frac{1}{n+1}R_t(2n). \tag{23}$$

We prove only Lemma 8. The proof of Lemma 9 is similar to the proof of Lemma 8. Therefore, the proof of Lemma 9 is omitted.

3.1 Proof of Lemma 8

Proof. Changing k to $n-k$ in Eq. (1), it follows that $P(n,m) = (-1)^n P(n,m)$. If n is odd, then it must be $P(n,m) = -P(n,m)$. This is equivalent to $P(n,m) = 0$ for an odd n. The Eq. (16) follows, as desired.

Similarly, changing k to $n-k$ in Eq. (4), it follows that $T(n,m) = (-1)^n T(n,m)$. If n is odd, then it must be $T(n,m) = -T(n,m)$ which is equivalent to $T(n,m) = 0$. This proves Eq. (18).

By Eq. (2), we know that

$$Q(2n,m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \binom{2k}{k} \binom{4n-2k}{2n-k} (2n-k).$$

Changing k to $2n-k$, the last equation above becomes

$$Q(2n,m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \binom{2k}{k} \binom{4n-2k}{2n-k} k.$$

By adding these two equations, Eq. (17) follows.

Let us now prove Eq. (19). By Eq. (3), we know that

$$R(2n,m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m C_k \binom{4n-2k}{2n-k}.$$

Changing k to $2n-k$, the last equation above becomes

$$R(2n,m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m C_{2n-k} \binom{2k}{k}.$$

By adding the last two equations above and using the fact [8, Eq. (18), p. 8] $C_k \binom{4n-2k}{2n-k} + C_{2n-k}\binom{2k}{k} = 2(n+1) C_k C_{2n-k}$, Eq. (19) follows. $\square$

4 Main lemmas, propositions and corollaries

Let n and t be non-negative integers such that $t \le \lfloor \frac{n}{2} \rfloor$. By Eqns. (1),(2), (3), (4), and Definition 7, we know that

$$P_t(n) = \sum_{k=t}^{n-t} (-1)^k \binom{n-2t}{k-t} \binom{2k}{k} \binom{2n-2k}{n-k}, \tag{24}$$

$$Q_t(n) = \sum_{k=t}^{n-t} (-1)^k \binom{n-2t}{k-t} (n-k) \binom{2k}{k} \binom{2n-2k}{n-k}, \tag{25}$$

$$R_t(n) = \sum_{k=t}^{n-t} (-1)^k \binom{n-2t}{k-t} C_k \binom{2n-2k}{n-k}, \tag{26}$$

$$T_t(n) = \sum_{k=t}^{n-t} (-1)^k \binom{n-2t}{k-t} C_k C_{n-k}. \tag{27}$$

We present the following five main lemmas for calculating the sums $P_t(2n)$, $Q_t(2n-1)$, $R_t(2n)$, $R_t(2n-1)$, and $T_t(2n)$.

Lemma 10. *Let n and t be non-negative integers such that $t \le n$. Then*

$$P_t(2n) = (-1)^t \frac{\binom{2n}{n}\binom{2t}{t}\binom{2n-2t}{n-t}}{\binom{2n-t}{t}}.$$

Lemma 11. *Let n be a positive integer, and let t be a non-negative integer such that $t \le n-1$. Then*

$$Q_t(2n-1) = (-1)^t \frac{2(2n-1)\binom{2(n-1)}{n-1}\binom{2t}{t}\binom{2(n-1-t)}{n-1-t}}{\binom{2n-1-t}{t}}.$$

Lemma 12. *Let n and t be non-negative integers such that $t \le n$. Then*

$$R_t(2n) = (-1)^t \frac{\binom{2n}{n}\binom{2t}{t}\binom{2n-2t}{n-t}}{\binom{2n+1-t}{t}}.$$

Lemma 13. *Let n and t be non-negative integers such that $t \le n$. Then*

$$T_t(2n) = (-1)^t \frac{C_n\binom{2t}{t}\binom{2n-2t}{n-t}}{\binom{2n+1-t}{t}}.$$

Lemma 14. *Let n be a positive integer, and let t be a non-negative integer such that $t \le n-1$. Then*

$$R_t(2n-1) = (-1)^t \frac{\binom{2n-1}{n}\binom{2t}{t}\binom{2n-2t-1}{n-t}}{\binom{2n-t}{t}}.$$

Sums $Q_t(n)$, $R_t(n)$, and $T_t(n)$ are auxiliary sums for the sum $P_t(n)$. We give recurrences between these sums. We start with two propositions.

Proposition 15. *Let n be a positive integer, and let t be a non-negative integer such that $t < \frac{n}{2}$. Then*

$$Q_t(n) = 4(n-2t)P_t(n-1) + 2(n-2t)(-1)^n R_t(n-1) + tP_t(n).$$

Proposition 16. *Let n be a positive integer, and let t be a non-negative integer such that $t < \frac{n}{2}$. Then*

$$(n-t+1)R_t(n) = P_t(n) + 4(n-2t)R_t(n-1) - 2(n-2t)T_t(n-1).$$

Proposition 15 has the following two consequences.

