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The Akiyama-Tanigawa algorithm for Bernoulli numbers

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Abstract

A direct proof is given for Akiyama and Tanigawa's algorithm for computing Bernoulli numbers. The proof uses a closed formula for Bernoulli numbers expressed in terms of Stirling numbers. The outcome of the same algorithm with different initial values is also briefly discussed.

1 The Algorithm

In their study of values at non-positive integer arguments of multiple zeta functions, S. Akiyama and Y. Tanigawa [1] found as a special case an amusing algorithm for computing Bernoulli numbers in a manner similar to "Pascal's triangle" for binomial coefficients.

Their algorithm reads as follows: Start with the 0-th row $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ and define the first row by $1\cdot(1-\frac{1}{2}),\ 2\cdot(\frac{1}{2}-\frac{1}{3}),\ 3\cdot(\frac{1}{3}-\frac{1}{4}), \ldots$ which produces the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Then define the next row by $1\cdot(\frac{1}{2}-\frac{1}{3}),\ 2\cdot(\frac{1}{3}-\frac{1}{4}),\ 3\cdot(\frac{1}{4}-\frac{1}{5}), \ldots$, thus giving $\frac{1}{6}, \frac{1}{6}, \frac{3}{20}, \ldots$ as the second row. In general, denoting the m-th ($m=0,1,2,\ldots$) number in the n-th $(n=0,1,2,\ldots)$ row by $a_{n,m}$, the m-th number in the $(n+1)$-st row $a_{n+1,m}$ is determined recursively by

$$a_{n+1,m} = (m+1)\cdot(a_{n,m} - a_{n,m+1}).$$

Then the claim is that the 0-th component $a_{n,0}$ of each row (the "leading diagonal") is just the n-th Bernoulli numbers B_n, where

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{xe^x}{e^x - 1} \left(= \frac{x}{e^x - 1} + x \right).$$

Note that we are using the definition of the Bernoulli numbers in which $B_1 = \frac{1}{2}$. This is the definition used by Bernoulli (and independently Seki, published one year prior to Bernoulli). Incidentally, this is more appropriate for the Euler formula $\zeta(1-k) = -B_k/k$ $(k = 1, 2, 3, \dots)$ for the values of the Riemann zeta function.

