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On pointwise conjugate homomorphisms of compact Lie groups

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Abstract. The question whether pointwise conjugate homomorphisms of compact Lie groups are conjugate is examined. The relation of this problem to maps of classifying spaces is explained. Examples are computed.

1. Introduction

The results in this paper were obtained jointly with Agnieszka Bojanowska (Warsaw University) (cf. [2]).

1.1. For G a Lie group, it is known that there exists a principal G-bundle $EG \to BG$ with the following property. Suppose X is a paracompact space, and $f{:}\,X \to BG$ is a continuous map. Then the assignment $f \mapsto f^*EG$, where f^*EG is the induced bundle over X, defines a bijection between the set of homotopy classes of maps $X \to BG$ and the set of principal G-bundles over X. The space BG is called the classifying space of G (see e.g. [4]). We should mention that B is a functor.

1.2. Throughout the paper assume that H and G are compact Lie groups, that G is connected, and that all maps $H \to G$ we consider are smooth homomorphisms. One wants to describe the set $[BH, BG]$ of homotopy classes of maps $BH \to BG$. We try to obtain information about this set without handling the classifying spaces themselves since in general they are very complicated objects. Since B is a functor, we have a natural function $\mathrm{Hom}(H,G) \to [BH, BG]$. Let f, $g{:}\,H \to G$ be homomorphisms which are conjugate: $f(h) = xg(h)x^{-1}$. Let γ_t be a path in G connecting x with the identity. Then $B(\gamma_t g \gamma_t^{-1})$ is a homotopy between Bf and Bg. Thus the function above factorizes through the set $\mathrm{Rep}(H,G)$ of conjugacy classes of homomorphisms $H \to G$. On the other hand, for any cohomology theory $\mathcal{H}^*$, each homotopy class of maps $BH \to BG$ induces a ring homomorphism $\mathcal{H}^*BG \to \mathcal{H}^*BH$. So we may ask whether the superposition

$$\mathrm{Rep}(H,G) \to [BH, BG] \to \mathrm{Hom}(\mathcal{H}^*BG, \mathcal{H}^*BH) \tag{1}$$

is injective.

1.3. There arises the question which cohomology theory will be suitable. We make use of a theorem which says that $R(G)\hat{}$ and $K^*(BG)$ are naturally isomorphic for any compact Lie group G (not necessarily connected, see [1]).

Here K^* denotes the wellknown $\mathbb{Z}_2$-graduated cohomology theory, and $R(G)\widehat{}$ is the completion of the representation ring of G in the I_G-adic topology, where I_G is the kernel of the dimension homomorphism $R(G) \to \mathbb{Z}$. Hence, setting $\mathcal{H}^* = K^*$ in 1.2, we may replace $\operatorname{Hom}(\mathcal{H}^*BG, \mathcal{H}^*BH)$ by $\operatorname{Hom}(R(G)\widehat{}, R(H)\widehat{})$.

1.4. It is known that the natural map

$$\operatorname{Hom}(H, G) \to \operatorname{Hom}(R(G), R(H))$$

factorizes through $\operatorname{Rep}(H, G)$. Thus we may consider the superposition

$$\operatorname{Rep}(H, G) \to \operatorname{Hom}(R(G), R(H)) \to \operatorname{Hom}(R(G)\widehat{}, R(H)\widehat{}) \tag{2}$$

which is in fact the same one as the composition (1) (modified using 1.3). Thus the problem from 1.2 splits into two questions: whether the first map of (2) is injective, and whether the second map, restricted to the image of the first one, is injective. The answer to the second part depends on whether the natural map $R(H) \to R(H)\widehat{}$ is a monomorphism. Two examples are given in 3.1.