Corollary 17. *Let n be a positive integer, and let t be a non-negative integer such that $t \le n-1$. Then*

$$R_t(2n-1) = \frac{1}{4}P_t(2n).$$

Corollary 18. *Let n be a positive integer, and let t be a non-negative integer such that $t \le n-1$. Then*

$$Q_t(2n-1) = 2(2n-1-2t)(2P_t(2n-2) - R_t(2n-2)).$$

Proposition 16 also has two consequences.

Corollary 19. *Let n and t be non-negative integers such that $t \le n$. Then*

$$R_t(2n) = \frac{2n-2t+1}{2n-t+1} P_t(2n).$$

Corollary 20. *Let n be a positive integer, and let t be a non-negative integer such that $t \le n-1$. Then*

$$R_t(2n-1) = \frac{2(2n-2t-1)(2n-1)}{n(2n-t)} R_t(2n-2).$$

Furthermore, by Corollaries 18 and 19, it follows that

Corollary 21. *Let n be a positive integer, and let t be a non-negative integer such that $t \le n-1$. Then*

$$Q_t(2n-1) = \frac{2(2n-1)(2n-1-2t)}{2n-1-t} P_t(2n-2).$$

Finally, by Corollaries 17, 19, and 20, we give the following recurrence for $P_t(2n)$.

Corollary 22. *Let n be a positive integer, and let t be a non-negative integer such that $t < n$. Then*

$$P_t(2n) = \frac{8(2n-1)(2n-2t-1)^2}{n(2n-t)(2n-t-1)} P_t(2n-2).$$

We prove Lemma 10 by telescoping the recurrence from Corollary 22. Once we obtain Lemma 10, proofs of Lemmas 11, 12, and 14 follow from Corollaries 21, 19, and 17, respectively. Note that Lemma 13 follows from Eq. (23) and Lemma 12. Therefore, proofs of Lemmas 11, 12, 13, and 14 are omitted. Since we do not use the sum $Q_t(2n-1)$ for calculating the sum $P_t(2n)$, proofs of Corollaries 18 and 21 are omitted too. The other corollaries are proved.

5 Proofs of main lemmas, propositions and corollaries

First, we prove Propositions 15 and 16. To do this, we use two known binomial identities. The first is [5, Eq. (1.2), p. 5]

$$(n-k)\binom{n}{k} = n\binom{n-1}{k}, \tag{28}$$

where k is an arbitrary integer. The second identity is

$$\binom{2k}{k} = 4\binom{2(k-1)}{k-1} - 2C_{k-1}, \tag{29}$$

where k is a positive integer. Eq. (29) follows from the recurrence relation for the central binomial coefficients [5, p. 26] and the definition of the Catalan numbers.

5.1 Proof of Proposition 15

Proof. By Eqns. (24) and (25), we have

$$\begin{aligned} Q_t(n) &= \sum_{k=t}^{n-t}(-1)^k\binom{n-2t}{k-t}((n-k-t)+t)\binom{2k}{k}\binom{2n-2k}{n-k} \\ &= \sum_{k=t}^{n-t}(-1)^k\binom{n-2t}{k-t}(n-k-t)\binom{2k}{k}\binom{2n-2k}{n-k} + tP_n(t). \end{aligned} \tag{30}$$

Note that the last term of the sum on the right-side of Eq. (30) equals zero. By using Eq. (28), it follows that $\binom{n-2t}{k-t}(n-k-t) = (n-2t)\binom{n-1-2t}{k-t}$. Combining these two facts, it follows that

$$\begin{aligned} &\sum_{k=t}^{n-t}(-1)^k\binom{n-2t}{k-t}(n-k-t)\binom{2k}{k}\binom{2n-2k}{n-k} \\ &= (n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\binom{2k}{k}\binom{2n-2k}{n-k}. \end{aligned} \tag{31}$$

Since $k \le n-1-t$ in Eq. (31), it follows that $k < n$. By using Eq. (29), we know that

$$\binom{2(n-k)}{n-k} = 4\binom{2(n-1-k)}{n-1-k} - 2C_{n-1-k}. \tag{32}$$

By using Eq. (32), we have

$$\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\binom{2k}{k}\binom{2n-2k}{n-k} \tag{33}$$

$$= 4\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\binom{2k}{k}\binom{2(n-1-k)}{n-1-k} \tag{34}$$

$$-2\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\binom{2k}{k}C_{n-1-k}. \tag{35}$$

By Eq. (24), Eq. (34) equals $4P_t(n-1)$. Changing k to $n-1-k$ and by using Eq. (26), Eq. (35) becomes $-2(-1)^{n-1}R_t(n-1)$. Therefore, combining these facts, Eq. (33) becomes

$$4P_t(n-1) + 2(-1)^n R_t(n-1). \tag{36}$$

By using Eqns. (33) and (36), Eq. (31) becomes

$$\begin{aligned}&\sum_{k=t}^{n-t}(-1)^k \binom{n-2t}{k-t}(n-k-t)\binom{2k}{k}\binom{2n-2k}{n-k} \\ &= 4(n-2t)P_t(n-1) + 2(-1)^n(n-2t)R_t(n-1).\end{aligned} \tag{37}$$

Finally, Eqns. (30) and (37) complete the proof of Proposition 15.

