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Combinatorial Interpretations of a Generalization of the Genocchi Numbers

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Abstract

We consider a natural generalization of the well-studied Genocchi numbers first proposed by Han. This generalization proves useful in enumerating the class of deterministic finite automata (DFA) that accept a finite language, and in enumerating a generalization of permutations counted by Dumont.

1 Introduction and Motivation

The study of Genocchi numbers and their combinatorial interpretations has received much attention [6, 7, 8, 9, 10, 12, 15]. In this paper, we consider combinatorial interpretations of a generalization of the Genocchi numbers due to Han [13].

The Genocchi numbers G_{2n} $(n \geq 1)$ may be defined in terms of the generating function

$$\frac{2t}{e^t + 1} = t + \sum_{n \geq 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}.$$

They may also be defined in the following way [12, 2, 15]. Let the Gandhi polynomials $A(n, X)$ be polynomials in X defined as follows:

$$\begin{aligned} A(n+1, X) &= X^2 A(n, X+1) - (X-1)^2 A(n, X) \quad \forall n \geq 1, \\ A(1, X) &= X^2 - (X-1)^2. \end{aligned}$$

Then $|G_{2n}| = A(n-1, 1)$. The first few values of $|G_{2n}|$ are 1,1,3,17,155 for $n = 1, 2, 3, 4, 5$, respectively.

Our motivation comes from automata theory. We are interested in the number of finite languages recognized by deterministic finite automata (DFAs) with n states. It is easy to see that if a DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepts a finite language (see Section 2 for definitions), then there exists an ordering of the elements of the set of states Q, say $Q = \{0, 1, 2, \ldots, n\}$ with $q_0 = 0$ such that $\delta(i, a) > i$ for all $i \in Q - \{n\}$ and $a \in \Sigma$ and $\delta(n, a) = n$ for all $a \in \Sigma$. Thus, we study directed graphs with labeled edges on n vertices that satisfy these properties.

In this paper, we consider an extension of the Genocchi numbers due to Han [13], and show its relation to enumerating DFAs accepting finite languages.

2 Definitions and Background

We first recall some definitions from automata theory and formal languages. For any terms not covered here, the reader may consult Hopcroft and Ullman [14] or Yu [17]. Let Σ denote a finite alphabet. Then Σ^* is the set of all finite strings over Σ. The empty string is denoted by ϵ. A *language* L over Σ is a subset of Σ^*. A *deterministic finite automaton* (DFA) is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is a finite alphabet of symbols, $q_0 \in Q$ is the initial state and $F \subseteq Q$ is a set of final states. The transition function δ is a function $\delta : Q \times \Sigma \to Q$ that may be extended to $Q \times \Sigma^* \to Q$ in the following manner: For all states $q \in Q$, $\delta(q, \epsilon) = q$. Further, for any $w \in \Sigma^*$ and $a \in \Sigma$, $\delta(q, wa) = \delta(\delta(q, w), a)$ for all states $q \in Q$.

A string $w \in \Sigma^*$ is accepted by M if $\delta(q_0, w) \in F$. The language accepted by a DFA M is the set of all strings accepted by M, denoted by $L(M)$:

$$L(M) = \{w \in \Sigma^* \ : \ \delta(q_0, w) \in F\}.$$

We say that a DFA M accepts a language $L \subseteq \Sigma^*$ if $L = L(M)$. In this paper, we are concerned primarily with finite languages, that is, those L with $|L| < \infty$. We use the notation $[n]$ to denote the set $\{1, 2, 3, \ldots, n\}$.

We now proceed with the generalization of the Genocchi numbers due to Han [13]. We define them in terms of a natural generalization of the Gandhi polynomials:

Definition 2.1 *Let $A^{(k)}_{n+1}(X)$ be the following Gandhi polynomials in X:*

$$\begin{aligned} A^{(k)}_{n+1}(X) &= X^k A^{(k)}_n(X+1) - (X-1)^k A^{(k)}_n(X) \quad \forall n \geq 0 \\ A^{(k)}_0(X) &= 1. \end{aligned} \tag{2.1}$$

Define the k-th generalized Genocchi numbers $\{G^{(k)}_{2n}\}_{n \geq 1}$ by $G^{(k)}_{2n} = A^{(k)}_{n-1}(1)$.