1.5. The rest of this paper is devoted to the question whether

$$\operatorname{Rep}(H, G) \to \operatorname{Hom}(R(G), R(H))$$

is injective. In other words, we ask if two homomorphisms f, $g\colon H \to G$ which induce the same map on representations are conjugate. Note that if $f^* = g^*\colon R(G) \to R(H)$ then f and g induce the same map on characters. But characters separate conjugacy classes, hence we have

$$(\forall h \in H)(\exists x_h \in G)\, f(h) = x_h g(h) x_h^{-1}.$$

In this case we say that f and g are *pointwise conjugate*. So our question is whether the following statement is true:

$(*)$ Any two pointwise conjugate homomorphisms $H \to G$ are conjugate.

2. General results

2.1. Obviously, if H is topologically cyclic (i.e., it has a dense cyclic subgroup) then $(*)$ holds for H and any G.

Proposition 2.2. *If H is connected, f, $g\colon H \to G$ are pointwise conjugate and* $\operatorname{rank} \operatorname{im} f = \operatorname{rank} G$ *then f and g are conjugate.*

Proof. Let T be a maximal torus of H. We can assume that f and g are monomorphisms, and $f|_T = g|_T$. Consider the adjoint representation of $f(T)$ on $L(G)$. Since any two non-trivial irreducible components of $L(G)$ are not isomorphic, we have $L\big(f(H)\big) = L\big(g(H)\big)$, hence $f(H) = g(H)$. Thus $fg^{-1}|_{g(H)} : g(H) \to f(H) = g(H)$ is an automorphism which is the identity on $f(T)$, hence is conjugation by some element of $f(T)$ (see [3], chap.IX, §4, Théorème 9). ∎

2.3. Suppose that $\pi : \widetilde{H} \to H$ is an epimorphism and $(*)$ holds for $\widetilde{H}$ and G. Then $(*)$ holds for H and G.

Proposition 2.4. *Suppose H to be connected and $\pi: \widetilde{G} \to G$ a covering homomorphism. If $(*)$ holds for H and G then $(*)$ holds for H and $\widetilde{G}$. The converse is true for simply connected H.*

Proof. Let $f, g: H \to \widetilde{G}$ be pointwise conjugate. Then πf and πg are pointwise conjugate, hence conjugate: $\pi f = x(\pi g)x^{-1}$ for some element $x \in G$. Let $y \in \widetilde{G}$ be an element of $\pi^{-1}(x)$. Then f and ygy^{-1} agree at the identity and both cover πf. Hence, by the connectedness of H, they are equal. The second part is proved analogously. ∎

2.5. For any G, there is a semisimple normal subgroup K such that $G = Z_0 K$ where Z_0 is the connected component of the identity of the center.

Proposition. *If H is simply connected and $(*)$ holds for H and K, then $(*)$ holds for H and G.*

Proof. The multiplication in G defines a covering $Z_0 \times K \to G$. Apply the previous result. ∎

Remark 2.6. For $G = G_1 \times G_2$ and arbitrary H, $(*)$ holds for H and G iff $(*)$ holds for H and both G_1 and G_2. In this way the problem for pairs with semisimle target group is reduced to the case that G is simple.

2.7. Let G be one of the classical groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$. Suppose f, $g: H \to G$ are pointwise conjugate. The arising representations have the same characters hence they are isomorphic. The isomorphism is given by conjugation by an element x of $\mathrm{SU}(n)$, $\mathrm{O}(n)$ or $\mathrm{Sp}(n)$, respectively. Thus we have the following

Proposition. *$(*)$ holds if $G = \mathrm{U}(n)$, $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$.* ∎

Proposition 2.8. *$(*)$ holds if $G = \mathrm{SO}(2n+1)$.*

Proof. In the situation of 2.7, replace the element $x \in \mathrm{O}(2n+1)$ by $-x$ if $\det x = -1$. ∎

2.9. Again, consider the situation of 2.7 for $G = \mathrm{SO}(2n)$. Obviously, if the element $x \in \mathrm{O}(2n)$ has negative determinant, it may be replaced by some element $y \in \mathrm{SO}(2n)$ iff the representation f has an automorphism with negative determinant.