□

5.2 Proof of Proposition 16

Proof. By Eq. (26), we have

$$\begin{aligned}R_t(n) &= \sum_{k=t}^{n-t}(-1)^k \binom{n-2t}{k-t} C_k \binom{2n-2k}{n-k} \\ &= \sum_{k=t}^{n-t}(-1)^k \binom{n-2t}{k-t} \frac{(1+k)-k}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k} \\ &= \sum_{k=t}^{n-t}(-1)^k \binom{n-2t}{k-t}(1-\frac{k}{k+1})\binom{2k}{k}\binom{2n-2k}{n-k}.\end{aligned} \tag{38}$$

By Eq. (24), Eq. (38) becomes

$$\begin{aligned}R_t(n) &= P_t(n) + \sum_{k=t}^{n-t}(-1)^k \binom{n-2t}{k-t}(-k)\, C_k \binom{2n-2k}{n-k} \\ &= P_t(n) + \sum_{k=t}^{n-t}(-1)^k \binom{n-2t}{k-t}((n-t-k)+(t-n))\, C_k \binom{2n-2k}{n-k}.\end{aligned} \tag{39}$$

By Eq. (26), Eq. (39) becomes

$$R_t(n) = P_t(n) + \sum_{k=t}^{n-t}(-1)^k \binom{n-2t}{k-t}(n-t-k)\, C_k \binom{2n-2k}{n-k} + (t-n)R_t(n). \tag{40}$$

The Eq. (40) is equivalent to the following equation

$$(n-t+1)R_t(n) = P_t(n) + \sum_{k=t}^{n-t}(-1)^k \binom{n-2t}{k-t}(n-t-k)\, C_k \binom{2n-2k}{n-k}. \tag{41}$$

Similarly as in Eq. (31), we have

$$\sum_{k=t}^{n-t}(-1)^k\binom{n-2t}{k-t}(n-t-k)\,C_k\binom{2n-2k}{n-k} \tag{42}$$

$$=(n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\,C_k\binom{2n-2k}{n-k}. \tag{43}$$

By using Eq. (32), it follows that

$$\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\,C_k\binom{2n-2k}{n-k} \tag{44}$$

$$=4\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\,C_k\binom{2(n-1-k)}{n-1-k} \tag{45}$$

$$-2\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}C_k\,C_{n-1-k}. \tag{46}$$

By Eq. (26), Eq. (45) is equal to $4R_t(n-1)$. By Eq. (27), Eq. (46) is equal to $-2T_t(n-1)$. Therefore, by Eqns. (44), (45), and (46), it follows that

$$\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\,C_k\binom{2n-2k}{n-k}=4R_t(n-1)-2T_t(n-1). \tag{47}$$

By Eqns. (47), (42), and (43), Eq. (41) becomes

$$\begin{aligned}(n-t+1)R_t(n)&=P_t(n)+(n-2t)(4R_t(n-1)-2T_t(n-1))\\&=P_t(n)+4(n-2t)R_t(n-1)-2(n-2t)T_t(n-1).\end{aligned}$$

This completes the proof of Proposition 16.

□

5.3 Proofs of Corollaries 17, 19, and 20

We begin with the proof of Corollary 17. Corollary 17 is a consequence of Proposition 15 and Lemma 9.

Proof. Let n be a positive integer. By setting $n := 2n$ in Proposition 15, we obtain that

$$Q_t(2n)=4(2n-2t)P_t(2n-1)+2(2n-2t)(-1)^{2n}R_t(2n-1)+tP_t(2n), \tag{48}$$

where $0\le t<\frac{2n}{2}=n$.

By Eqns. (20) and (21) from Lemma 9, Eq. (48) is equivalent to

$$\begin{aligned} nP_t(2n) &= 2(2n-2t)R_t(2n-1) + tP_t(2n), \\ 4(n-t)R_t(2n-1) &= (n-t)P_t(2n). \end{aligned} \tag{49}$$

Since $t < n$, we can divide both sides of Eq. (49) by $4(n-t)$. Therefore, Eq. (49) becomes

$$R_t(2n-1) = \frac{1}{4}P_t(2n).$$

This completes the proof of Corollary 17. □

Corollary 19 is a consequence of Proposition 16, Corollary 17, and Lemma 9. We now prove Corollary 19.

Proof. Let n be a positive integer. By setting $n := 2n$ in Proposition 16, we obtain

$$(2n-t+1)R_t(2n) = P_t(2n) + 4(2n-2t)R_t(2n-1) - 2(2n-2t)T_t(2n-1), \tag{50}$$

where $0 \le t < \frac{2n}{2} = n$.

By Eq. (22) from the Lemma 9, the integer $T_t(2n-1)$ vanishes. Then Eq. (50) is equivalent to

$$(2n-t+1)R_t(2n) = P_t(2n) + 4(2n-2t)R_t(2n-1). \tag{51}$$

By Corollary 17, Eq. (51) becomes as follows:

$$\begin{aligned} (2n-t+1)R_t(2n) &= P_t(2n) + 4(2n-2t)\frac{1}{4}P_t(2n), \\ (2n-t+1)R_t(2n) &= (2n-2t+1)P_t(2n). \end{aligned}$$

Since $t < n$, we can divide both sides of the last equation above by $2n-t+1$. It follows that

$$R_t(2n) = \frac{2n-2t+1}{2n-t+1}P_t(2n). \tag{52}$$

Therefore, we proved Corollary 19 for all positive integers n and for all non-negative integers t such that $t < n$.