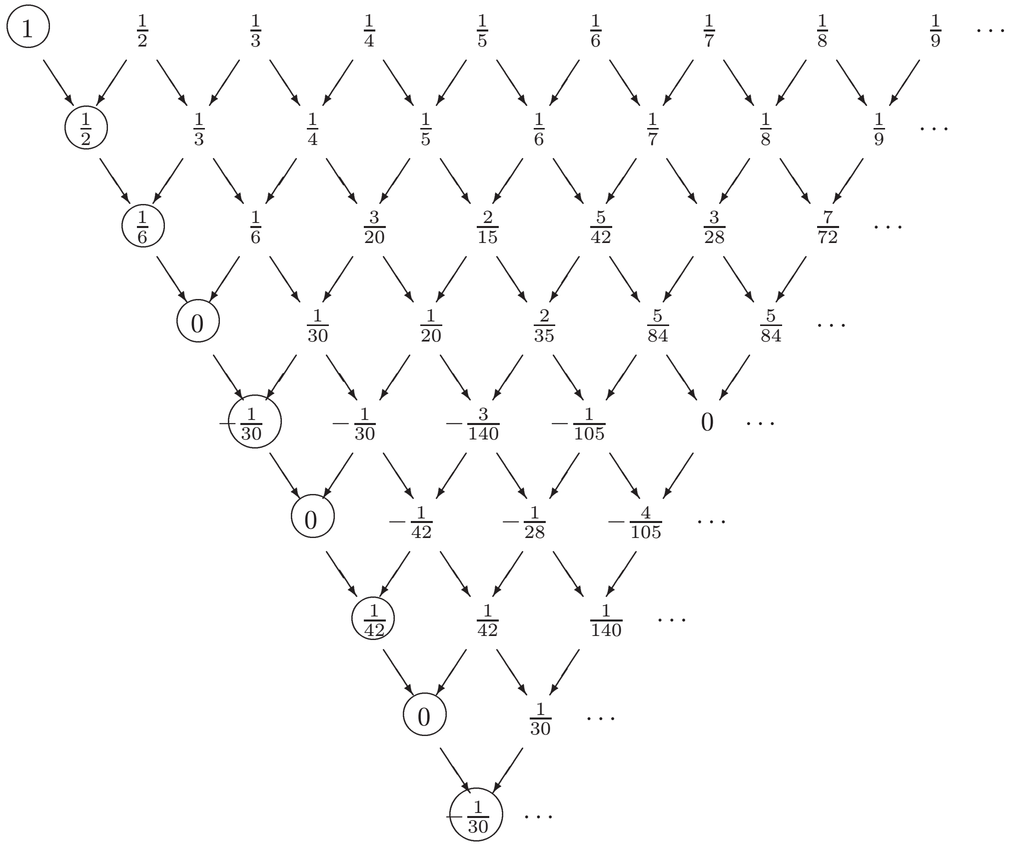

Figure 1: Akiyama-Tanigawa triangle

2 Proof

The proof is based on the following identity for Bernoulli numbers, a variant of which goes as far back as Kronecker (see [4]). Here we denote by $\left\{ {n \atop m} \right\}$ the Stirling number of the second kind:

$$x^n = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} x^{\underline{m}},$$

where $x^{\underline{m}} = x(x-1)\cdots(x-m+1)$ for $m \geq 1$ and $x^{\underline{0}} = 1$. (We use Knuth's notation [7]. For the Stirling number identities that we shall need, the reader is referred for example to [5].)

Theorem 1

$$B_n = \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n+1 \atop m+1} \right\}}{m+1}, \qquad \forall n \geq 0.$$

We shall give later a proof of this identity for the sake of completeness. Once we have this, the next proposition ensures the validity of our algorithm.

Proposition 2 *Given an initial sequence* $a_{0,m}$ $(m = 0, 1, 2, \ldots)$*, define the sequences* $a_{n,m}$ $(n \geq 1)$ *recursively by*

$$a_{n,m} = (m+1) \cdot (a_{n-1,m} - a_{n-1,m+1}) \quad (n \geq 1, m \geq 0). \tag{1}$$

Then

$$a_{n,0} = \sum_{m=0}^{n} (-1)^m m! \left\{ {n+1 \atop m+1} \right\} a_{0,m}. \tag{2}$$

Proof. Put

$$g_n(t) = \sum_{m=0}^{\infty} a_{n,m} t^m.$$

By the recursion (1) we have for $n \geq 1$

$$\begin{aligned}
g_n(t) &= \sum_{m=0}^{\infty} (m+1)(a_{n-1,m} - a_{n-1,m+1}) t^m \\
&= \frac{d}{dt}(\sum_{m=0}^{\infty} a_{n-1,m} t^{m+1}) - \frac{d}{dt}(\sum_{m=0}^{\infty} a_{n-1,m+1} t^{m+1}) \\
&= \frac{d}{dt}(t g_{n-1}(t)) - \frac{d}{dt}(g_{n-1}(t) - a_{n-1,0}) \\
&= g_{n-1}(t) + (t-1)\frac{d}{dt}(g_{n-1}(t)) \\
&= \frac{d}{dt}((t-1) g_{n-1}(t)).
\end{aligned}$$

Hence, by putting $(t-1)g_n(t) = h_n(t)$ we obtain

$$h_n(t) = (t-1)\frac{d}{dt}(h_{n-1}(t)) \quad (n \geq 1),$$

and thus

$$h_n(t) = \left((t-1)\frac{d}{dt}\right)^n (h_0(t)).$$

Applying the formula (*cf.* [5, Ch. 6.7 Exer. 13])

$$\left(x\frac{d}{dx}\right)^n = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} x^m \left(\frac{d}{dx}\right)^m$$

for $x = t-1$, we have

$$h_n(t) = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (t-1)^m \left(\frac{d}{dt}\right)^m h_0(t).$$

