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Chains with Small Intervals in the Lattice of Binary Paths

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Abstract

We call an interval $[x, y]$ in a poset *small* if y is the join of some elements covering x. In this paper, we study the chains of paths from a given arbitrary (binary) path P to the maximum path having only small intervals. More precisely, we obtain and use several formulas for the enumeration of chains having only small intervals and minimal length. For this, we introduce and study the notions of filling and degree of a path, giving in addition some related statistics.

1 Introduction

Let $\mathcal{P}_n$ be the set of all (binary) paths P of length $|P| = n$, i.e., lattice paths $P = p_1 p_2 \cdots p_n$, starting from the origin of a pair of axes, where each *step* p_i, $i \in [n]$, is either an *upstep* $u = (1, 1)$ or a *downstep* $d = (1, -1)$, connecting two consecutive points of the path. We denote by $|P|_u$ (resp., $|P|_d$) the number of upsteps (resp., downsteps) of P. An *ascent* (resp., *descent*) of P is a maximal sequence of u's (resp., d's) in P. A *peak* (resp., *valley*) of the path is the last point of an ascent (resp., descent). Clearly, every peak (resp., valley) is either the middle point of an occurrence of ud (resp., du), or the endpoint of an occurrence of u (resp., d) at the end of the path. The *height* of a point of the path P is its y-coordinate. We denote by $\mathrm{lv}(P)$ (resp., $\mathrm{hv}(P)$) the height of the lowest (resp., highest) valley of P. A *low* valley of

P is a valley of P with height $\mathrm{lv}(P)$. We set $\mathcal{P} = \bigcup_{n\geq 0} \mathcal{P}_n$, where $\mathcal{P}_0$ consists of only the empty path ε (the path which has no steps).

A *Dyck path* is a path that starts and ends at the same height and lies weakly above this height. In this paper, we will denote Dyck paths using lower case letters. The set of Dyck paths of length $2n$ is denoted by $\mathcal{D}_n$, and we set $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$, where $D_0 = \{\varepsilon\}$. It is well known that $|D_n| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number, (sequence A000108 in the OEIS [10]). Every Dyck path of the form $u^n d^n$, where $n \geq 0$, is called *pyramid*. A *Dyck prefix* (resp., *Dyck suffix*) is a path which is a prefix (resp., suffix) of a Dyck path. Every non-initial point of a Dyck prefix having height zero is called *return*. A *prime* Dyck path is a Dyck path with only one return point. It is well known that every non-empty Dyck path a is the product of prime Dyck paths, i.e., $a = ua_1dua_2d\cdots ua_kd$, where $a_i \in \mathcal{D}$, $i \in [k]$. Every Dyck prefix (resp., Dyck suffix) P can be uniquely decomposed in the form $P = a_0ua_1\cdots ua_k$ (resp., $P = a_0da_1\cdots da_k$), where $a_i \in \mathcal{D}$, $i \in [0, k]$, $k \geq 0$.

A natural (partial) ordering on $\mathcal{P}_n$ is defined via the geometric representation of the paths $P, Q \in \mathcal{P}_n$, where $P \leq Q$ whenever P lies (weakly) below Q. Obviously, Q covers P whenever Q is obtained from P by turning exactly one valley of P into a peak. This ordering is better understood by considering the following alternative encoding of binary paths: Every $P \in \mathcal{P}_n$ can be described uniquely by the sequence $(h_i(P))_{i\in[n]}$ of the heights of its points, so that $P \leq Q$ iff $h_i(P) \leq h_i(Q)$, $i \in [n]$. Then, the join and meet of P, Q are given by:

$$h_i(P \vee Q) = \max\{h_i(P), h_i(Q)\} \text{ and } h_i(P \wedge Q) = \min\{h_i(P), h_i(Q)\}.$$

From these relations, it follows immediately that the poset $(\mathcal{P}_n, \leq)$, or simply $\mathcal{P}_n$, is a finite distributive lattice. Clearly, $\mathcal{P}_n$ is self-dual, with minimum and maximum elements the paths $\mathbf{0}_n = d^n = \underbrace{dd\cdots d}_{n \text{ times}}$ and $\mathbf{1}_n = u^n = \underbrace{uu\cdots u}_{n \text{ times}}$ respectively.

We note that the length of every maximal chain of the interval $[P, Q]$, where $P = p_1p_2\cdots p_n$ and $Q = q_1q_2\cdots q_n$, is equal to

$$l(P, Q) = \frac{1}{2}\sum_{i=1}^{n}(h_i(Q) - h_i(P)) = \sum_{i=1}^{n}(n - i + 1)\cdot([q_i = u] - [p_i = u]),$$

where $[S]$ is the Iverson binary notation, i.e., for every proposition S, $[S] = 1$ if S is true, and 0 if S is false. Hence, the lattice $\mathcal{P}_n$ is graded with rank equal to $\binom{n+1}{2}$, and its rank function is

$$\rho(P) = \frac{1}{2}\left(\sum_{i=1}^{n} h_i(P) + \binom{n+1}{2}\right) = \sum_{i=1}^{n}(n - i + 1)[p_i = u].$$

This lattice appears in the literature in various equivalent forms (e.g., binary words [3, p. 92], subsets of $[n]$ [5], permutations of $[n]$ [12, p. 402], partitions of n into distinct parts [11], threshold graphs [7]). The sublattice $\mathcal{D}_n$ of Dyck paths has been studied by several authors (e.g., [2, 9]). Manes et al. [6] have recently presented a bijection between comparable pairs of paths of this lattice and Dyck prefixes of odd length.

Every path P can be decomposed as

$$P = u^{k_1} d u^{k_2-k_1} d u^{k_3-k_2} d \cdots u^{k_m-k_{m-1}} d u^{k_{m+1}-k_m}, \tag{1}$$

where $m = |P|_d$ and $(k_i)_{i\in[m+1]}$ is a non-decreasing sequence of integers. For $i \in [m]$, the term k_i is the number of u's before the i-th downstep of P and $k_{m+1} = |P|_u$. In the sequel, we will write $P = (k_i)_{i\in[m+1]}$ to denote the encoding of P by this sequence. This sequence is an extension of the notion of P-sequences defined by Pallo and Racca [8] for binary trees, and used by Germain and Pallo [2] in an equivalent form for Dyck paths, in order to prove that $\mathcal{D}_n$ is a distributive graded lattice. It is well known (e.g., see [4, Theorem 10.7.1]) that, using this encoding, the cardinality of every interval $[P, Q]$ in $\mathcal{P}_n$ can be evaluated for every pair of paths P, Q when the two paths end at the same point (i.e., $|P|_d = |Q|_d$). Indeed, we have

$$|[P, Q]| = \det_{i,j\in[m]} \left(\binom{\mu_i - k_j + 1}{j - i + 1} \right), \tag{2}$$

where $P = (k_i)$, $Q = (\mu_i)$, $i \in [m + 1]$.

In order to evaluate the cardinality $|[P, Q]|$ for two paths $P, Q \in \mathcal{P}_n$ that do not end at the same point (i.e., $|Q|_d < |P|_d$), we partition the interval $[P, Q]$ into intervals $[P_i, Q_i]$, $i \in [0, |P|_d - |Q|_d]$, of paths ending at the same point, where P_i (resp., Q_i) is the path obtained by turning the last $|P|_d - |Q|_d - i$ d's of P into u's (resp., the last i u's of Q into d's). Thus, we obtain

$$|[P, Q]| = \sum_{i=0}^{|P|_d - |Q|_d} |[P_i, Q_i]|. \tag{3}$$

In this work, we will mainly deal with the intervals $[a, u^{|a|/2} d^{|a|/2}]$ and $[P, u^{|P|}]$ for $a \in \mathcal{D}$ and $P \in \mathcal{P}$, using the notation $I(a) = |[a, u^{|a|/2} d^{|a|/2}]|$ and $J(P) = |[P, u^{|P|}]|$.

In this paper we study chains from a certain path P to the maximum path such that each member of the chain is the join of some covers of the previous element, i.e., chains with small intervals only. More formally, we say that a chain (or more generally a multichain) $C : P_0 \leq P_1 \leq \cdots \leq P_k$ in $\mathcal{P}$ has (only) small intervals if P_i is obtained by turning some valleys of P_{i-1} into peaks, for every $i \in [k]$.

In Section 2, we introduce and study the notions of filling and degree of a path $P \in \mathcal{P}_n$, which will be used for the evaluation of the number $f(P)$ of minimal $P-\mathbf{1}_n$ chains with small intervals. Apart from this, the filling and the degree are of independent interest and they are related to some interesting statistics. Although Sapounakis et al. [9] have already defined them for the sublattice $\mathcal{D}_n$ of Dyck paths, these new notions are not simple extensions of the old ones. Furthermore, we give a connection between minimal chains with small intervals and the powers of the Möbius function.

In Section 3, which is the main part of this paper, we evaluate the number $f(P)$ for an arbitrary path $P \in \mathcal{P}_n$. We do this by producing several formulas concerning special classes of paths, the combination of which completes the general case. Finally, we show that for specific classes of paths the map f is related to the zeta function.