Proposition 2.10. *A real representation V of the group H has an automorphism with negative determinant iff it has an odd-dimensional component.*

Proof. If V has an odd-dimensional component then $-\mathbf{id}$ on this component and $\mathbf{id}$ on the complement defines the required automorphism. Now suppose φ is an automorphism of V and $\det \varphi = -1$. Since φ preserves the isotypical summands, the determinant must be equal to -1 on one of them. We may assume V is isotypical with irreducible summands W_i. If W_1 is the realization of a complex or quaternionic representation then φ is the realization of a complex or quaternionic map, hence $\det \varphi = 1$. Assume W_1 is "purely real". Then φ determines maps $W_i \to W_j$ which are homotheties with coefficients λ_{ij}. One checks that $\det \varphi = \det[\lambda_{ij}]^{\dim W_1}$ which may be negative only if W_1 is odd-dimensional. ∎

Remark 2.11. We conclude that pairs of pointwise conjugate homomorphisms $f, g\colon H \to \mathrm{SO}(2n)$ are obtained in a unique way: Take a representation

$$f\colon H \to \mathrm{SO}(2n)$$

with the following properties:

(A) All irreducible components of f are even-dimensional.

(B) For any element $h \in H$, $f(h)$ has an eigenvalue equal to 1 or -1.

Define $g = xfx^{-1}$ for any element $x \in \mathrm{O}(2n) \setminus \mathrm{SO}(2n)$. Then f and g are pointwise conjugate but not conjugate.

Proof. By (A) and 2.10, f and g are not conjugate. The closure of the subgroup generated by an element $h \in H$ is a compact abelian group. Hence f restricted to this subgroup has 1- and 2-dimensional irreducible components only. Condition (B) ensures that there is a 1-dimensional component. So by 2.10, f and g are pointwise conjugate. ∎

Definition 2.12. A representation $f\colon H \to \mathrm{SO}(2n)$ which fulfils conditions (A) and (B) we will shortly call a *counterexample* for H.

Corollary 2.13. *Let H be connected with maximal torus T. Then any counterexample for H is a representation f with even-dimensional components such that $f|_T$ has a trivial component. Moreover, if f is minimal (i.e., such that no proper subrepresentation is a counterexample) then f is irreducible, and the center of $f(H)$ is trivial.*

Proof. The first part follows from the fact that T intersects all conjugacy classes of H. If $f = f_1 \oplus \cdots \oplus f_k$ is a counterexample then all f_i are even-dimensional, and $f|_T$ has a trivial component, hence for some i, $f_i|_T$ has a trivial component. Consequently, one of the f_i is a counterexample. Now let f be a minimal counterexample, and suppose h is an element of H such that $f(h)$ is central in $f(H)$. Then the 1-eigenspace of $f(h)$ is a subrepresentation which is non-trivial by condition (B). Hence by irreducibility it is the whole space, so $f(h)$ is the unity. ∎

Corollary 2.14. *Let H be finite. Any counterexample for H is a real representation f such that the following conditions are fulfilled.*

(A) *All irreducible components of f are even-dimensional.*

(B) *Let χ denote the character of f, and $o(h)$ is the order of h. Then $\sum_{k=1}^{o(h^2)} \chi(h^{2k}) > 0$ for all $h \in H$.*

(C) *The number $o(h)^{-1} \sum_{k=1}^{o(h)} (-1)^k \chi(h^k)$ is even for all $h \in H$.*

Proof. Conditions (A) and (B) are equivalent to the conditions in 2.11. Condition (C) ensures that the image of f lies in $\mathrm{SO}(2n)$ since the formula yields the dimension of the (-1)-eigenspace of $f(h)$. ∎

Remark 2.15. The preceding corollaries show that we can describe all counterexamples if we know the irreducible characters.

3. Examples

3.1. *Injectivity of completion.* We consider two examples of important classes of groups for which the completion map of the representation ring is a monomorphism. Hence if we take such groups as source group H, the answer to the second part of the problem in 1.4 is positive.