Putting $t = 0$ we obtain

$$\begin{aligned}
-a_{n,0} &= \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (-1)^m m! (a_{0,m-1} - a_{0,m}) \\
&= \sum_{m=0}^{n-1} \left\{ {n \atop m+1} \right\} (-1)^{m+1} (m+1)! a_{0,m} - \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (-1)^m m! a_{0,m} \\
&= -\sum_{m=0}^{n} (-1)^m m! a_{0,m} \left((m+1) \left\{ {n \atop m+1} \right\} + \left\{ {n \atop m} \right\} \right) \\
&= -\sum_{m=0}^{n} (-1)^m m! \left\{ {n+1 \atop m+1} \right\} a_{0,m}.
\end{aligned}$$

(We have used the recursion $\left\{ {n+1 \atop m+1} \right\} = (m+1)\left\{ {n \atop m+1} \right\} + \left\{ {n \atop m} \right\}$.) This proves the proposition.

Proof of Theorem 1. We show the generating series of the right hand side coincide with that of B_n. To do this, we use the identity

$$\frac{e^x(e^x-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n+1 \atop m+1} \right\} \frac{x^n}{n!} \tag{3}$$

which results from the well-known generating series for the Stirling numbers (*cf.* [5, (7.49)])

$$\frac{(e^x-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!}$$

by replacing m with $m+1$ and differentiating with respect to x. With this, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{(-1)^m m!\left\{{n+1 \atop m+1}\right\}}{m+1}\right)\frac{x^n}{n!}\\
&= \sum_{m=0}^{\infty}\frac{(-1)^m m!}{m+1}\sum_{n=m}^{\infty}\left\{{n+1 \atop m+1}\right\}\frac{x^n}{n!} = \sum_{m=0}^{\infty}\frac{(-1)^m m!}{m+1}\frac{e^x(e^x-1)^m}{m!}\\
&= e^x\sum_{m=0}^{\infty}\frac{(1-e^x)^m}{m+1} = \frac{e^x}{1-e^x}\sum_{m=1}^{\infty}\frac{(1-e^x)^m}{m}\\
&= \frac{e^x}{1-e^x}\left(-\log\left(1-(1-e^x)\right)\right) = \frac{xe^x}{e^x-1}.
\end{aligned}
$$

This proves Theorem 1.

Remark. A referee suggested the following interpretation of the algorithm using generating function:

Suppose the first row is $a_0, a_1, a_2, \dots$, with ordinary generating function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Let the leading diagonal be $b_0 = a_0, b_1, b_2, \dots$, with exponential generating function

$$\mathbb{B}(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}.$$

Then we have

$$\mathbb{B}(x) = e^x A(1-e^x).$$

This follows from (2) and (3), the calculation being parallel to that of the proof of Theorem 1. To get the Bernoulli numbers we take $a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{3}, \dots$ with $A(x) = -\log(1-x)/x$, and find $\mathbb{B}(x) = xe^x/(e^x-1)$.

3 Poly-Bernoulli numbers

If we replace the initial sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ by $1, \frac{1}{2^k}, \frac{1}{3^k}, \frac{1}{4^k}, \dots$ and apply the same algorithm, the resulting sequence is $(-1)^n D_n^{(k)}$ $(n = 0, 1, 2, \dots)$, where $D_n^{(k)}$ is a variant of "poly-Bernoulli numbers" ([6], [2], [3]): For any integer k, we define a sequence of numbers $D_n^{(k)}$ by

$$\frac{Li_k(1-e^{-x})}{e^x-1} = \sum_{n=0}^{\infty} D_n^{(k)} \frac{x^n}{n!},$$

where $Li_k(t) = \sum_{m=1}^{\infty} \frac{t^m}{m^k}$ (k-th polylogarithm when $k \geq 1$). The above assertion is then a consequence of the following (or, is just a special case of the preceding remark)

Proposition 3 *For any $k \in \mathbf{Z}$ and $n \geq 0$, we have*

$$D_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n+1 \atop m+1} \right\}}{(m+1)^k}.$$

Proof. The proof can be given completely in the same way as the proof of Theorem 1 using generating series, and hence will be omitted.

Acknowledgements

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(The Bernoulli numbers are A027641/A027642. The table in Figure 1 yields sequences A051714/A051715. Other sequences which mention this paper are A000367, A002445, A026741, A045896, A051712, A051713, A051716, A051717, A051718, A051719, A051720, A051721, A051722, A051723.)

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