2 Filling and degree of a path

For every path $P \in \mathcal{P}_n \setminus \{\mathbf{1}_n\}$, we call the join of all elements covering P *filling* of P, and we denote it by $\widetilde{P}$. We also define $\widetilde{\mathbf{1}_n} = \mathbf{1}_n$, for $n \geq 0$. Obviously, the filling of P is obtained by turning every valley of P into a peak. For example, if $P = dduudududdd$, then $\widetilde{P} = duduudududddu$.

We note that for every $P, Q \in \mathcal{P}_n$ with $P \leq Q < \mathbf{1}_n$ we have that $P < \widetilde{P} \leq \widetilde{Q}$.

It is easy to check that the interval $[P, \widetilde{P}]$ is isomorphic to the Boolean lattice B_k, where k is the number of valleys of P, for each $P \in \mathcal{P}_n$.

In the following result we characterize the set of fillings of $\mathcal{P}_n$. For the proof, we can easily show by induction, using the first valley decomposition, that a path P is a filling iff every valley of P is adjacent to a peak.

Proposition 1. *A path P is a filling of some path in $\mathcal{P}$ iff $P \neq d$ and it satisfies all of the following conditions:*

(i) P avoids d^2u^2;

(ii) P does not start with du^2;

(iii) P does not end with d^2.

In the next result, we use the above characterization in order to enumerate the fillings of $\mathcal{P}_n$.

Proposition 2. *The number a_n of fillings of $\mathcal{P}_n$ is the $(n+1)$-th tribonacci number (sequence A000213 in the OEIS [10]), given by*

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}, \ n \geq 3, \tag{4}$$

and $a_0 = a_1 = 1$, $a_2 = 3$.

Proof. Let $\widetilde{\mathcal{P}}_n$ be the set of fillings in $\mathcal{P}_n$. For $n = 0, 1, 2$ we have that $\widetilde{\mathcal{P}}_0 = \{\varepsilon\}$, $\widetilde{\mathcal{P}}_1 = \{u\}$, $\widetilde{\mathcal{P}}_2 = \{uu, ud, du\}$ and hence, $|\widetilde{\mathcal{P}}_0| = |\widetilde{\mathcal{P}}_1| = 1$ and $|\widetilde{\mathcal{P}}_2| = 3$. For $n \geq 3$, $\widetilde{\mathcal{P}}_n$ can be partitioned into the following three sets:

$$\begin{aligned}
\widetilde{\mathcal{P}}_{n,1} &= \{P \in \widetilde{\mathcal{P}}_n : P \text{ starts with } uu \text{ or } dd\},\\
\widetilde{\mathcal{P}}_{n,2} &= \{P \in \widetilde{\mathcal{P}}_n : P \text{ starts with } ud\},\\
\widetilde{\mathcal{P}}_{n,3} &= \{P \in \widetilde{\mathcal{P}}_n : P \text{ starts with } du\}.
\end{aligned}$$

By deleting the first step of each path of $\widetilde{\mathcal{P}}_{n,1}$, we can easily check that $\left|\widetilde{\mathcal{P}}_{n,1}\right| = \left|\widetilde{\mathcal{P}}_{n-1}\right| = a_{n-1}$. Similarly, by deleting the first two (resp., three) steps of every path in $\widetilde{\mathcal{P}}_{n,2}$ (resp.,

$\widetilde{\mathcal{P}}_{n,3}$), we obtain that $\left|\widetilde{\mathcal{P}}_{n,2}\right| = \left|\widetilde{\mathcal{P}}_{n-2}\right| = a_{n-2}$ (resp., $\left|\widetilde{\mathcal{P}}_{n,3}\right| = \left|\widetilde{\mathcal{P}}_{n-3}\right| = a_{n-3}$), which gives relation (4). $\square$

Using the notion of the filling, we restate that a chain $C : P_0 \leq P_1 \leq \cdots \leq P_k$ has small intervals iff $P_i \leq \widetilde{P_{i-1}}$, for every $i \in [k]$.

For every path $P \in \mathcal{P}_n$ we define inductively a finite sequence of paths $P^{(i)}$ in $\mathcal{P}_n$, as follows: $P^{(0)} = P$, and $P^{(i)} = \widetilde{P^{(i-1)}}$ whenever $P^{(i-1)} \neq \mathbf{1}_n$. The number $\delta(P)$ for which $P^{(\delta(P))} = \mathbf{1}_n$ is called the *degree* of P. Clearly, the chain $C : P_0 = P^{(0)} \leq P^{(1)} \leq \cdots \leq P^{(\delta(P))} = \mathbf{1}_n$ is a $P - \mathbf{1}_n$ chain with small intervals and length $\delta(P)$. In the following result, we establish the minimality of $\delta(P)$ with respect to this property.

Proposition 3. *The length of every chain from a path $P \in \mathcal{P}_n$ to $\mathbf{1}_n$, with small intervals is greater than or equal to $\delta(P)$.*

Proof. Let $C : P = P_0 \leq P_1 \leq \cdots \leq P_k = \mathbf{1}_n$ be a chain with $P_i \leq \widetilde{P_{i-1}}$, for every $i \in [k]$. For every $j \in [\delta(P)]$, we denote by i_j the greatest element of $[k]$ such that $P_{i_j-1} \leq P^{(j-1)}$. It follows that $P^{(j-1)} < P_{i_j} \leq \widetilde{P_{i_j-1}} \leq P^{(j)}$, so that every interval $[P^{(j-1)}, P^{(j)}]$ contains an element of the chain C, giving automatically that $\delta(P) \leq k$. $\square$

We now come to the evaluation of $\delta(P)$ for every path $P \in \mathcal{P}_n$. Clearly, we have $\delta(\mathbf{1}_n) = 0$, $\delta(\mathbf{0}_n) = n - 1$ and $\delta(\widetilde{P}) = \delta(P) - 1$ for $P \neq \mathbf{1}_n$. In the general case, we will see that $\delta(P)$ is closely related to $\mathrm{lv}(P)$ (the height of the lowest valley of P). Indeed, since $\widetilde{P}$ is obtained by turning all valleys of P into peaks, it follows that the heights of their low valley points will differ by exactly one, i.e., $\mathrm{lv}(\widetilde{P}) = \mathrm{lv}(P) + 1$, for $P \neq u^{|P|}, u^{|P|-1}d$. It follows that

$$\mathrm{lv}(P^{(i)}) = \mathrm{lv}(P^{(i-1)}) + 1,\ 1 \leq i \leq \delta(P) - 1.$$

Summing for all i, we obtain that

$$\mathrm{lv}(P^{(\delta(P)-1)}) = \mathrm{lv}(P^{(0)}) + \delta(P) - 1,$$

so that

$$|P| - 2 = \mathrm{lv}(P) + \delta(P) - 1,$$

giving the following result.

Lemma 4. *For every path $P \in \mathcal{P}$ with $P \neq u^{|P|}$ the degree $\delta(P)$ is given by the formula*

$$\delta(P) = |P| - 1 - \mathrm{lv}(P).$$

Proposition 5. *The number of paths of $\mathcal{P}_n$ having degree k equals*

$$\binom{\min\{n, k\}}{\lfloor \frac{k+2}{2} \rfloor}$$

for $1 \leq k \leq 2n - 1$, $n \in \mathbb{N}^*$.

Proof. Let $\Delta_{n,k}$ be the set of all $P \in \mathcal{P}_n$ with $\delta(P) = k$, and let $M_{n,k}$, $N_{n,k}$ be its subsets of paths which start with u and d respectively.

We first prove that for $n \geq 2$ we have that

$$|M_{n,k}| = |\Delta_{n-1,k}|\,, \tag{5}$$

and

$$|N_{n,k}| = \begin{cases} 0, & \text{if } k < n; \\ |\mathcal{DP}_{n-1}|\,, & \text{if } k = n; \\ |\Delta_{n-1,k-2}|\,, & \text{if } k > n, \end{cases} \tag{6}$$

where $\mathcal{DP}_n$ is the set of Dyck prefixes of length n, which is well known that it is enumerated by the binomial $\binom{n}{\lfloor n/2 \rfloor}$, (sequence A001405 in the OEIS [10]). Indeed, by deleting the first u from each path of $M_{n,k}$ we obtain a path in $\Delta_{n-1,k}$, giving a bijection between the sets $M_{n,k}$ and $\Delta_{n-1,k}$ which justifies relation (5).

On the other hand, for the proof of relation (6), we consider the following cases:

(i) if $k < n$, then $\mathrm{lv}(P) \geq 0$, so that every $P \in \Delta_{n,k}$ starts with u, giving $N_{n,k} = \emptyset$.

(ii) $k \geq n$; by deleting the first d from each path $P \in N_{n,k}$ we obtain a path Q such that if $k = n$, then $\mathrm{lv}(P) = -1$, or equivalently $Q \in \mathcal{DP}_{n-1}$, whereas if $k > n$, then $\delta(Q) = k - 2$, i.e., $Q \in \Delta_{n-1,k-2}$; this gives a bijection between the sets $N_{n,k}$ and $\mathcal{DP}_{n-1}$ if $k = n$, and between the sets $N_{n,k}$ and $\Delta_{n-1,k-2}$ if $k > n$, justifying relation (6).

Clearly, the result holds for $n = 1$, whereas for $n \geq 2$, using (5) and (6), we proceed by induction on n.