Figure 2.1 gives values of $G^{(k)}_{2n}$ for small values of k. The sequence $G^{(2)}_{2n}$ appears as A001469 in Sloane [16]. The sequences $G^{(3)}_{2n}$, $G^{(4)}_{2n}$ and $G^{(5)}_{2n}$ appear in Sloane as A064624, A064625 and A065756, respectively.

Han [13, Thm. 3] studied polynomials that are even more general than those given in Definition 2.1, but the definition of $A^{(k)}_n$ will suffice in what follows. We now apply the results of Han to the specific case of the polynomials $A^{(k)}_n$.

n	1	2	3	4	5	6	7
$G_{2n}^{(2)}$	1	1	3	17	155	2073	38227
$G_{2n}^{(3)}$	1	1	7	145	6631	566641	81184327
$G_{2n}^{(4)}$	1	1	15	1025	209134	100482849	97657699279
$G_{2n}^{(5)}$	1	1	31	6721	5850271	15060446401	94396946822431

Figure 2.1: Small values of $G_{2n}^{(k)}$.

2.1 Properties of $G_{2n}^{(k)}$

Following Dumont [6], we first show some algebraic properties of $G_{2n}^{(k)}$. First, we translate the polynomials $A_n^{(k)}$ as follows. Define the polynomials $B_k(X,n)$ in X as

$$B_k(X,n) = X^k A_{n-1}^{(k)}(X+1) \tag{2.2}$$

for any $n \geq 1$. Then (2.1) becomes

$$\begin{aligned} B_k(X,n) &= X^k(B_k(X+1,n-1) - B_k(X,n-1)) \\ B_k(X,1) &= X^k. \end{aligned} \tag{2.3}$$

We may compare this with the work of Dumont [6, Eq. (3), p. 323]. Let

$$B_k(X,n) = \sum_{j=0}^{(k-1)n+1} B_{n,j}^{(k)} X^j. \tag{2.4}$$

It is easy to see that $B_k(X,n)$ is a polynomial of degree $(k-1)n+1$ in X. Now equating coefficients in (2.3) gives the recurrence:

$$B_{n,j}^{(k)} = \sum_{\ell=j-k+1}^{(n-1)(k-1)+1} \binom{\ell}{j-k} B_{n-1,\ell}^{(k)} \tag{2.5}$$

for $j \geq k$. Iterating (2.5) gives us our most useful definition:

Lemma 2.1 *For all $n \geq 2$, and $k \geq 2$,*

$$B_{n,j}^{(k)} = \sum \binom{k}{j_1}\binom{2k-j_1}{j_2-j_1}\binom{3k-j_2}{j_3-j_2}\cdots\binom{k(n-1)-j_{n-2}}{j_{n-1}-j_{n-2}}, \tag{2.6}$$

where the sum is taken over all integers $j_1, j_2, \ldots, j_{n-1}$ satisfying $1 \leq j_1 < j_2 < \cdots < j_{n-2} < j_{n-1} \leq kn - j$ and $j_i \leq ki$ for all $1 \leq i \leq n-2$.

Proof. The proof is by induction on n. To prove the base case, note that for all k,

$$\begin{aligned} B_k(X,2) &= X^k(X+1)^k - X^{2k} \\ &= \sum_{\ell=0}^{k-1} \binom{k}{\ell} X^{\ell+k}, \end{aligned}$$

whence $B_{2,j}^{(k)} = \binom{k}{k-j}$. Thus, the condition holds for $n=2$. Let the formula hold for $n-1$ and for all j. Then we have

$$\begin{aligned} B_{n,j}^{(k)} &= \sum_{\ell=j-k+1}^{n(k-1)} \binom{\ell}{j-k} B_{n-1,\ell}^{(k)} \\ &= \sum_{\ell=j-k+1}^{n(k-1)} \binom{\ell}{\ell-j+k} \sum \binom{k}{j_1}\binom{2k-j_1}{j_2-j_1}\cdots\binom{k(n-2)-j_{n-3}}{j_{n-2}-j_{n-3}}. \end{aligned}$$