By Eq. (24), we know that

$$P_n(2n) = (-1)^n \binom{2n}{n}^2, \tag{53}$$

where n is a non-negative integer. By Eq. (26), it follows that

$$R_n(2n) = (-1)^n C_n \binom{2n}{n}, \tag{54}$$

where n is a non-negative integer. Therefore, by using Eqns. (53) and (54), it follows that Eq. (52) holds for $t = n$. Also, this proves the case $n = 0$. Therefore, Corollary 19 follows, as desired. □

Finally, let us prove Corollary 20. Corollary 19 is a consequence of Proposition 16 and Lemma 9.

Proof. Let n be a positive integer. By setting $n := 2n-1$ in Proposition 16, we obtain that

$$(2n-t)R_t(2n-1) = P_t(2n-1) + 4(2n-1-2t)R_t(2n-2) - 2(2n-1-2t)T_t(2n-2); \quad (55)$$

where $0 \le t < \frac{2n-1}{2}$.

By Eq. (20) from Lemma 9, the integer $P_t(2n-1)$ vanishes in Eq. (55). By Eq. (23) from Lemma 9, the integer $T_t(2n-2)$ is equal to $\frac{1}{n}R_t(2n-2)$. Therefore, Eq. (54) becomes as follows:

$$\begin{aligned}
(2n-t)R_t(2n-1) &= 4(2n-1-2t)R_t(2n-2) - 2(2n-1-2t)\frac{1}{n}R_t(2n-2) \\
(2n-t)R_t(2n-1) &= 2(2n-1-2t)R_t(2n-2)(2-\frac{1}{n}) \\
(2n-t)R_t(2n-1) &= 2(2n-1-2t)R_t(2n-2)\,\frac{2n-1}{n}. \qquad (56)
\end{aligned}$$

Since $t \le n-1$, we can divide the both sides of Eq. (56) by $2n-t$. The Eq. (56) becomes

$$R_t(2n-1) = \frac{2(2n-1)(2n-1-2t)}{n(2n-t)}R_t(2n-2).$$

This completes the proof of Corollary 20. □

5.4 Proof of Corollary 22

Corollary 22 is a consequence of Corollaries 17, 19, and 20.

Proof. Let n be a positive integer.

By setting $n := n-1$ in Corollary 19, we obtain

$$R_t(2n-2) = \frac{2n-2t-1}{2n-t-1}P_t(2n-2), \quad (57)$$

where t is a non-negative integer such that $t \le n-1$. □

By using Corollary 17 and Eq. (57), Corollary 20 becomes as follows:

$$\begin{aligned}
\frac{1}{4}P_t(2n) &= \frac{2(2n-1)(2n-1-2t)}{n(2n-t)} \cdot \frac{2n-2t-1}{2n-t-1}P_t(2n-2), \\
P_t(2n) &= \frac{8(2n-1)}{n} \cdot \frac{(2n-2t-1)^2}{(2n-t)(2n-1-t)}P_t(2n-2). \qquad (58)
\end{aligned}$$

This completes the proof of Corollary 22.

5.5 Proof of Lemma 10

Proof. By Eq. (53), we know that

$$P_t(2t) = (-1)^t \binom{2t}{t}^2, \tag{59}$$

where t is a non-negative integer. Let n be a positive integer, and let t be a non-negative integer such that $t < n$. We telescope Eq. (58) from Corollary 22.

We have

$$\begin{aligned} P_t(2n) &= \frac{8(2n-1)}{n} \cdot \frac{(2n-2t-1)^2}{(2n-t)(2n-1-t)} P_t(2n-2) \\ &= 8^{n-t} \frac{(2n-1)(2n-3)\cdots(2t+1)}{n(n-1)\cdots(t+1)} \cdot \frac{\big((2n-2t-1)!!\big)^2}{(2n-t)\cdots(t+1)} P_t(2t). \end{aligned}$$

By using Eq. (59), the last equation above becomes

$$P_t(2n) = 8^{n-t} \frac{(2n-1)(2n-3)\cdots(2t+1)}{n(n-1)\cdots(t+1)} \cdot \frac{\big((2n-2t-1)!!\big)^2}{(2n-t)\cdots(t+1)} (-1)^t \frac{((2t)!)^2}{(t!)^4}. \tag{60}$$

Then Eq. (60) becomes as follows:

$$\begin{aligned} P_t(2n) &= (-1)^t 8^{n-t} \frac{(2n-1)(2n-3)\cdots(2t+1)}{n!\,(2n-t)!} \cdot \frac{\big((2n-2t-1)!!\big)^2}{(t!)^2} ((2t)!)^2, \\ &= (-1)^t 2^{n-t} \frac{(2n-1)(2n-3)\cdots(2t+1)}{n!\,(2n-t)!} \cdot \frac{\big(2^{n-t}(2n-2t-1)!!\big)^2 ((2t)!)^2}{(t!)^2} \\ &= (-1)^t 2^{n-t} \frac{(2n-1)(2n-3)\cdots(2t+1)}{n!\,(2n-t)!} \cdot \frac{\big(2^{n-t}(n-t)!\,(2n-2t-1)!!\big)^2 ((2t)!)^2}{((n-t)!)^2 \cdot (t!)^2} \\ &= (-1)^t 2^{n-t} \frac{(2n-1)(2n-3)\cdots(2t+1)}{n!\,(2n-t)!} \cdot \frac{\big((2n-2t)!!\,(2n-2t-1)!!\big)^2 ((2t)!)^2}{((n-t)!)^2 \cdot (t!)^2} \\ &= (-1)^t 2^{n-t} \frac{(2n-1)(2n-3)\cdots(2t+1)}{n!\,(2n-t)!} \cdot \frac{((2n-2t)!)^2 ((2t)!)^2}{((n-t)!)^2 \cdot (t!)^2} \\ &= (-1)^t 2^{n-t} \frac{n(n-1)\cdots(t+1)}{n(n-1)\cdots(t+1)} \cdot \frac{(2n-1)(2n-3)\cdots(2t+1)\big((2n-2t)!\big)^2 ((2t)!)^2}{n! \cdot (2n-t)! \cdot (t!)^2 \cdot ((n-t)!)^2} \\ &= (-1)^t \frac{2n(2n-2)\cdots(2t+2)}{n(n-1)\cdots(t+1)} \cdot \frac{(2n-1)(2n-3)\cdots(2t+1)\big((2n-2t)!\big)^2 ((2t)!)^2}{n! \cdot (2n-t)! \cdot (t!)^2 \cdot ((n-t)!)^2} \\ &= (-1)^t \frac{2n(2n-1)\cdots(2t+1) \cdot (2t)!}{n(n-1)\cdots(t+1) \cdot t!} \cdot \frac{\big((2n-2t)!\big)^2 (2t)!}{n! \cdot (2n-t)! \cdot t! \cdot ((n-t)!)^2}. \end{aligned}$$

$$
\begin{aligned}
P_t(2n) &= (-1)^t \frac{(2n)!}{n!} \cdot \frac{\big((2n-2t)!\big)^2 (2t)!}{n! \cdot (2n-t)! \cdot t! \cdot ((n-t)!)^2} \\
&= (-1)^t \binom{2n}{n} \cdot \frac{(2n-2t)!}{((n-t)!)^2} \cdot \frac{(2n-2t)!(2t)!}{(2n-t)! \cdot t!} \\
&= (-1)^t \binom{2n}{n} \cdot \binom{2n-2t}{n-t} \cdot \frac{(2n-2t)! \cdot t!}{(2n-t)!} \cdot \frac{(2t)!}{(t!)^2} \\
&= (-1)^t \binom{2n}{n} \cdot \binom{2n-2t}{n-t} \cdot \frac{1}{\binom{2n-t}{t}} \cdot \binom{2t}{t}.
\end{aligned}
$$

Therefore, it follows that

$$
P_t(2n) = (-1)^t \frac{\binom{2n}{n}\binom{2t}{t}\binom{2n-2t}{n-t}}{\binom{2n-t}{t}}, \tag{61}
$$

where $n > t$. Eqns. (59) and (61) complete the proof of Lemma 10. $\square$

Now Eq. (12) directly follows from Lemma 10.

Remark 23. Note that Lemma 13 generalizes Eq. (7). By setting $t = 0$ in Lemma 13, we obtain Eq. (7). Similarly, Lemma 12 generalizes Eq. (6) for even n, and Lemma 14 generalizes Eq. (6) for odd n.

6 Proofs of main results

We begin with the proof of Theorem 1.

6.1 Proof of Theorem 1

Proof. By Eq. (5), we know that Theorem 1 is true for $m = 1$. Therefore, let us suppose $m \geq 2$.

Let n be a fixed non-negative integer. Let j be a non-negative integer such that $j \leq n$. We prove that $D_P(2n, j, 0)$ is divisible by $\binom{2n}{n}$ for all j such that $j \leq n$.

By Relation 15, it follows that

$$
\begin{aligned}
D_P(2n, j, 0) &= \sum_{u=0}^{\lfloor \frac{2n-2j}{2} \rfloor} \binom{2n-j}{u} \binom{2n-j-u}{j+u} P_{j+u}(2n) \\
&= \sum_{u=0}^{n-j} \binom{2n-j}{u} \binom{2n-j-u}{j+u} P_{j+u}(2n).
\end{aligned} \tag{62}
$$

Obviously, $0 \leq j+u \leq n$ in Eq. (62). By setting $t := j+u$ in Lemma 10, it follows that

$$P_{j+u}(2n) = (-1)^{j+u} \frac{\binom{2n}{n}\binom{2(j+u)}{j+u}\binom{2n-2j-2u}{n-j-u}}{\binom{2n-j-u}{j+u}}. \tag{63}$$

By using Eq. (63), Eq. (62) becomes as follows

$$\begin{aligned} D_P(2n, j, 0) &= \sum_{u=0}^{n-j} \binom{2n-j}{u}\binom{2n-j-u}{j+u}(-1)^{j+u} \frac{\binom{2n}{n}\binom{2(j+u)}{j+u}\binom{2n-2j-2u}{n-j-u}}{\binom{2n-j-u}{j+u}} \\ &= (-1)^j \binom{2n}{n} \sum_{u=0}^{n-j} (-1)^u \binom{2n-j}{u}\binom{2(j+u)}{j+u}\binom{2n-2j-2u}{n-j-u}. \end{aligned} \tag{64}$$

By Eq. (64), it follows that $D_P(2n, j, 0)$ is divisible by $\binom{2n}{n}$ for all j such that $j \leq n$.