If $k < n$, then $|\Delta_{n,k}| = |\Delta_{n-1,k}| = \binom{k}{\lfloor \frac{k+2}{2} \rfloor}$.

If $k = n$, then $|\Delta_{n,n}| = |\Delta_{n-1,n}| + |\mathcal{DP}_{n-1}| = \binom{\min\{n-1,n\}}{\lfloor \frac{n+2}{2} \rfloor} + \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = \binom{n-1}{\lfloor \frac{n+2}{2} \rfloor} + \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = \binom{n}{\lfloor \frac{n+2}{2} \rfloor}$.

If $k > n$, then $|\Delta_{n,k}| = |\Delta_{n-1,k}| + |\Delta_{n-1,k-2}| = \binom{n-1}{\lfloor \frac{k+2}{2} \rfloor} + \binom{n-1}{\lfloor \frac{k}{2} \rfloor} = \binom{n}{\lfloor \frac{k+2}{2} \rfloor}$. $\square$

The minimal $P - \mathbf{1}_n$ chains with small intervals which, according to Proposition 3, must have length equal to $\delta(P)$, are closely related to the powers of the Möbius function μ of $\mathcal{P}$. Indeed, if f is the map on $\mathcal{P}$ defined by

$$f(P) = \#\ P - \mathbf{1}_n \text{ chains of length } \delta(P) \text{ with small intervals,}$$

then we have the following result.

Proposition 6. *For every path $P \in \mathcal{P}$ we have that*

$$\mu^k(P, \mathbf{1}_{|P|}) = \begin{cases} 0, & \textit{if } k < \delta(P); \\ (-1)^{l(P,\mathbf{1}_{|P|})} f(P), & \textit{if } k = \delta(P). \end{cases}$$

Proof. Since $\mathcal{P}$ is a distributive locally finite lattice, it follows (e.g., see [1, Proposition 3.7, p. 90]) that the Möbius function of $\mathcal{P}$ has the following formula

$$\mu(P,Q) = \begin{cases} (-1)^{l(P,Q)}, & \text{if } P \leq Q \leq \widetilde{P}; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for $k \geq 1$, we can easily check that

$$\mu^k(P,Q) = \sum \prod_{i=1}^{k} \mu(P_{i-1}, P_i),$$

where the sum is taken over all multichains $C : P = P_0 \leq P_1 \leq \cdots \leq P_k = Q$ of length k with small intervals, so that

$$\mu^k(P,Q) = (-1)^{l(P,Q)} \cdot (\#\ P - Q \text{ multichains of length } k \text{ with small intervals}).$$

Then, for $Q = \mathbf{1}_{|P|}$, using Proposition 3, the required formula follows automatically. ☐

3 Counting minimal chains with small intervals

In this section, we evaluate the number $f(P)$ of minimal $P - \mathbf{1}_n$ chains of length $\delta(P)$ with small intervals, for every path $P \in \mathcal{P}$. We will use the notation P' (resp., P^*) for the path obtained by turning every low valley of P (resp., every valley of P, except the last one if P ends with d) into a peak.

Note that $f(\mathbf{1}_n) = 1$, for every $n \geq 0$. Clearly, we have $f(uP) = f(P)$ for every $P \in \mathcal{P}$, so that it is enough to evaluate $f(P)$ when P is a Dyck prefix. In the following result, we give a recursive formula for the map f.

Proposition 7. *For every path $P \in \mathcal{P}_n \setminus \{\mathbf{1}_n\}$ we have that*

$$f(P) = \sum_{Q \in [P', \widetilde{P}]} f(Q).$$

Proof. Let $C : P = P_0 \leq P_1 \leq \cdots \leq P_k = \mathbf{1}_n$, where $k = \delta(P)$, be a k-chain from P to $\mathbf{1}_n$ with small intervals. Then, since $P_{i-1} \leq P_i \leq \widetilde{P_{i-1}}$ for every $i \in [k]$, we can easily prove by induction that $P_i^{(k-i)} = \mathbf{1}_n$, i.e., $\delta(P_i) = k - i$, for every $i \in [k]$. In particular, $\delta(P_1) = k - 1$, which by Lemma 4 gives that $\operatorname{lv}(P_1) = \operatorname{lv}(P) + 1$. This shows that $P_1 \in [P', \widetilde{P}]$. Moreover, if we delete P from C we obtain a $(k-1)$-chain from P_1 to $\mathbf{1}_n$ with small intervals.

On the other hand, given $Q \in [P', \widetilde{P}]$, by adding P in the beginning of every $\delta(Q)$-chain from Q to $\mathbf{1}_n$ with small intervals, we obtain a $\delta(P)$-chain from P to $\mathbf{1}_n$ with small intervals.

Thus, the result follows automatically by decomposing the $\delta(P)$-chains from P to $\mathbf{1}_n$ with small intervals according to their second member. ☐

Corollary 8. *For every $P \in \mathcal{P}$ we have that*

$$f(P) = 1 \textit{ iff all valleys of } P \textit{ have the same height.}$$

Every path $P \in \mathcal{P}$ can be decomposed (not necessarily uniquely) as a product of a Dyck suffix P_1 followed by a Dyck prefix P_2. In the following result we give a recursive formula of f that utilizes this decomposition.

Proposition 9. *If $P = P_1 P_2$, where P_1 is a Dyck suffix and P_2 is a Dyck prefix, then $f(P) = f(P_1) f(dP_2)$.*

Proof. Without loss of generality we may assume that P starts with d, $P_1 \neq d$ and $P_2 \neq \varepsilon$.

We use induction with respect to the length and to the (dual) partial order, i.e., assuming that the result holds for every path Q (that starts with d) for which $|Q| < |P|$, or $|Q| = |P|$ and $Q > P$, we will prove that the result holds for P.

We decompose P_1, P_2 as follows:

$$P_1 = R_1 a d, \qquad P_2 = u b R_2,$$

where R_1 (resp., R_2) is a Dyck suffix (resp., Dyck prefix), a (resp., b) may be either empty, or a path that starts with d (resp., ends with u), ends at height 0 and it is bounded by the line $y = -1$ (see Figure 1).

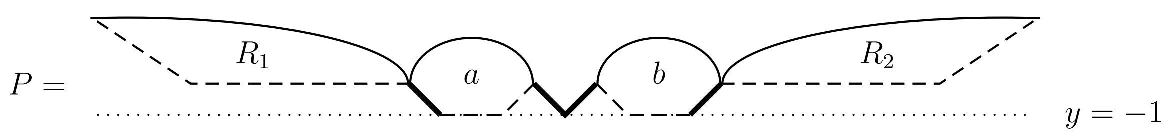


Figure 1: The decomposition of $P = P_1 P_2$

Then, we have that

$$P_1' = R_1 a' u, \qquad (dP_2)' = udb' R_2, \qquad P' = R_1 a' u d b' R_2,$$

$$\widetilde{P_1} = R_1^* a^* u, \qquad \widetilde{dP_2} = ud\,\widetilde{b}\,\widetilde{R_2}, \qquad \widetilde{P} = R_1^* a^* u d\,\widetilde{b}\,\widetilde{R_2}.$$

It follows that every $Q \in [P', \widetilde{P}]$ can be uniquely decomposed as

$$Q = Q_1 u d Q_2,$$

where $Q_1 \in [R_1 a', R_1^* a^*]$, $Q_2 \in [b' R_2, \widetilde{\widetilde{b}R_2}]$. Clearly, since Q_1 is a Dyck suffix and Q_2 is a Dyck prefix with $Q > P$ and $|Q_1|, |Q_2| \leq |P| - 2$, we have that

$$\begin{aligned} f(Q) &= f(Q_1 u d Q_2) = f(Q_1 u d) f(d Q_2) = f(Q_1) f(dud) f(u d Q_2) \\ &= f(Q_1) f(du) f(u d Q_2) = f(Q_1 u) f(u d Q_2), \end{aligned}$$

and therefore

$$\begin{aligned}
f(P) &= \sum_{Q\in[P',\widetilde{P}]} f(Q) = \sum_{\substack{Q_1\in[R_1a',R_1^*a^*]\\ Q_2\in[b'R_2,\widetilde{b\widetilde{R_2}}]}} f(Q_1u)f(udQ_2)\\
&= \sum_{Q_1u\in[R_1a'u,R_1^*a^*u]} f(Q_1u) \sum_{udQ_2\in[udb'R_2,ud\widetilde{b\widetilde{R_2}}]} f(udQ_2)\\
&= \sum_{Q_1u\in[P_1',\widetilde{P_1}]} f(Q_1u) \sum_{udQ_2\in[(dP_2)',\widetilde{dP_2}]} f(udQ_2)\\
&= f(P_1)f(dP_2),
\end{aligned}$$

completing the proof. □

Corollary 10. *If P is a Dyck prefix with at least one return point, then $f(dP) = f(P)$.*

Proof. We use induction with respect to the length of the path. Clearly, since P has at least one return point, it can be written as $P = uadR$, where $a \in \mathcal{D}$ and R is a Dyck prefix. Then, using Propositions 7, 9 and the induction hypothesis, we have that

$$\begin{aligned}
f(duad) &= \sum_{Q\in[udau,uda^*u]} f(Q) = \sum_{b\in[a,a^*]} f(udbu) = \sum_{b\in[a,a^*]} f(udb)f(du)\\
&= \sum_{b\in[a,a^*]} f(ub)f(du) = \sum_{b\in[a,a^*]} f(ubu) = \sum_{Q\in[uau,ua^*u]} f(Q) = f(uad).
\end{aligned}$$

It follows that $f(dP) = f(duad)f(dR) = f(uad)f(dR) = f(P)$. □

From the two previous results it follows that the map f on the set of Dyck paths is multiplicative. Moreover, since every Dyck prefix can be uniquely decomposed in the form aQ, where $a = \varepsilon$ or $a = ua_1d\cdots ua_kd$, $a_i \in \mathcal{D}$, $i \in [k]$ and $Q = \varepsilon$ or $Q = uP$ for some Dyck prefix P, for the evaluation of f it is enough to restrict ourselves to the two cases $f(uad)$ and $f(duP)$, where $a \in \mathcal{D}$ and P is a Dyck prefix. For this, we introduce a new kind of multichains for Dyck paths, based on the heights of the valleys of the paths.

