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A New Fibonacci-like Sequence of Composite Numbers

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Abstract

We find a new Fibonacci-like sequence of composite numbers and reduce the current record for the starting values to $A_0 = 106276436867$ and $A_1 = 35256392432$.

In this note we consider Fibonacci-like sequences, i.e., the sequences $\{A_n\}$ satisfying the recurrence relation

$$A_n = A_{n-1} + A_{n-2}, \qquad n \geq 2. \tag{1}$$

Ronald Graham [1] proved that there exist relatively prime positive integers a and b such that the sequence $\{A_n\}$ defined by (1) and the initial conditions

$$A_0 = a, A_1 = b \tag{2}$$

contains no prime number. He also provided an example of such a pair (a, b). Both numbers had 34 decimal digits. Donald Knuth [2] described an improvement of Graham's method and found a 17-digit pair. Soon after, Herbert Wilf [4] discovered a smaller 17-digit pair.

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Knuth's paper [2] ends with a challenge asking for substantially smaller starting values, say fewer than ten digits each. Although this bound has not been achieved yet, John Nicol [3] made a significant progress. Developing Knuth's idea and using an extensive computer search he found the following 12-digit pair:

$$a = 407389224418, \quad b = 76343678551.$$

The aim of this note is to provide a new Fibonacci-like sequence with even smaller initial values. We prove

Theorem 1. *Let $\{A_n\}$, $n \geq 0$, be defined by (1)–(2) with the following relatively prime initial values:*

$$a = 106276436867, \quad b = 35256392432.$$

Then $\{A_n\}$ contains no prime number.

Proof. The idea of the proof in its essence is Knuth's modification of Graham's method. First, we find 17 quadruples (p_i, m_i, r_i, c_i) with the following properties:

(i) each p_i is a prime;

(ii) p_i divides F_{m_i}, the m_i-th Fibonacci number;

(iii) the congruences
$$x \equiv r_i \pmod{m_i} \tag{3}$$
cover the integers, i.e., for any integer x there is some i, $1 \leq i \leq 17$, such that $x \equiv r_i \pmod{m_i}$.

Next, we define a and b as follows:

$$\begin{aligned} a &\equiv c_i F_{m_i - r_i} \pmod{p_i}, \quad i = 1, \ldots, 17; \\ b &\equiv c_i F_{m_i - r_i + 1} \pmod{p_i}, \quad i = 1, \ldots, 17. \end{aligned}$$

In particular,
$$A_n \equiv c_i F_{n + m_i - r_i} \pmod{p_i}.$$

Using (ii) and the divisibility property $F_m \mid F_{mk}$ we deduce that p_i divides A_n if $n \equiv r_i \pmod{m_i}$ Since A_n is an increasing sequence for $n \geq 1$ and all prime divisors p_i are relatively small, condition (iii) implies that $\{A_n\}$ contains only composites.

The role of the parameters c_i is to minimize the solution corresponding to a given covering system. They were found by a method similar to that described by Knuth [2].

Now we list the desired quadruples. We arrange them into three groups, namely,

$(3, 4, 3, 2)$	$(2, 3, 1, 1)$	$(5, 5, 4, 2)$
$(7, 8, 5, 3)$	$(17, 9, 2, 5)$	$(11, 10, 6, 6)$
$(47, 16, 9, 34)$	$(19, 18, 14, 14)$	$(61, 15, 12, 29)$
$(23, 24, 17, 6)$	$(107, 36, 8, 19)$	$(31, 30, 0, 21)$
$(1103, 48, 33, 9)$	$(181, 90, 80, 58)$	$(41, 20, 18, 11)$
	$(541, 90, 62, 185)$	$(2521, 60, 48, 306)$

As these quadruples are listed, the rest of the proof is a straightforward verification. Notice that the least common multiple of m_i, $i = 1, \dots, 17$, equals 720. Thus, to prove (iii) it is enough to check that any number between 1 and 720 is covered by at least one congruence (3). $\square$

Graham [1], Knuth [2] and Wilf [4] used similar covering systems with primes 2207, 1087, 4481, 53, 109, 5779 instead of 23, 1103, 107, 181, 541, while Nicol's construction involved primes 53, 109, 5779 instead of 107, 181, 541. We notice that the above congruences in our first group cover all integers $x \equiv 3, 5 \pmod 6$, and the congruences in the third group cover all integers $x \equiv 0 \pmod 6$. The second group covers $x \equiv 1, 4 \pmod 6$, but in contrast to the previous covering systems the second group itself is not enough to cover all integers $x \equiv 2 \pmod 6$. However, our congruences (3) based on the three groups all together still cover the set of integers.

The interesting problem is how far from the optimal solution we are. It is possible to keep primes p_i and vary residues r_i still preserving condition (iii). Further analysis of these cases gives no better choice of (a, b). However, it might be possible to find a smaller pair (a, b) exploiting a covering system based on another set of primes.

References

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