We assert that $D_P(2n, j, t)$ is divisible by $\binom{2n}{n}$ for all non-negative integers j and t such that $j \leq n$.

We assume that n is a fixed non-negative integer. We use induction on t. For $t = 0$, we proved that $D_P(2n, j, t)$ is divisible by $\binom{2n}{n}$ for all non-negative j such that $j \leq n$. Let s be a non-negative integer. Let us assume that $D_P(2n, j, t)$ is divisible by $\binom{2n}{n}$ for $t = s$ and for all non-negative integers j such that $j \leq n$. What happens with $D_P(2n, j, s+1)$? By Relation 10, it follows that

$$D_P(2n, j, s+1) = \sum_{u=0}^{n-j} \binom{2n}{j+u}\binom{2n-j}{u} D_P(2n, j+u, s). \tag{65}$$

Obviously, $0 \leq j+u \leq n$ in Eq. (65). By the induction hypothesis, $D_P(2n, j+u, s)$ is divisible by $\binom{2n}{n}$. By Eq. (65), it follows that $D_P(2n, j, s+1)$ is divisible by $\binom{2n}{n}$.

This proves the induction step. Therefore, $D_P(2n, j, t)$ is divisible by $\binom{2n}{n}$ for all non-negative integers j and t such that $j \leq n$.

By Relation 9, it follows that

$$P(2n, m) = D_P(2n, 0, m-2), \tag{66}$$

where m is a positive integer such that $m \geq 2$.

Since $D_P(2n, 0, m-2)$ is divisible by $\binom{2n}{n}$, by Eq. (66), it follows that $P(2n, m)$ is divisible by $\binom{2n}{n}$ for all $m \geq 2$. This completes the proof of Theorem 1. □

Remark 24. By setting $j = 0$ in Eq. (64) and by using Eq. (9), it follows that

$$P(2n, 2) = \binom{2n}{n} \sum_{u=0}^{n} (-1)^u \binom{2n}{u}\binom{2u}{u}\binom{2n-2u}{n-u}. \tag{67}$$

Remark 25. Let us suppose that n is a positive integer. Then, at least one of binomial coefficients $\binom{2(j+u)}{j+u}$ and $\binom{2n-2j-2u}{n-j-u}$ must be even in Eq. (64). By Eq. (64), we obtain $D_P(2n, j, 0)$ is divisible by $2\binom{2n}{n}$ for all positive integers n and all non-negative integers j such that $j \leq n$. By the method of D sums and Eq. (5), we can conclude that $P(2n, m)$ is divisible by $2\binom{2n}{n}$ for all positive integers m and n.

Remark 26. The proof of Theorem 2 is similar to the proof of Theorem 1. We use Relation 15 and Lemma 11. If n is a positive integer greater than 1, it follows that $Q(2n-1, m)$ is divisible by $4(2n-1)\binom{2(n-1)}{n-1}$ for all positive integers m.

6.2 Proof of Theorem 3

By Eqns. (3) and (6), we know that $R(2n, 1) = \binom{2n}{n}^2$. By definition of the Catalan numbers, it follows that $n+1$ divides $\binom{2n}{n}$. Therefore, Theorem 3 is true for $m = 1$. Let us suppose that m is a positive integer greater than 2. Let n be a fixed non-negative integer. Let j be a non-negative integer such that $j \leq n$. We use the method of D sums, Lemma 12, and one additional result. Namely, let a and b be positive integers. It is known [3, Corollary 1.5] that

$$\frac{a}{2(a+b)}\binom{2a}{a}\binom{2b}{b} \text{ is always an integer.} \tag{68}$$

In other words, the integer $a\binom{2a}{a}\binom{2b}{b}$ is divisible by $2(a+b)$. Gessel gave a combinatorial interpretation of Eq. (68). See [2, Section 7].

Our proof of Theorem 3 consists of two parts.

In the first part, we show Theorem 3 is true for $m = 2$ by using Eqns. (9), (15), (68), and Lemma 12.

In the second part, we show that $D_R(2n, j, 1)$ is divisible by $(n+1)\binom{2n}{n}$ for all non-negative integers j such that $j \leq n$. By Relation 10, Eq. (68), and induction, it follows that $D_P(2n, j, t)$ is divisible by $(n+1)\binom{2n}{n}$ for all positive integers t and for all non-negative integers j such that $j \leq n$. By Relation 9, we obtain Theorem 3 is true for all $m \geq 3$.

The first part:

Proof. By Relation 15, it follows that

$$D_R(2n, j, 0) = \sum_{u=0}^{n-j} \binom{2n-j}{u}\binom{2n-j-u}{j+u} R_{j+u}(2n). \tag{69}$$

Obviously, $0 \leq j+u \leq n$ in Eq. (69). By setting $t := j+u$ in Lemma 12, it follows that

$$R_{j+u}(2n) = (-1)^{j+u} \frac{\binom{2n}{n}\binom{2(j+u)}{j+u}\binom{2n-2j-2u}{n-j-u}}{\binom{2n+1-j-u}{j+u}}. \tag{70}$$