We say that a multichain of Dyck paths $C : \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_h$, where $h = \mathrm{hv}(\sigma_0)$ (the height of the highest valley of σ_0) is *of type* v iff for every $j \in [h]$ the paths σ_j, σ_{j-1} have the same valleys at every height at most $h - j$.

Example 11. The multichain $C : \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \sigma_3$ with

$$\begin{aligned}
\sigma_0 &= u^2du^2du^2d^2u^2d^3ud^2u^3du^2dud^2udud^3,\\
\sigma_1 &= u^2du^2du^2d^2u^2d^3ud^2u^3du^3d^3udud^3,\\
\sigma_2 &= u^2du^4d^2udud^3ud^2u^5du^2d^4ud^3,\\
\sigma_3 &= u^4du^2d^2ududud^4u^6dud^4ud^3
\end{aligned}$$

is of type v, whereas the multichain $\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq s$ with $s = u^5du^3d^3ud^3udu^4dudud^3ud^4$ is not of type v (see Figure 2).

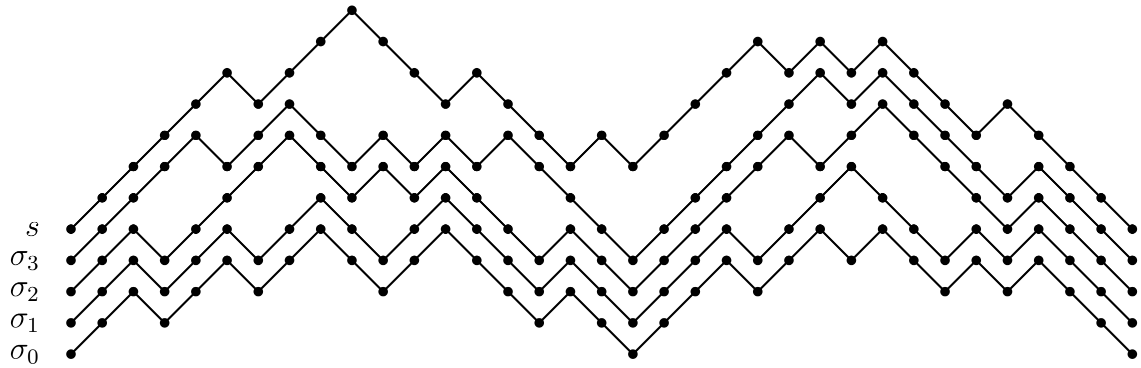


Figure 2: The multichain of example 11

For $a, s \in \mathcal{D}$ with $a \leq s$ we denote by $\mathrm{v}(a, s)$ the number of all $a - s$ multichains of type v. Clearly, we have $\mathrm{v}(a, s) \neq 0$ iff a, s have exactly the same low valleys. Furthermore, we can easily check that

$$\mathrm{v}(ua_1d \cdots ua_kd, us_1d \cdots us_kd) = \prod_{i=1}^{k} \mathrm{v}(ua_id, us_id)$$

when $a_i \leq s_i$ for every $i \in [k]$, and that

$$\mathrm{v}(uad, usd) = \sum_{t \leq s} \mathrm{v}(a, t). \tag{7}$$

In the following Proposition, we give an alternative formula of v on the pairs of prime Dyck paths, which will be used in the sequel.

Proposition 12. *For every $a, s \in \mathcal{D}$ with $a \leq s$ we have that*

$$\mathrm{v}(uad, usd) = \sum_{w \in [a, a^*]} \mathrm{v}(w, s).$$

Proof. In view of formula (7), it is enough to construct a bijection between the set of $a - t$ multichains of type v for all $t \leq s$, and the set of $w - s$ multichains of type v for all $w \in [a, a^*]$.

A key property of Dyck paths that we will use throughout the proof is the following: If two Dyck paths t, r of length $2n$ have the same valleys for every height at most j, then they coincide up to height $j + 1$, and if $t \leq r$ then every valley of r at height $j + 1$ is necessarily also a valley of t. Note also that given a multichain $\sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_h$ of type v and if $0 \leq j \leq i \leq h$, then σ_j, σ_i have the same valleys at every height at most $h - i$.

In order to exhibit the required bijection, for a multichain $a = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_h = t$ of type v with $t \leq s$, we set $\sigma_{h+1} = s$ and we first construct a sequence of Dyck paths τ_i, $i \in [0, h + 1]$, where τ_i is the Dyck path obtained by turning into peaks all valleys of σ_i at height j that are not also valleys of σ_{h+1-j}, for every $j \leq h-i$. Clearly, we have $\tau_0 \in [a, a^*]$ and $\tau_{h+1} = s$.

Furthermore, $\tau_{i-1} \leq \tau_i$ for every $i \in [h+1]$. Indeed, all valleys of σ_{i-1}, σ_i at height $j \leq h-i$ which turn into peaks for the construction of τ_{i-1}, τ_i are the same, whereas the valleys of σ_{i-1} at height $h-i+1$ that turn into peaks are not valleys of $\sigma_{h+1-(h-i+1)} = \sigma_i$, so that σ_i (let alone τ_i) passes at least two units above these valleys and hence, τ_{i-1} is weakly below τ_i.

Set $w = \tau_0$; hence, $w \in [a, a^*]$. Moreover, $\tau_{h+1-k} = w$, where $k = \mathrm{hv}(w)$. Indeed, clearly σ_0, σ_{h+1-k} have the same valleys at every height at most $k-1$. It follows that the valleys of τ_0, τ_{h+1-k} that are either at height at most $k-1$, or at height k and have been created by the above construction are the same. Furthermore, if there exists a valley of τ_0 at height k that has not been created by the construction, then this valley is also a valley of both σ_0, σ_{h+1-k} so that it is also a valley of τ_{h+1-k}. Thus, since the height of the highest valley of τ_0 is k, the paths τ_0, τ_{h+1-k} have the same valleys, which gives $\tau_0 = \tau_{h+1-k}$.

We define $s_i = \tau_{i+(h+1-k)}$, $i \in [0, k]$. It is easy to check that $w = s_0 \leq s_1 \leq \cdots \leq s_k = s$ is a multichain of type v.

For the converse we note that the peaks of τ_i at height $j+2$ generated according to the above construction from σ_i for $j \leq h-i$, are exactly these peaks of τ_i that are peaks of τ_0 and not of a.

Now, for a multichain $w = s_0 \leq s_1 \leq \cdots \leq s_k = s$ with $\mathrm{hv}(w) = k$, we define a multichain

$$w = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_{h+1} = s$$

with

$$\tau_i = \begin{cases} s_0, & i \in [0, h+1-k]; \\ s_{i-(h+1-k)}, & i \in [h+2-k, h+1]. \end{cases}$$

Finally, we define σ'_i, $i \in [0, h]$ to be the Dyck path obtained by turning all peaks of τ_i at height $j+2$ that are also peaks of w but not peaks of a into valleys, for every $j \leq h-i$.

Clearly, we have $\sigma'_0 = a$ and $\sigma'_h \leq s$.

Furthermore, we can analogously prove that $a = \sigma'_0 \leq \sigma'_1 \leq \cdots \leq \sigma'_h \leq s$ is a multichain of type v and $\sigma_i = \sigma'_i$ for every $i \in [0, h]$. □

In order to illustrate the bijection in the proof of Proposition 12 we give the following example.

Example 13. Let $a = u^2du^2du^2d^2u^2d^3ud^2u^3du^2dud^2udud^3$, $t = u^4du^2d^2ududud^4u^6dud^4ud^3$ and $s = u^5du^3d^3ud^3udu^4dudud^3ud^4$. For the multichain $a = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \sigma_3 = t$ of Example 11 (see Figure 2), using the construction in the proof of Proposition 12, we have that

$$w = \tau_0 = \tau_1 = u^3du^2dududud^2ud^2udu^3du^2d^2ud^2ud^3,$$

$$\tau_2 = u^3du^3d^2udud^2ud^2udu^4du^2d^4ud^3,$$

$$\tau_3 = u^4du^2d^2ududud^3udu^5dud^4ud^3$$

and

$$\tau_4 = s.$$

Then, the multichain $w = s_0 \leq s_1 \leq s_2 \leq s_3 = s$, where $s_i = \tau_{i+1}$, $i \in [3]$, is the corresponding $w - s$ multichain of type v (see Figure 3).