If we now choose $j_{n-1} = k(n-1)+k-j$ we will see that the conditions on the j_i are satisfied. This gives $B_{n,j}^{(k)} = \sum \binom{k}{j_1}\binom{2k-j_1}{j_2-j_1}\binom{3k-j_2}{j_3-j_2}\cdots\binom{k(n-1)-j_{n-2}}{j_{n-1}-j_{n-2}}$. ■

We now note that $G_{2n}^{(k)}$ is given by $G_{2n}^{(k)} = B_{n,k}^{(k)}$, which follows directly from the definition and the translation given by (2.2). Thus, we note the formula

$$G_{2n}^{(k)} = \sum_{\ell=1}^{(n-1)(k-1)+1} B_{n-1,\ell}^{(k)}. \tag{2.7}$$

This follows from (2.5). Tables of $B_{n,j}^{(k)}$ are given in Appendix A. It will be useful to rewrite (2.6) when $j=k$ to give an expression for $G_{2n}^{(k)}$ as follows:

$$\begin{aligned} &\sum_{i_1=1}^{k} \binom{k}{i_1} \sum_{i_2=1}^{2k-i_1} \binom{2k-i_1}{i_2} \sum_{i_3=1}^{3k-(i_1+i_2)} \binom{3k-(i_1+i_2)}{i_3} \cdots \\ &\cdots \sum_{i_j=1}^{kj-\sum_{\ell=1}^{j-1} i_\ell} \binom{kj-\sum_{\ell=1}^{j-1} i_\ell}{i_j} \cdots \sum_{i_{n-2}=1}^{k(n-2)-\sum_{\ell=1}^{n-3} i_\ell} \binom{k(n-2)-\sum_{\ell=1}^{n-3} i_\ell}{i_{n-2}}. \end{aligned} \tag{2.8}$$

We obtain this through the change of of variables $i_1 = j_1$ and $i_\ell = j_\ell - j_{\ell-1}$ for $2 \le \ell \le n-3$. Equation (2.8) will prove particularly useful for our enumeration of automata, which we investigate in Section 3.

2.2 Generalized Central Factorial Numbers

Definition 2.2 *Define $T_k(n,i)$ for all $k \ge 2$, $n \ge 1$ and all integers i as follows:*

$$\begin{aligned} T_k(1,1) &= 1, \\ T_k(n,i) &= 0 \quad \forall i \notin [n], \\ T_k(n,i) &= i^k T_k(n-1,i) + T_k(n-1,i-1) \quad \forall i \in [n]. \end{aligned}$$

When $k = 2$, Definition 2.2 gives the *central factorial numbers* (see Carlitz and Riordan [3]) and used in the proof of the equivalence of the Genocchi numbers and the Gandhi polynomials by Carlitz [2] and Riordan and Stein [15].

By Han [13, p. 7], we can relate $T_k(n, i)$ with the generalized Genocchi numbers, which is of independent interest (compare with Riordan and Stein [15, Eq. (2), p. 382]):

$$G_{2n+2}^{(k)} = \sum_{\ell=1}^{n} (-1)^{\ell+1} (\ell!)^k T_k(n, \ell). \tag{2.9}$$

Figure 2.2 gives the value of $T_3(n, i)$ for $1 \le i \le n \le 7$. Note that expressions for $T_k(n, i)$

$n \setminus i$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	9	1				
4	1	73	36	1			
5	1	585	1045	100	1		
6	1	4681	28800	7445	255	1	
7	1	37449	782281	505280	35570	441	1

Figure 2.2: Small values of $T_3(n, i)$

numbers are given by Comtet [4] and Bach [1] as generalizations of Stirling numbers.