Let H be connected and T a maximal torus. The injectivity of the map $R(H) \to R(H)\hat{}$ follows from the fact that the maps $R(H) \to R(T)$ and $R(T) \to R(T)\hat{}$ are monomorphisms.

Now consider a finite abelian p-group H. Let I_H be the kernel of the dimension homomorphism $R(H) \to \mathbb{Z}$ We have to show that $\bigcap_{n=1}^{\infty} I_H^n$ is trivial. The ideal I_H^n is generated by products of n elements of the form $\chi - 1$, where χ is an irreducible character of H. Note that if $\chi^{p^k} = 1$ then the element $(\chi - 1)^{p^k}$ is divisible by p. Hence the elements of I_H^n are divisible by an arbitrarily high power of p if only n is large enough.

Next we check whether $(*)$ holds for some groups H and $G = \mathrm{SO}(2n)$. First consider connected H.

3.2. *A counterexample.* (cf. [5]) The adjoint representation

$$\mathrm{Ad}\colon \mathrm{SU}(3) \to \mathrm{SO}(8)$$

is a counterexample in the sense of 2.12. It is irreducible, and obviously, any maximal torus $T \subset \mathrm{SU}(3)$ acts trivially on $L(T)$.

Proposition 3.3. *If* $\operatorname{rank} H = 1$ *or* $H = \mathrm{SO}(4)$, *and* $G = \mathrm{SO}(2n)$ *then* $(*)$ *holds.*

Proof. For $\operatorname{rank} H = 1$ we have to check $H = \mathrm{SO}(2)$, $\mathrm{SO}(3)$ and $\mathrm{Sp}(1)$. The circle is topologically cyclic. For $\mathrm{Sp}(1)$, any counterexample would factorize through $\mathrm{SO}(3)$ by 2.13. All irreducible representations of $\mathrm{SO}(3)$ are odd-dimensional. Finally, note that the same holds for $\mathrm{SO}(3) \times \mathrm{SO}(3)$, and $\mathrm{SO}(4)$ maps onto $\mathrm{SO}(3) \times \mathrm{SO}(3)$ by the adjoint representation. The kernel of this map is the center. Apply 2.13. ∎

Now we consider discrete H.

Lemma 3.4. *Finite p-groups. For p an odd prime, H a non-cyclic p-group, and $G = \mathrm{SO}(2n)$, condition $(*)$ holds iff $n \leq p$.*

Proof. First consider the case $H = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Any counterexample is a sum $f = f_1 \oplus \cdots \oplus f_n$ of compositions $f_i = \alpha\varphi_i$, where $\varphi_i\colon \mathbb{Z}_p \oplus \mathbb{Z}_p \to \mathbb{Z}_p$ is a functional and $\alpha\colon \mathbb{Z}_p \to \mathrm{SO}(2)$ is an inclusion. Condition (B) says that $H = \bigcup \ker \varphi_i$. But $\ker \varphi_i$ is a line in the vector space H, and H consists of $p+1$ lines. Thus for $n \leq p$ condition (B) cannot be fulfilled, and $(*)$ holds. For $n = p+1$ we give a counterexample defining $\varphi_k(x,y) = x + ky$ if $k = 1, \ldots, p$, and $\varphi_{p+1}(x,y) = y$ for any pair $(x,y) \in \mathbb{Z}_p \oplus \mathbb{Z}_p$. Now let H be arbitrary. Note that H maps onto $\mathbb{Z}_p \oplus \mathbb{Z}_p$. If H is non-abelian, divide it by its center. The quotient is non-cyclic. Repeat this procedure until you obtain an abelian group. This group maps onto $\mathbb{Z}_p \oplus \mathbb{Z}_p$. The superposition of this epimorphism and the preceding counterexample for $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is a counterexample for H.