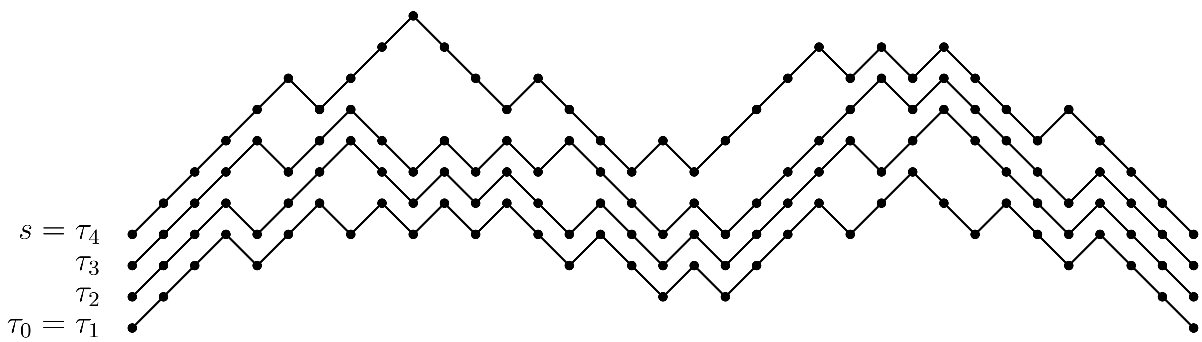


Figure 3: The multichain produced in Example 13 from the multichain of type v of Example 11

In the following result, we give a formula for the evaluation of f on prime Dyck paths. In the proof, we use the following obvious consequence of Proposition 7:

$$f(uad) = \sum_{w \in [a, a^*]} f(w), \text{ for every Dyck path } a. \tag{8}$$

Proposition 14. *For every Dyck path $a \in \mathcal{D}$ we have that*

$$f(uad) = \sum_{s \geq a} \mathrm{v}(a, s) I(s). \tag{9}$$

Proof. We prove the required formula by induction on the length of a.

Clearly, it holds for $a = \varepsilon$. Now, let $a = ua_1dua_2d \cdots ua_kd$, where $a_i \in \mathcal{D}$, for $i \in [k]$, $k \in \mathbb{N}^*$. Each $w \in [a, a^*]$ has a common low valley with a. Assume that the r-th low valley of a, $r \in [k]$, is the leftmost such valley. Then, $w = uw_1udw_2 \cdots udw_rd\beta$, where $w_i \in [a_i, a_i^*]$, for $i \in [r]$, $\beta \in [b_r, b_r^*]$ and $b_r = ua_{r+1}dua_{r+2}d \cdots ua_kd$, $r \in [k]$; (note that $b_k = \varepsilon$). Then, using the induction hypothesis, equation (8) and Proposition 12, we have that

$$
\begin{aligned}
f(uad) &= \sum_{w\in[a,a^*]} f(w) = \sum_{r=1}^{k} \sum_{\substack{w_i\in[a_i,a_i^*]\\ i\in[r]}} f(uw_1udw_2\cdots udw_rd) \sum_{\beta\in[b_r,b_r^*]} f(\beta)\\
&= \sum_{r=1}^{k} \sum_{\substack{w_i\in[a_i,a_i^*]\\ i\in[r]}} \sum_{s\geq w_1udw_2\cdots udw_r} \mathrm{v}(w_1udw_2\cdots udw_r, s) I(s) f(ub_rd)\\
&= \sum_{r=1}^{k} \sum_{\substack{w_i\in[a_i,a_i^*]\\ i\in[r]}} \sum_{\substack{s_i\geq w_i\\ i\in[r]}} \left(\prod_{i=1}^{r} \mathrm{v}(w_i,s_i)\right) I(s_1uds_2\cdots uds_r) f(ub_rd)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{r=1}^{k} \sum_{\substack{s_i\geq a_i\\ i\in[r]}} \left(\prod_{i=1}^{r} \sum_{w_i\in[a_i,a_i^*]} \mathrm{v}(w_i,s_i)\right) I(s_1uds_2\cdots uds_r) f(ub_rd)\\
&= \sum_{r=1}^{k} \sum_{\substack{s_i\geq a_i\\ i\in[r]}} \left(\prod_{i=1}^{r} \mathrm{v}(ua_id,us_id)\right) I(s_1uds_2\cdots uds_r) f(ub_rd). \qquad (10)
\end{aligned}
$$

On the other hand, given a path $s\in\mathcal{D}$ with $s\geq a$, having the same low valleys with a, we set $s=us_1dus_2d\cdots us_kd$, where $s_i\in\mathcal{D}$ and $s_i\geq a_i$ for every $i\in[k]$. Clearly, as before, we can decompose every path $c\in[s,u^{|s|/2}d^{|s|/2}]$ with respect to the leftmost common low valley with a, i.e., $c=u\phi d\chi$ where $\phi,\chi\in\mathcal{D}$, $\phi\geq s_1uds_2\cdots uds_r$ and $\chi\geq us_{r+1}dus_{r+2}d\cdots us_kd$, for some $r\in[k]$.

It follows that

$$
\begin{aligned}
\sum_{s\geq a} \mathrm{v}(a,s)I(s) &= \sum_{r=1}^{k} \sum_{\substack{s_i\geq a_i\\ i\in[k]}} \left(\prod_{i=1}^{k} \mathrm{v}(ua_id,us_id)\right) I(s_1uds_2\cdots uds_r) I(us_{r+1}dus_{r+2}d\cdots us_kd)\\
&= \sum_{r=1}^{k} \sum_{\substack{s_i\geq a_i\\ i\in[r]}} \left(\prod_{i=1}^{r} \mathrm{v}(ua_id,us_id)\right) I(s_1uds_2\cdots uds_r) \sum_{t\geq b_r} \mathrm{v}(b_r,t)I(t)\\
&= \sum_{r=1}^{k} \sum_{\substack{s_i\geq a_i\\ i\in[r]}} \left(\prod_{i=1}^{r} \mathrm{v}(ua_id,us_id)\right) I(s_1uds_2\cdots uds_r) f(ub_rd). \qquad (11)
\end{aligned}
$$

The required formula follows from (10) and (11). □

Example 15. For $a = u^2d^2u^3dudud^3$, from formula (9) we have that $f(u^3d^2u^3dudud^4) = f(uad) = \sum_{s\geq a} \mathrm{v}(a,s)I(s)$.

Clearly, there are five $s \geq a$ which have the same low valleys with a, namely: $s_i = u^2d^2a_i$, $i \in [5]$ where $a_1 = u^3dudud^3$, $a_2 = u^4d^2ud^3$, $a_3 = u^3du^2d^4$, $a_4 = u^4dud^4$ and $a_5 = u^5d^5$.

We can easily check by the definition of v that $\mathrm{v}(a,s_1) = 1$, $\mathrm{v}(a,s_2) = \mathrm{v}(a,s_3) = 2$, $\mathrm{v}(a,s_4) = 4$ and $\mathrm{v}(a,s_5) = 5$.

Furthermore, using formula (2) we have that

$$
\begin{aligned}
I(s_1) &= |\,[(2,2,5,6,7,7,7),(7,7,7,7,7,7,7)]\,| = |\,[(2,2,5,6),(7,7,7,7)]\,| \\
&= \begin{vmatrix} 6 & \binom{6}{2} & \binom{3}{3} & \binom{2}{4} \\ 1 & 6 & \binom{3}{2} & \binom{2}{3} \\ 0 & 1 & 3 & \binom{2}{2} \\ 0 & 0 & 1 & 2 \end{vmatrix} = 71, \\
I(s_2) &= |\,[(2,2,6,6,7,7,7),(7,7,7,7,7,7,7)]\,| = |\,[(2,2,6,6),(7,7,7,7)]\,| \\
&= \begin{vmatrix} 6 & \binom{6}{2} & \binom{2}{3} & \binom{2}{4} \\ 1 & 6 & \binom{2}{2} & \binom{2}{3} \\ 0 & 1 & 2 & \binom{2}{2} \\ 0 & 0 & 1 & 2 \end{vmatrix} = 51, \\
I(s_3) &= |\,[(2,2,5,7,7,7,7),(7,7,7,7,7,7,7)]\,| = |\,[(2,2,5),(7,7,7)]\,| = \begin{vmatrix} 6 & \binom{6}{2} & \binom{3}{3} \\ 1 & 6 & \binom{3}{2} \\ 0 & 1 & 3 \end{vmatrix} = 46, \\
I(s_4) &= |\,[(2,2,6,7,7,7,7),(7,7,7,7,7,7,7)]\,| = |\,[(2,2,6),(7,7,7)]\,| = \begin{vmatrix} 6 & \binom{6}{2} & \binom{2}{3} \\ 1 & 6 & \binom{2}{2} \\ 0 & 1 & 2 \end{vmatrix} = 36, \\
I(s_5) &= |\,[(2,2,7,7,7,7,7),(7,7,7,7,7,7,7)]\,| = |\,[(2,2),(7,7)]\,| = \begin{vmatrix} 6 & \binom{6}{2} \\ 1 & 6 \end{vmatrix} = 21.
\end{aligned}
$$

From the above we obtain that $f(u^3d^2u^3dudud^4) = 514$.