3 Combinatorial Interpretations

In this section, we discuss some combinatorial interpretations of the generalized Genocchi numbers, as well as the generalization of the central factorial numbers given in Section 2.2. These interpretations includes a new graph theoretic combinatorial interpretation for the standard Genocchi numbers.

3.1 Quasi-Permutations

Consider the following definition [7, p. 306]:

Definition 3.1 *A set $P \subseteq [n] \times [n]$ is a quasi-permutation of $[n]$ if there exists a permutation p of $[n]$ such that P is a subset of the following set*

$$\{(i, p(i)) \ : \ i \in [n], p(i) > i\}.$$

Let $|P|$ denote the cardinality of P as a set. For any subset $P \subseteq [n] \times [n]$, let $Y(P) = \{i \ : \exists i'$ such that $(i', i) \in P\}$, the projection of P on the second component.

We can generalize a theorem of Dumont [7, Thm. 1, p. 309] (which is itself inspired by a theorem of Foata and Schützenberger [11, Prop. 2.8., p. 38]) concerning combinatorial interpretations of the central factorial numbers as follows:

Theorem 3.2 *The quantity $T_k(n,i)$ is equal to the number of k-tuples of quasi-permutations of $[n]$ $(Q_1, Q_2, \ldots, Q_k)$ such that*

- $|Q_j| = n - i$ *for all* j *with* $1 \leq j \leq k$
- *for all* $1 \leq j, j' \leq k$, $Y(Q_j) = Y(Q_{j'})$.

Proof. The proof is a simple generalization of the proof of Dumont [7, Thm. 1, p. 309]. ■

Note that a simple calculation will show that the result of Dumont concerning tuples of permutations [7, Thm. 2, p. 310] does not generalize to k-th generalized Genocchi numbers.

3.2 Finite language DFAs over 2 letters

We start by defining a set of directed graphs that will be of interest:

Definition 3.3 *Let $\mathcal{G}_{n,k}$ define the set of digraphs satisfying the following conditions: For all $G = (V, E) \in \mathcal{G}_{n,k}$,*

(a) There are n vertices, labeled with integers from the set $[n]$.

(b) The edges of E are labeled with integers from the set $[k]$. Thus an edge of E is given by an element of $[n] \times [k] \times [n]$.

(c) All the edges of E are directed and satisfy the following: if $e = (u, a, v) \in E$ and $u \neq n$ then e is directed from u to v and $u < v$. If $u = n$ then necessarily $v = n$.

(d) G is initially connected, that is, for each vertex v, there exists a directed path from 1 to v.

(e) For each vertex v and each integer i $(1 \leq i \leq k)$, there exists an edge with source v and label i.

Given (2.8), we can prove the following:

Theorem 3.4 *For all $n \geq 1$, $|\mathcal{G}_{n,2}| = G_{2n}$.*

Proof. The sum given in (2.8) represents the number of ways of connecting each of the vertices $2, \ldots, n$ with a lower numbered vertex. We can see this as follows. Consider vertex 2. In order for vertex 2 to be connected to vertex 1, at least one of the 2 edges leaving vertex 1 must enter vertex 2. We let i_1 of them enter 2, and account for all possible combinations.

Now for vertex 3, at least 1 of the $4 - i_1$ edges leaving vertex 1 and 2 that have yet to be assigned must enter vertex 3; let i_2 of them enter vertex 3.

We continue this process for the first $n-1$ vertices. The result is the sum (2.8). The vertex n is initially connected since by definition all edges leaving vertex $n-1$ must enter vertex n. ■

We can also give a direct proof of Theorem 3.4. A *surjective step function (SSF) of size* $2n$ is a function $f : [2n] \to [2n]$ such that $f(i) \geq i$, and the image of f is exactly $\{2, 4, 6, \ldots, 2n\}$. The following is due to Dumont [7]:

Theorem 3.5 *The number of surjective step functions of size* $2n$ *is* $G_{2(n+1)}$.