By using Eq. (70), Eq. (69) becomes as follows:

$$\begin{aligned} D_R(2n,j,0) &= \sum_{u=0}^{n-j} \binom{2n-j}{u} \binom{2n-j-u}{j+u} (-1)^{j+u} \frac{\binom{2n}{n}\binom{2(j+u)}{j+u}\binom{2n-2j-2u}{n-j-u}}{\binom{2n+1-j-u}{j+u}} \\ &= (-1)^j \binom{2n}{n} \sum_{u=0}^{n-j} (-1)^u \frac{\binom{2n-j}{u}\binom{2n-j-u}{j+u}}{\binom{2n+1-j-u}{j+u}} \binom{2(j+u)}{j+u} \binom{2n-2j-2u}{n-j-u}. \end{aligned} \tag{71}$$

By Eq. (28), it follows that

$$\frac{\binom{2n-j-u}{j+u}}{\binom{2n+1-j-u}{j+u}} = \frac{2n+1-2j-2u}{2n+1-j-u}. \tag{72}$$

By Eqns. (72) and (28), we have

$$\binom{2n-j}{u} \frac{\binom{2n-j-u}{j+u}}{\binom{2n+1-j-u}{j+u}} = \frac{2n+1-2j-2u}{2n+1-j} \binom{2n+1-j}{u}. \tag{73}$$

By using Eq. (72), Eq. (71) becomes

$$D_R(2n,j,0) =$$

$$\frac{(-1)^j\binom{2n}{n}}{2n+1-j} \sum_{u=0}^{n-j} (-1)^u (2n+1-2j-2u) \binom{2n+1-j}{u} \binom{2(j+u)}{j+u} \binom{2n-2j-2u}{n-j-u}. \tag{74}$$

Let $M(n,j)$ denote the sum

$$\sum_{u=0}^{n-j} (-1)^u (2n+1-2j-2u) \binom{2n+1-j}{u} \binom{2(j+u)}{j+u} \binom{2n-2j-2u}{n-j-u}. \tag{75}$$

Then Eq. (74) can be written as

$$D_R(2n,j,0) = \frac{(-1)^j\binom{2n}{n}}{2n+1-j} M(n,j). \tag{76}$$

Let us prove that $M(n,j)$ is divisible by $n+1$ for all non-negative integers j such that $j \le n$. It is readily verified that

$$(2n+1-2j-2u)\binom{2n-2j-2u}{n-j-u} = \frac{n+1-j-u}{2} \binom{2(n+1-j-u)}{n+1-j-u}. \tag{77}$$

By using Eq. (77), Eq. (75) becomes

$$M(n,j) = \sum_{u=0}^{n-j} (-1)^u \binom{2n+1-j}{u} \frac{n+1-j-u}{2} \binom{2(j+u)}{j+u} \binom{2(n+1-j-u)}{n+1-j-u}. \tag{78}$$

By Eq. (68), it follows that $\frac{n+1-j-u}{2}\binom{2(n+1-j-u)}{n+1-j-u}\binom{2(j+u)}{j+u}$ is divisible by $n+1$. By Eqns. (68) and (78), the sum $M(n,j)$ is divisible by $n+1$. In particular, the sum $M(n,0)$ is divisible by $n+1$.

Let us prove that the sum $M(n,0)$ is divisible by $2n+1$. By Eq. (75), we obtain

$$M(n,0)=\sum_{u=0}^{n}(-1)^u(2n+1-2u)\binom{2n+1}{u}\binom{2u}{u}\binom{2n-2u}{n-u}. \tag{79}$$

We have

$$\begin{aligned}(2n+1-2u)\binom{2n+1}{u} &= (2n+1)\binom{2n+1}{u}-2u\binom{2n+1}{u}\\ &= (2n+1)\binom{2n+1}{u}-2(2n+1)\binom{2n}{u-1}\\ &= (2n+1)(\binom{2n+1}{u}-2\binom{2n}{u-1}).\end{aligned} \tag{80}$$

Note that we used the identity $k\binom{n}{k}=n\binom{n-1}{k-1}$. See [5, Eq. (1.1), p. 5].

By Eq. (80), it follows that $(2n+1-2u)\binom{2n+1}{u}$ is divisible by $2n+1$. By Eq. (79), it follows that $M(n,0)$ is divisible by $2n+1$. Note that numbers $n+1$ and $2n+1$ are relatively prime. Therefore, we conclude that $M(n,0)$ is divisible by $(n+1)(2n+1)$.

By Relation 9, we know that

$$R(2n,2)=D_R(2n,0,0). \tag{81}$$

By setting $j=0$ in Eq. (76), it follows that

$$D_R(2n,0,0)=\frac{\binom{2n}{n}}{2n+1}M(n,0). \tag{82}$$

We know that $\frac{M(n,0)}{2n+1}$ is an integer divisible by $n+1$.

Therefore, by using Eqns. (81) and (82), we conclude $R(2n,2)$ is divisible by $(n+1)\binom{2n}{n}$. This proves the first part of Theorem 3. $\square$

We note that the integers $\binom{2n}{n}$ and $2n+1$ are not relatively prime in general.

The second part:

Proof. By setting $t=0$ and $S=R$, Relation 10 becomes

$$D_R(2n,j,1)=\sum_{u=0}^{n-j}\binom{2n}{j+u}\binom{2n-j}{u}D_R(2n,j+u,0). \tag{83}$$

By setting $j := j + u$ in Eq. (76), Eq. (76) becomes

$$D_R(2n, j+u, 0) = \frac{(-1)^{j+u}\binom{2n}{n}}{2n+1-j-u} M(n, j+u), \tag{84}$$

where $0 \le j + u \le n$.