In the following result we show that, for specific Dyck paths, the map f is related to the zeta function. We recall that the zeta function of a poset X is given by the formula

$$\zeta(x,y) = \begin{cases} 1, & x \leq y; \\ 0, & \text{otherwise}, \end{cases}$$

for $x, y \in X$. Moreover, it is well known (e.g., see [12, p. 263]) that $\zeta^k(x,y)$ counts the number of $x - y$ multichains in X of length k, for $k \geq 0$.

Corollary 16. *If a is a product of pyramids, then*

$$f(u^k a d^k) = \zeta^{k+1}(a, u^{|a|/2} d^{|a|/2}),$$

for every $k \geq 0$.

Proof. For $k = 0$ the result obviously holds.

We now prove that for $k \geq 0$ we have that

$$\mathrm{v}(u^k a d^k, u^k s d^k) = \zeta^k(a, s).$$

Indeed, we can easily see that $\mathrm{hv}(u^k a d^k) = k$, and that every k-multichain from $u^k a d^k$ to $u^k s d^k$ of type v is of the form

$$u^k a d^k = u^k \sigma_0 d^k \leq \cdots \leq u^k \sigma_{k-1} d^k \leq u^k \sigma_k d^k = u^k s d^k,$$

producing the $a - s$ multichain

$$a = \sigma_0 \leq \cdots \leq \sigma_{k-1} \leq \sigma_k = s.$$

Then, for $k \geq 1$, by Proposition 14 we have that

$$\begin{aligned} f(u^k a d^k) &= \sum_{s \geq u^{k-1} a d^{k-1}} \mathrm{v}(u^{k-1} a d^{k-1}, s) I(s) \\ &= \sum_{s \geq a} \mathrm{v}(u^{k-1} a d^{k-1}, u^{k-1} s d^{k-1}) I(s) \\ &= \sum_{s \geq a} \zeta^{k-1}(a, s) I(s) = \zeta^{k+1}(a, u^{|a|/2} d^{|a|/2}). \end{aligned}$$

□

Since f is multiplicative on $\mathcal{D}$, by using formula (9) we can evaluate f for every Dyck path. For Dyck prefixes that are not Dyck paths, it is enough to evaluate $f(duP)$, where P is a Dyck prefix. We achieve this in formula (13) of the following Proposition, the proof of which, although it shares some common ideas with the proof of formula (9), it is much more complicated. The difficulty lies in the fact that it is not possible to prove (13) directly by induction, so that we introduce and prove the more general equality (14) which concerns a pair of paths (P, R) where a lexicographic induction applies. We also note that in this proof, we use several times the following obvious consequence of Proposition 7:

$$f(duP) = \sum_{Q \in [P, \widetilde{P}]} f(dQ), \text{ for every Dyck prefix } P. \tag{12}$$

Proposition 17. *For every Dyck prefix $P = a_0 u a_1 \cdots u a_k$, $k \geq 0$, $a_i \in \mathcal{D}$, $i \in [0, k]$, we have that*

$$f(duP) = \sum \prod_{i=0}^{k} \mathrm{v}(a_i, s_i) J(s_0 V_1), \tag{13}$$

where the sum is taken over all sequences (s_i), $i \in [0, k]$, of Dyck paths with $s_i \geq a_i$, and over all sequences of Dyck prefixes (V_i), $i \in [k+1]$, with $V_i \geq u s_i V_{i+1}$, $i \in [k]$, and $V_{k+1} = \varepsilon$.

Proof. Let a Dyck prefix $R = b_0ub_1 \cdots ub_\lambda$, $\lambda \geq 0$, $b_i \in \mathcal{D}$, $i \in [0, \lambda]$.

For the pair (P, R) we consider the equality

$$\sum_{Q \in [R, \widetilde{R}]} f(duPQ) = \sum \prod_{i=0}^{k} \mathrm{v}(a_i, s_i) \prod_{i=0}^{\lambda} \mathrm{v}(b_i, t_i) J(s_0 V_1), \tag{14}$$

where the sum is taken over all sequences (s_i), $i \in [0, k]$, (t_i), $i \in [0, \lambda]$, of Dyck paths with $s_i \geq a_i$ and $t_i \geq b_i$, and over all sequences of Dyck prefixes (V_i), $i \in [k + \lambda + 2]$, with $V_i \geq us_iV_{i+1}$, $i \in [k]$, $V_{k+1} \geq t_0V_{k+2}$, $V_{k+i+1} \geq ut_iV_{k+i+2}$, $i \in [\lambda]$, and $V_{k+\lambda+2} = \varepsilon$.

Clearly, formula (13) is a special case of equality (14) for $R = \varepsilon$.

We prove equality (14) by induction. More precisely, assuming that (14) holds for every pair (P_1, R_1) with either $|P_1| + |R_1| < |P| + |R|$, or $|P_1| + |R_1| = |P| + |R|$ with $|R_1| < |R|$, we prove that (14) also holds for the pair (P, R).

We restrict ourselves to the case $P \notin \mathcal{D}$, since the case $P = a_0 \in \mathcal{D}$ is similar and easier to prove. We consider the following cases:

1. Assume that $R = u^\lambda$.

We first note that if $\lambda > 0$, then by the induction hypothesis we deduce that equality (14) holds for the pair $(Pu^\lambda, \varepsilon)$, from which we can easily obtain that (14) holds also for the pair (P, u^λ). Thus, we restrict ourselves to the case $R = \varepsilon$, so that now it is enough to prove formula (13).

We set $H = a_1ua_2 \cdots ua_k$ and we consider two subcases:

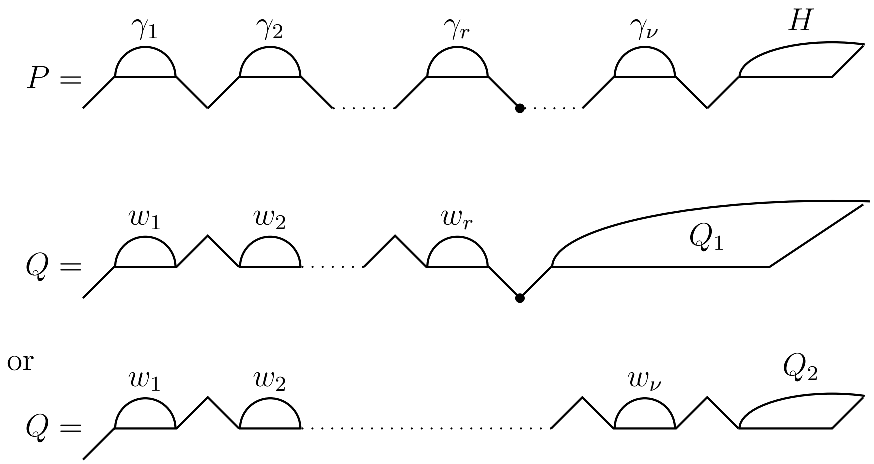
1(i). Assume that $a_0 = \varepsilon$, i.e., $P = uH$. Then, by formula (12), and by equality (14) for the pair (ε, H), it follows that

$$f(duP) = \sum_{Q \in [H, \widetilde{H}]} f(duQ) = \sum \prod_{i=1}^{k} \mathrm{v}(a_i, s_i) J(V_1),$$

where the sum it taken over all sequences (s_i), $i \in [k]$, of Dyck paths with $s_i \geq a_i$ and over all sequences (V_i), $i \in [k+1]$ with $V_1 \geq s_1V_2$, $V_i \geq us_iV_{i+1}$, $i \in [2, k]$, and $V_{k+1} = \varepsilon$. Since $J(V_1) = J(uV_1)$, we can replace V_1 by uV_1, thus verifying formula (13).

1(ii). Assume that $a_0 \neq \varepsilon$. We set $a_0 = u\gamma_1du\gamma_2d \cdots u\gamma_\nu d$ and $\delta_r = u\gamma_{r+1}du\gamma_{r+2}d \cdots u\gamma_\nu d$, $\nu \in \mathbb{N}^*$, $r \in [\nu]$.

Every $Q \in [P, \widetilde{P}]$ can be decomposed according to the leftmost common low valley with P (see Figure 4).



$$w_i \in [\gamma_i, \gamma_i^*],\ i \in [\nu], \qquad Q_1 \in [\delta_r u H, \widetilde{\delta_r u H}], \qquad Q_2 \in [H, \widetilde{H}].$$

Figure 4: The decomposition of $Q \in [P, \widetilde{P}]$ according to the leftmost common low valley of P, Q.