Finally, assume $f: H \to \mathrm{SO}(2n)$ and $n \leq p$. There are two cases. If the image of f is abelian then one shows in a fashion similar to the one used for the treatment of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ that f cannot fulfil condition (B). Now suppose $f(H)$ is non-abelian. It follows from representation theory that in this case f is irreducible and $n = p$. If f is a counterexample and some element $f(h)$ is central in $f(H)$, then $f(h)$ has a non-trivial 1-eigenspace by condition (B). On the other hand, this eigenspace is a subrepresentation, hence the whole space. Thus the center of $f(H)$ would be trivial which is impossible. ■

Remark 3.5. If H is a non-cyclic 2-group counterexamples need not necessarily exist. For example, consider the groups $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and Q_8.

But $(*)$ is not fulfilled for groups H which map onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $G = \mathrm{SO}(2n)$, $n \geq 3$. We construct a counterexample $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \to \mathrm{SO}(6)$ as above defining maps $\varphi_i: \mathbb{Z}_4 \oplus \mathbb{Z}_4 \to \mathbb{Z}_4$. Set $\varphi_1(x,y) = x$, $\varphi_2(x,y) = y$, $\varphi_3(x,y) = x + y$.

Remark 3.6. The group $G = \mathrm{SO}(4)$ does not admit counterexamples at all. ■

3.7. *Groups of order pq.* It might seem that for a finite group to admit a counterexample, it must map onto some $\mathbb{Z}_m \oplus \mathbb{Z}_m$. This is not true: Let H be the semidirect product of $\mathbb{Z}_q$ by $\mathbb{Z}_p$, where q and p odd primes. Interpret H as a subgroup of $\mathrm{Aff}(\mathbb{Z}_q)$. The real group ring $E = \mathbb{R}(\mathbb{Z}_q)$ of $\mathbb{Z}_q$ is a representation space for a representation f of H. Obviously, the vector $\sum_{x \in \mathbb{Z}_q} x$ spans an invariant line in E. Let f_1 be the complement of this trivial subrepresentation in f. Furthermore, let g be the superposition of an epimorphism $H \to \mathbb{Z}_p$ and an inclusion $\mathbb{Z}_p \to \mathrm{SO}(2)$. Then $f_1 \oplus g$ is a counterexample. For $h \in H$ with order q, $\chi_f(h) = 0$, and $\chi_f(0) = q$, hence the trivial component of $f(h)$ has dimension $(0 + \cdots 0 + q)/q = 1$. Thus $f_1(h)$ has 2-dimensional components only, so all components of f_1 are even-dimensional. Furthermore, g is 2-dimensional, so condition (A) holds. For $h \in H$ with order p we have $\chi_f(h) \geq 1$ since the action of $\mathbb{Z}_p$ on $\mathbb{Z}_q$ fixes $0 \in \mathbb{Z}_q$. Thus $\chi_{f_1}(h) \geq 0$ for such h, and these elements have a trivial component in f_1. For $h \in H$ with order q, $g(h)$ is trivial. Thus condition (B) is fulfilled.

3.8. *An example how to use the character table.* Consider the permutation group S_5.

Condition (A) is fulfilled for 4- and 6-dimensional representations. Condition (B) holds for the 6-dimensional representation, but in the 4-dimensional representations the 5-cycle has no invariant line. The cycle (12) changes orientation in each of the even-dimensional representations, hence the sum of an odd number of them does not fulfil condition (C). Hence counterexamples for S_5 are sums of 4-and 6-dimensional representations with an even number of irreducible summands in which at least one has dimension 6. Note that there are two minimal counterexamples with characters $\chi_3 + \chi_7$ and $\chi_4 + \chi_7$.

Character table of S_5.
Elements are presented in cycle decomposition

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
id	1	1	4	4	5	5	6
(12)	1	-1	2	-2	1	-1	0
(12)(34)	1	1	0	0	1	1	-2
(123)	1	1	1	1	-1	-1	0
(123)(45)	1	-1	-1	1	1	-1	0
(1234)	1	-1	0	0	-1	1	0
(12345)	1	1	-1	-1	0	0	1

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