By using Eq. (84), Eq. (83) becomes

$$D_R(2n, j, 1) = (-1)^j \binom{2n}{n} \sum_{u=0}^{n-j} (-1)^u \frac{1}{2n+1-j-u} \binom{2n}{j+u} \binom{2n-j}{u} M(n, j+u). \tag{85}$$

It is readily verified that

$$\frac{1}{2n+1-j-u} \binom{2n}{j+u} = \frac{1}{2n+1} \binom{2n+1}{j+u}. \tag{86}$$

By using Eq. (86), Eq. (85) becomes

$$D_R(2n, j, 1) = \frac{(-1)^j \binom{2n}{n}}{2n+1} \sum_{u=0}^{n-j} (-1)^u \binom{2n+1}{j+u} \binom{2n-j}{u} M(n, j+u). \tag{87}$$

Let $N(n, j)$ denote the sum

$$\sum_{u=0}^{n-j} (-1)^u \binom{2n+1}{j+u} \binom{2n-j}{u} M(n, j+u). \tag{88}$$

By Eq. (88), Eq. (87) becomes

$$D_R(2n, j, 1) = \frac{(-1)^j \binom{2n}{n}}{2n+1} N(n, j). \tag{89}$$

Recall that $M(n, j)$ is divisible by $n+1$ for all integers j such that $0 \le j \le n$. Therefore, $M(n, j+u)$ is divisible by $n+1$ for all non-negative integers u such that $u \le n - j$. By Eq. (88), it follows that $N(n, j)$ is divisible by $n+1$ for all non-negative integers j such that $j \le n$. Let us prove that $N(n, j)$ is divisible by $2n + 1$.

By Eq. (88), it is sufficient to prove that $\binom{2n+1}{j+u} M(n, j+u)$ is divisible by $2n + 1$. By setting $j := j + u$ in Eq. (75), it follows that

$$M(n, j+u) =$$

$$\sum_{v=0}^{n-j-u} (-1)^v (2n+1-2j-2u-2v) \binom{2n+1-j-u}{v} \binom{2(j+u+v)}{j+u+v} \binom{2n-2j-2u-2v}{n-j-u-v}.$$

Let $I(n,j,u,v)$ denote $\binom{2(j+u+v)}{j+u+v}\binom{2n-2j-2u-2v}{n-j-u-v}$. Then $\binom{2n+1}{j+u}M(n,j+u)$ is equal to

$$\sum_{v=0}^{n-j-u}(-1)^v(2n+1-2j-2u-2v)\binom{2n+1}{j+u}\binom{2n+1-j-u}{v}I(n,j,u,v).$$

Let us prove that $(2n+1-2j-2u-2v)\binom{2n+1}{j+u}\binom{2n+1-j-u}{v}$ is divisible by $2n+1$. It is readily verified that

$$\binom{2n+1}{j+u}\binom{2n+1-j-u}{v} = \binom{2n+1}{j+u+v}\binom{j+u+v}{v}. \tag{90}$$

By Eq. (90), it follows that

$$\begin{aligned}&(2n+1-2j-2u-2v)\binom{2n+1}{j+u}\binom{2n+1-j-u}{v}\\ &= (2n+1-2j-2u-2v)\binom{2n+1}{j+u+v}\binom{j+u+v}{v}.\end{aligned} \tag{91}$$

Recall that, by Eq. (80), it follows that $(2n+1-2u)\binom{2n+1}{u}$ is divisible by $2n+1$. Therefore, $(2n+1-2(j+u+v))\binom{2n+1}{j+u+v}$ is divisible by $2n+1$. This means that the right-side of Eq. (91) is divisible by $2n+1$. Then the left-side of Eq. (91) is divisible by $2n+1$. This implies that $\binom{2n+1}{j+u}M(n,j+u)$ is divisible by $2n+1$. By Eq. (88), it follows that $N(n,j)$ is divisible by $2n+1$. Hence we have proved that $N(n,j)$ is divisible by the integers $n+1$ and $2n+1$. Since they are relatively prime, it follows that $N(n,j)$ is divisible by $(n+1)(2n+1)$.

Finally, by using Eq. (89), we obtain that $D_R(2n,j,1)$ is divisible by $(n+1)\binom{2n}{n}$ for all non-negative integers j such that $j \leq n$. We assert that $D_R(2n,j,t)$ is divisible by $(n+1)\binom{2n}{n}$ for all non-negative integers j and for all positive integers t such that $j \leq n$. We use Relation 10 and induction as in the proof of Theorem 1. See Eq. (65).

By Relation 9, we have

$$R(2n,t+2) = D_R(2n,0,t). \tag{92}$$

Since $t \geq 1$, this implies that $t+2 \geq 3$. By Eq. (92), it follows that $R(2n,m)$ is divisible by $(n+1)\binom{2n}{n}$ for all integers m such that $m \geq 2$. This proves the second part of Theorem 3. Theorem 3 follows, as desired. □

Remark 27. Corollary 4 follows from Theorem 3 and Eq. (19) from Lemma 8.

Remark 28. The proof of Theorem 5 is similar to the proof of Theorem 3. We use Lemma 14 instead of Lemma 12. We do not use Eq. (68).

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