Using formula (12) and the above decomposition we have that

$$\begin{aligned}
f(duP) &= \sum_{Q \in [P, \widetilde{P}]} f(dQ) \\
&= \sum_{r=1}^{\nu} \sum_{\substack{w_i \in [\gamma_i, \gamma_i^*] \\ i \in [r]}} \sum_{Q \in [\delta_r u H, \widetilde{\delta_r u H}]} f(duw_1 ud \cdots w_{r-1} udw_r dQ) \\
&+ \sum_{\substack{w_i \in [\gamma_i, \gamma_i^*] \\ i \in [\nu]}} \sum_{Q \in [H, \widetilde{H}]} f(duw_1 ud \cdots w_\nu udQ) \\
&= \sum_{r=1}^{\nu} \sum_{\substack{w_i \in [\gamma_i, \gamma_i^*] \\ i \in [r]}} f(uw_1 ud \cdots w_{r-1} udw_r d) \sum_{Q \in [\delta_r u H, \widetilde{\delta_r u H}]} f(dQ) \\
&+ \sum_{\substack{w_i \in [\gamma_i, \gamma_i^*] \\ i \in [\nu]}} \sum_{Q \in [H, \widetilde{H}]} f(duw_1 ud \cdots w_\nu udQ).
\end{aligned} \tag{15}$$

Then, using Proposition 14, and equality (14) for the pair $(w_1 ud \cdots w_\nu ud, H)$ we have that

$$f(duP) = \sum_{r=1}^{\nu} \sum_{\substack{w_i\in[\gamma_i,\gamma_i^*]\\ i\in[r]}} \sum_{\substack{t_i\geq w_i\\ i\in[r]}} \prod_{i=1}^{r} \mathrm{v}(w_i,t_i) I(t_1ud\cdots t_{r-1}udt_r) f(du\delta_r uH)$$

$$+ \sum_{w_i\in[\gamma_i,\gamma_i^*]} \sum_{\substack{t_i\geq w_i\\ i\in[\nu]}} \sum_{\substack{s_i\geq a_i\\ i\in[k]}} \sum_{\substack{W_1\geq s_1V_2\\ V_i\geq us_iV_{i+1}, i\in[2,k]\\ V_{k+1}=\varepsilon}} \prod_{i=1}^{\nu} \mathrm{v}(w_i,t_i) \prod_{i=1}^{k} \mathrm{v}(a_i,s_i) J(t_1ud\cdots t_\nu udW_1)$$

$$= \sum_{r=1}^{\nu} \sum_{\substack{t_i\geq\gamma_i\\ i\in[r]}} \left(\prod_{i=1}^{r} \sum_{\substack{w_i\in[\gamma_i,\gamma_i^*]\\ i\in[r]}} \mathrm{v}(w_i,t_i) \right) I(t_1ud\cdots t_{r-1}udt_r) f(du\delta_r uH)$$

$$+ \sum_{\substack{t_i\geq\gamma_i, i\in[\nu]\\ s_i\geq a_i, i\in[k]}} \sum_{\substack{W_1\geq s_1V_2\\ V_i\geq us_iV_{i+1}, i\in[2,k]\\ V_{k+1}=\varepsilon}} \prod_{i=1}^{k} \mathrm{v}(a_i,s_i) \left(\prod_{i=1}^{\nu} \sum_{\substack{w_i\in[\gamma_i,\gamma_i^*]\\ i\in[\nu]}} \mathrm{v}(w_i,t_i) \right) J(t_1ud\cdots t_\nu udW_1).$$

Finally, using formula (13) for the path $\delta_r uH$ and Proposition 12, we obtain that

$$f(duP)$$
$$= \sum_{r=1}^{\nu} \sum_{\substack{t_i\geq\gamma_i\\ i\in[r]}} \prod_{i=1}^{r} \mathrm{v}(u\gamma_i d, ut_id) I(t_1ud\cdots t_{r-1}udt_r)\cdot$$
$$\sum_{\substack{t_i\geq\gamma_i, i\in[r+1,\nu]\\ s_i\geq a_i, i\in[k]}} \sum_{\substack{V_i\geq us_iV_{i+1}, i\in[k]\\ V_{k+1}=\varepsilon}} \mathrm{v}(\delta_r, ut_{r+1}d\cdots ut_\nu d) \prod_{i=1}^{k} \mathrm{v}(a_i,s_i) J(ut_{r+1}d\cdots ut_\nu dV_1)$$
$$+ \sum_{\substack{t_i\geq\gamma_i, i\in[\nu]\\ s_i\geq a_i, i\in[k]}} \sum_{\substack{W_1\geq s_1V_2\\ V_i\geq us_iV_{i+1}, i\in[2,k]\\ V_{k+1}=\varepsilon}} \prod_{i=1}^{k} \mathrm{v}(a_i,s_i) \prod_{i=1}^{\nu} \mathrm{v}(u\gamma_i d, ut_id) J(t_1ud\cdots t_\nu udW_1)$$
$$= \sum_{\substack{s_i\geq a_i, i\in[0,k]\\ s_0=ut_1d\cdots ut_\nu d}} \prod_{i=0}^{k} \mathrm{v}(a_i,s_i)\cdot$$
$$\sum_{\substack{W_1\geq s_1V_2\\ V_i\geq us_iV_{i+1},\\ i\in[2,k]\\ V_{k+1}=\varepsilon}} \left(\sum_{r=1}^{\nu} I(t_1ud\cdots t_{r-1}udt_r) J(ut_{r+1}d\cdots ut_\nu duW_1) + J(t_1ud\cdots t_\nu udW_1) \right). \tag{16}$$

Clearly, since the number $\sum_{r=1}^{\nu} I(t_1ud\cdots t_{r-1}udt_r)J(ut_{r+1}d\cdots ut_\nu duW_1)$ (resp., the number $J(t_1ud\cdots t_\nu udW_1)$) counts the set of all paths greater than or equal to s_0uW_1 that have (resp., do not have) low valleys we obtain that

$$\sum_{r=1}^{\nu} I(t_1ud\cdots t_{r-1}udt_r)J(ut_{r+1}d\cdots ut_\nu duW_1) + J(t_1ud\cdots t_\nu udW_1) = J(s_0uW_1),$$

so that, by setting $V_1 = uW_1$ in formula (16), we obtain (13).

2. Assume that $R \neq u^\lambda$. Then, we set $R = u^\mu b_\mu ub_{\mu+1}\cdots ub_\lambda$, where μ is the least integer such that $b_\mu \neq \varepsilon$.

It is enough to restrict ourselves to the case where $\mu = 0$ (i.e., $b_0 \neq \varepsilon$), since the general case follows easily by applying equality (14) for the pair $(Pu^\mu, b_\mu ub_{\mu+1}\cdots ub_\lambda)$.

Furthermore, we can restrict ourselves to the case where $R \notin \mathcal{D}$ (i.e., $\lambda > 0$), since the case where $R \in \mathcal{D}$ is similar and easier to prove.

Set $b_0 = u\eta_1du\eta_2d\cdots u\eta_\xi d$, $\theta_r = u\eta_{r+1}du\eta_{r+2}d\cdots u\eta_\xi d$, $\xi \in \mathbb{N}^*$, $r \in [\xi]$, and $T = b_1ub_2\cdots ub_\lambda$.

By decomposing each $Q \in [R, \widetilde{R}]$ according to the leftmost low common valley with R, as before we have that

$$\begin{aligned}\sum_{Q\in[R,\widetilde{R}]} f(duPQ) =& \sum_{r=1}^{\xi} \sum_{\substack{w_i\in[\eta_i,\eta_i^*]\\ i\in[r]}} \sum_{Q\in[\theta_ruT,\widetilde{\theta_ruT}]} f(duPuw_1ud\cdots w_{r-1}udw_rdQ)\\ &+ \sum_{\substack{w_i\in[\eta_i,\eta_i^*]\\ i\in[\xi]}} \sum_{Q\in[T,\widetilde{T}]} f(duPuw_1ud\cdots w_\xi udQ).\end{aligned} \tag{17}$$

It is enough to prove that the first (resp., second) sum on the RHS of the above equality is equal to the part of the second sum in equality (14) which has (resp., does not have) low valleys.

Clearly, by applying (14) for the pair $(Puw_1ud\cdots w_{r-1}udw_r, \theta_ruT)$ for $r \in [\xi]$, we have that

$$\begin{aligned}&\sum_{Q\in[\theta_ruT,\widetilde{\theta_ruT}]} f(duPuw_1ud\cdots w_{r-1}udw_rdQ)\\ &= \sum \prod_{i=0}^{k} \mathrm{v}(a_i,s_i)\,\mathrm{v}(uw_1ud\cdots w_{r-1}udw_rd, usd)\,\mathrm{v}(\theta_r,z_r) \prod_{i=1}^{\lambda} \mathrm{v}(b_i,t_i) J(s_0V_1),\end{aligned} \tag{18}$$

where the sum is taken over all sequences (s_i), $i \in [0,k]$, (t_i), $i \in [\lambda]$, s, z_r and $(V_1,\ldots,V_k, W_{k+1}, V_{k+2},\ldots, V_{k+\lambda})$ with $s_i \geq a_i$, $t_i \geq b_i$, $s \geq w_1udw_2ud\cdots w_{r-1}udw_r$, $z_r \geq \theta_r$, $V_i \geq us_iV_{i+1}$, $i \in [k-1]$, $V_k \geq us_kusdW_{k+1}$, $W_{k+1} \geq z_rV_{k+2}$, $V_{k+i+1} \geq ut_iV_{k+i+2}$, $i \in [\lambda]$, and $V_{k+\lambda+2} = \varepsilon$.