We show a bijection between all SSFs of size $2(n-1)$ and $\mathcal{G}_{n,2}$. Let $f : [2n-2] \to [2n-2]$ be a surjective step function of size $2(n-1)$. Then define the graph $G_f = (V_f, E_f)$ as follows: $V_f = [n]$, and

$$
\begin{aligned}
E_f &= \{(n, a, n) \;:\; a \in \{1, 2\}\} \\
&\cup \{(i, 1, \frac{f(2i)}{2} + 1) \;:\; 1 \leq i < n\} \\
&\cup \{(i, 2, \frac{f(2i-1)}{2} + 1) \;:\; 1 \leq i < n\}.
\end{aligned}
$$

Thus, we have a direct bijection demonstrating Theorem 3.4. We now return to our motivation: DFAs that recognize finite languages. The following lemma [5, Prop. 18] states that the underlying structure of DFAs that accept finite languages corresponds exactly to $\mathcal{G}_{n,2}$:

Lemma 3.1 *Let M be a minimal n-state DFA with $L(M)$ finite. Then M is isomorphic (up to renaming of the states) to a DFA $M' = (Q, \Sigma, \delta, q_0, F)$ satisfying $Q = \{q_0, q_1, \ldots, q_{n-1}\}$ and the following conditions:*

(a) $\delta(q_{n-1}, a) = q_{n-1}$ *for all* $a \in \Sigma$.

(b) If $n \geq 2$ *then* $\delta(q_{n-2}, a) = q_{n-1}$ *for all* $a \in \Sigma$.

(c) $q_{n-1} \notin F$.

(d) If $n \geq 2$, *then* $q_{n-2} \in F$.

(e) If $\delta(q_i, a) = q_j$ *for* $i < n-1$ *then* $i < j$.

Thus, we may give an upper bound for the number of distinct DFAs on n states accepting a finite language. Adding final states in all possible ways (subject to $q_{n-1} \notin F$ and $q_{n-2} \in F$), we have the following corollary of Theorem 3.4:

Corollary 3.6 *The number of finite languages over a two letter alphabet accepted by a DFA with* n *states is at most* $2^{n-2}G^{(2)}_{2n}$.

Unfortunately, this bound is not tight. This is due to the fact that many of the languages recognized by distinct labeled DFAs will be the same, and what is needed is an unlabeled enumeration of DFAs.

3.3 k-th Surjective step functions

We may adapt the combinatorial interpretation of the Genocchi numbers in terms of surjective step functions, due to Dumont [7], to generalized Genocchi numbers:

Definition 3.7 *A k-th surjective step function (k-SSF) of size kn is an increasing surjective function $f : [kn] \to \{k, 2k, 3k, \dots, kn\}$.*

The following is a corollary of Han [13].

Theorem 3.8 *There are $G^{(k)}_{2(n+1)}$ k-SSFs of size kn.*

3.4 Finite language DFAs over k letters

The argument of Theorem 3.4 can be easily extended to graphs over a k letter alphabet. In fact, if we repeat the same argument we get the following result:

Theorem 3.9 $|\mathcal{G}_{n,k}| = G^{(k)}_{2n}$.

This allows us to extend our upper bound to automata over arbitrary sized alphabets:

Corollary 3.10 *The number of finite languages over a k letter alphabet accepted by a DFA with n states is at most $2^{n-2}G^{(k)}_{2n}$.*

We may also extend the isomorphism between $\mathcal{G}_{n,2}$ and 2-SSFs of size $2(n-1)$ to $\mathcal{G}_{n,k}$ and k-SSFs of size $k(n-1)$. Let f be a k-SSF of size $k(n-1)$. Then define $G_f = (V_f, E_f)$ as follows: $V_f = [n]$, and

$$\begin{aligned} E_f &= \{(n, a, n) \; : \; 1 \le a \le k\} \\ &\cup \; \{(i, a, \frac{f(ki-a+1)}{k} + 1) \; : \; 1 \le i < n, 1 \le a \le k\}. \end{aligned}$$