Clearly, we have

$$\mathrm{v}(uw_1ud\cdots w_{r-1}udw_rd, usd) = \sum_{\phi\le s}\mathrm{v}(w_1ud\cdots w_{r-1}udw_r, \phi) = \sum_{\substack{\phi_i\ge w_i\\ i\in[r]}}\prod_{i=1}^{r}\mathrm{v}(w_i,\phi_i),$$

where $\phi_i \in \mathcal{D}$, $i \in [r]$, and $\phi_1ud\cdots\phi_{r-1}ud\phi_r \le s$. Then, by substituting in equality (18), summing in terms of (W_i), $i \in [r]$, changing the order of summation and using Proposition 12, in an analogous way as the one used in the proof of formula (15), we can easily deduce that

$$\begin{aligned}\sum_{\substack{w_i\in[\eta_i,\eta_i^*]\\ i\in[r]}}\ \sum_{Q\in[\theta_r uT,\widetilde{\theta_r uT}]} & f(duPuw_1ud\cdots w_{r-1}udw_rdQ)\\ &= \sum\prod_{i=0}^{k}\mathrm{v}(a_i,s_i)\prod_{i=1}^{r}\mathrm{v}(u\eta_i d,u\phi_i d)\,\mathrm{v}(\theta_r,z_r)\prod_{i=1}^{\lambda}\mathrm{v}(b_i,t_i)J(s_0V_1),\end{aligned} \tag{19}$$

where $s_i \ge a_i$, $i \in [0,k]$, $t_i \ge b_i$, $i \in [\lambda]$, $\phi_i \ge \eta_i$, $i \in [r]$, $s \ge \phi_1ud\cdots\phi_{r-1}ud\phi_r$, $z_r \ge \theta_r$, $V_i \ge us_iV_{i+1}$, $i \in [k-1]$, $V_k \ge us_kusdW_{k+1}$, $W_{k+1} \ge z_rV_{k+2}$, $V_{k+i+1} \ge ut_iV_{k+i+2}$, $i \in [\lambda]$, and $V_{k+\lambda+2} = \varepsilon$.

Then, by setting $t_0 = u\phi_1d\cdots u\phi_rdz_r$ and $V_{k+1} = usdW_{k+1}$, the RHS of equality (19) becomes

$$\sum\prod_{i=0}^{k}\mathrm{v}(a_i,s_i)\prod_{i=0}^{\lambda}\mathrm{v}(b_i,t_i)J(s_0V_1),$$

where the sum is taken over all sequences (s_i), (t_i), (V_i) with the same restrictions as in equality (14), with the extra condition that the r-th low valley of t_0 (and b_0) coincides with the first low valley of V_{k+1}.

Next, by summing in terms of $r \in [\xi]$, we deduce that the first sum on the RHS of equality (17) coincides with the part of the second sum in (14) for which V_{k+1} has low valleys.

Finally, for the second sum on the RHS of equality (17), by applying (14) for the pair $(Puw_1ud\cdots w_\xi ud, T)$, we have that

$$\sum_{\substack{w_i\in[\eta_i,\eta_i^*]\\ i\in[\xi]}}\ \sum_{Q\in[T,\widetilde{T}]} f(duPuw_1ud\cdots w_\xi udQ) = \sum_{\substack{w_i\in[\eta_i,\eta_i^*]\\ i\in[\xi]}}\sum\prod_{i=0}^{k}\mathrm{v}(a_i,s_i)\prod_{i=1}^{\xi}\mathrm{v}(w_i,\phi_i)\prod_{i=1}^{\lambda}\mathrm{v}(b_i,t_i)J(s_0V_1),$$

where the sum is taken over all sequences (s_i), (ϕ_i), (b_i) with $s_i \ge a_i$, $i \in [0,k]$, $\phi_i \ge w_i$, $i \in [\xi]$, $b_i \ge t_i$, $i \in [\lambda]$, and over all sequences $(V_1,\ldots,V_{k+1},W_{k+2},V_{k+3},\ldots,V_{k+\lambda+2})$ with $V_i \ge us_iV_{i+1}$, $i \in [k]$, $V_{k+1} \ge u\phi_1ud\cdots\phi_\xi udW_{k+2}$, $W_{k+2} \ge t_1V_{k+3}$, $V_{k+i+1} \ge ut_iV_{k+i+2}$, $i \in [2,\lambda]$, and $V_{k+\lambda+2} = \varepsilon$.

By changing the order of summation on the RHS of the above equality, using Proposition 12, as before, and setting $t_0 = u\phi_1d\cdots u\phi_\xi d$, $V_{k+2} = uW_{k+2}$ we deduce easily that the second sum on the RHS of equality (17) coincides with the part of the second sum in (14) for which V_{k+1} has no valleys, which completes the proof of equality (14). $\square$

Example 18. For $P = u^2du^2d^3u^2d$, by formula (13) we have that $f(du^3du^2d^3u^2d) = f(duP) = \sum \mathrm{v}(u^2du^2d^3, s_0)\,\mathrm{v}(ud, s_1)J(s_0V_1)$, where the sum is taken over all prime Dyck paths s_0, s_1 with $s_0 \geq u^2du^2d^3$ and $s_1 \geq ud$, and all Dyck prefixes V_1 with $V_1 \geq us_1$.

It is easy to check that $s_0 = u^2du^2d^3$ or $s_0 = u^4d^4$, $s_1 = ud$ and $V_1 = u^2d$ or $V_1 = u^3$. Furthermore, for these s_0, s_1 we have that $\mathrm{v}(u^2du^2d^3, s_0) = 1$ and $\mathrm{v}(ud, s_1) = 1$ so that

$$\begin{aligned} f(du^3du^2d^3u^2d) &= J(u^2du^2d^3u^2d) + J(u^2du^2u^3u^3) + J(u^3dud^3u^2d) + J(u^3dud^3u^3) \\ &\quad + J(u^4d^4u^2d) + J(u^4d^4u^3). \end{aligned}$$

Then, by formulas (3) and (2) we obtain that

$$\begin{aligned} &J(u^2du^2d^3u^2d) \\ &= |[u^{11}, u^{11}]| + |[u^2du^8, u^{10}d]| + |[u^2du^2du^5, u^9d^2]| + |[u^2du^2d^2u^4, u^8d^3]| \\ &\quad + |[u^2du^2d^3u^3, u^7d^4]| + |[u^2du^2d^3u^2d, u^6d^5]| \\ &= 1 + |[(2), (10)]| + |[(2,4), (9,9)]| + |[(2,4,4), (8,8,8)]| \\ &\quad + |[(2,4,4,4), (7,7,7,7)]| + |[(2,4,4,4,6), (6,6,6,6,6)]| \\ &= 1 + 11 - 2 + \begin{vmatrix} 8 & \binom{6}{2} \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 7 & \binom{5}{2} & \binom{5}{3} \\ 1 & 5 & \binom{5}{2} \\ 0 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 6 & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\ 1 & 4 & \binom{4}{2} & \binom{4}{3} \\ 0 & 1 & 4 & \binom{4}{2} \\ 0 & 0 & 1 & 4 \end{vmatrix} + \begin{vmatrix} 5 & \binom{3}{2} & \binom{3}{3} & \binom{3}{4} & \binom{1}{5} \\ 1 & 3 & \binom{3}{2} & \binom{3}{3} & \binom{1}{4} \\ 0 & 1 & 3 & \binom{3}{2} & \binom{1}{3} \\ 0 & 0 & 1 & 3 & \binom{1}{2} \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \\ &= 1 + 9 + 33 + 65 + 75 + 35 = 218. \end{aligned}$$

Similarly, we find that $J(u^2du^2d^3u^3) = 183$, $J(u^3dud^3u^2d) = 166$, $J(u^3dud^3u^3) = 141$, $J(u^4d^4u^2d) = 114$ and $J(u^4d^4u^3) = 99$, so that we obtain that $f(du^3du^2d^3u^2d) = 921$.

In the last result, we show that for specific paths, the map f is also related to the zeta function.

Corollary 19. *If a is a product of pyramids, then*

$$f(du^ka) = \zeta^{k+1}(a, u^{|a|}),$$

for every $k \geq 1$.

Proof. We apply Proposition 17 for the Dyck prefix $P = u^{k-1}a$. Firstly, for $k = 1$ we obtain that

$$f(dua) = J(a) = \zeta^2(a, u^{|a|}).$$

Next, for $k \geq 2$ we obtain that

$$f(dua) = \sum J(V_1),$$

where the sum is taken over all sequences (V_i), $i \in [k]$ of Dyck prefixes with $V_i \geq uV_{i+1}$, $i \in [k-1]$, $V_{k-1} \geq uaV_k$ and $V_k = \varepsilon$.

Each such sequence (V_i), $i \in [k]$, produces a unique multichain (W_i), $i \in [0, k-1]$, of length $k-1$ from $u^{k-1}a$ to V_1, defined by $W_0 = u^{k-1}a$ and $W_i = u^{k-i-1}V_{k-i}$, $i \in [k-1]$.

Then, formula (13) gives that

$$f(dua) = \sum_{V_1 \geq u^{k-1}a} \zeta^{k-1}(u^{k-1}a, V_1)J(V_1) = \zeta^{k+1}(u^{k-1}a, u^{k-1+|a|}) = \zeta^{k+1}(a, u^{|a|}). \qquad \square$$

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