3.5 Generalizations of Dumont Permutations

Dumont has also given several interpretations of the usual Genocchi numbers $G^{(2)}_{2n}$ in terms of permutations, including the following theorem [7, Thm. 5, p. 315]:

Theorem 3.11 *Let P_{2n} be the set of permutations π of $[2n]$ such that $\pi(i) \ge i$ iff $\pi(i)$ is even. Then $|P_{2n}| = G^{(2)}_{2n+2}$.*

By a direct application of a result due to Dumont [7, Thm. 4, p. 313] and Theorem 3.8, we can generalize Theorem 3.11 naturally:

Theorem 3.12 *Let P_{kn} be the set of permutations π of $[kn]$ such that $\pi(i) \ge i$ iff $\pi(i) \equiv 0 \,(mod\ k)$. Then $|P_{kn}| = G^{(k)}_{2n+2}$.*

Example 3.13 *Consider $k = 3$, $n = 2$. Thus, we are concerned with permutations π of $[6]$ such that $\pi(i) \geq i$ if and only if $\pi(i) \equiv 0 \,(mod\, 3)$. Then of the 720 permutations, we find that the following permutations satisfy our conditions:*

$$\begin{array}{llll} (1\,3\,2)(4\,6\,5) & (1\,3\,6\,5\,4\,2) & (1\,3)(2\,6\,5\,4) & (1\,3\,2\,6\,5\,4) \\ (1\,6\,5\,4\,2)(3) & (1\,6\,5\,4\,2\,3) & (1\,6\,5\,4)(2\,3) & \end{array}$$

This agrees with $G^{(3)}_{2n+2} = 7$.

Dumont also gives the following corollary to Theorem 3.11 [7, Cor. 1,p. 316]:

Theorem 3.14 *Let P'_{2n} be the set of permutations π of $[2n]$ such that $\pi(i) \geq i$ iff i is odd. Then $|P'_{2n}| = G^{(2)}_{2n+2}$.*

However, the ability to conclude Theorem 3.14 from Theorem 3.11 does not naturally generalize to permutations of $[kn]$ for $k > 2$.

4 Conclusions

In this paper, we have considered a new generalization of the Genocchi numbers. This generalization has proved useful in our attempts to enumerate the number of finite languages recognized by DFAs with n states.

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Appendix A: Tables

The triangle $B^{(3)}_{n,j}$ is A065747 in Sloane [16]. The column $B^{(3)}_{n,4}$ is A065753. These are given in Figure 4.3. The triangle $B^{(4)}_{n,j}$ is A065748 in Sloane, while column $B^{(4)}_{n,5}$ is A065754. These are given in Figure 4.4. Figure 4.5 gives the triangle $B^{(5)}_{n,j}$, which is A065755. Column $B^{(5)}_{n,6}$ is given by A065757 in Sloane.

$n \setminus j$	3	4	5	6	7	8	9	10	11
1	1								
2	1	3	3						
3	7	30	51	42	15				
4	145	753	1656	1995	1410	567	105		
5	6631	39048	100704	149394	140475	86562	34566	8316	945

Figure 4.3: Values of $B_{n,j}^{(3)}$ for $1 \le n \le 5$ and $1 \le j \le 11$.

$n \setminus j$	4	5	6	7	8	9	10	11	12	13
1	1									
2	1	4	6	4						
3	15	88	220	304	250	120	28			
4	1025	7308	23234	43420	52880	43880	25088	9680	2340	280

Figure 4.4: Values of $B_{n,j}^{(4)}$ for $1 \le n \le 4$ and $1 \le j \le 13$.

$n \setminus j$	5	6	7	8	9	10	11	12	13
1	1								
2	1	5	10	10	5				
3	31	230	755	1440	1760	1430	770	260	45

Figure 4.5: Values of $B_{n,j}^{(5)}$ for $1 \le n \le 3$ and $1 \le j \le 13$.

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(Concerned with sequences A001469 A064624 A064625 A065747 A065748 A065753 A065754 A065755 A065756 and A065757 .)

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