Seminar Sophus Lie
2 (1991) 31–38

A note on the oscillator group

Norbert Dörr*

Abstract. In this note we give an alternative parametrization of the oscillator algebra. We determine the semigroup of the oscillator group that is infinitesimally generated by an invariant cone whose exponential image is not dense in the semigroup.

1. An alternative parametrization for the oscillator algebra

Let $M(2)$ be the *universal covering group of the group of Euclidian motions of the plane*. Then $M(2)$ may be realized as the group of 3×3-matrices

$$\begin{pmatrix} e^{ir} & c & 0 \\ 0 & e^{-ir} & 0 \\ 0 & 0 & e^r \end{pmatrix}, \quad r \in \mathbb{R}, c \in \mathbb{C}.$$

If we write (c, r) for these matrices, the multiplication is given by

$$(c, r)\cdot(c', r') = (e^{ir}c' + e^{-ir'}c, r + r'). \tag{1}$$

The corresponding Lie algebra $\mathfrak{m}(2)$ may be identified with $\mathbb{C} \oplus \mathbb{R}$ with Lie bracket

$$[(c, r), (c', s')] = (2i\cdot \det \left(\begin{smallmatrix} r & r' \\ c & c' \end{smallmatrix}\right), 0). \tag{2}$$

We prefer this parametrization of $M(2)$ because all one-parameter groups are "flat", i. e., they are contained in a two-dimensional subspace.

Lemma 1.1. *Let $M(2)$ be the group of Euclidian motions of the plane with multiplication* (1)*. Then the one-parameter groups are given by*

$$\varphi(t) = (\rho\cdot \sin r, r), \tag{3}$$

where $\rho \in \mathbb{C}$.

Proof. This may be verified by direct computation. ■

*The author thanks Prof. Dr. Karl H. Hofmann for his support.

Let H be the *Heisenberg group*. It can be realized as $\mathbb{R}\times\mathbb{C}$ with multiplication

$$(z,c)\cdot(z',c') = \Big(z+z'+\frac{1}{2}\,Im(\bar{c}c'),\, c'+c\Big), \tag{4}$$

where $z,z'\in\mathbb{R},\ c,c'\in\mathbb{C}$. The *Heisenberg algebra* $\mathfrak{h}=L(H)=\mathbb{R}\oplus\mathbb{C}$ has Lie bracket

$$[(z,c),(z',c')] = \Big(Im(\bar{c}c'),0\Big) \tag{5}$$

for $z,z'\in\mathbb{R},\ c,c'\in\mathbb{C}$.

We consider the four-dimensional *oscillator group* O_4 with corresponding Lie algebra $\mathfrak{o}_4 = L(O_4)$. The oscillator group is the semidirect product of $\mathbb{R}$ and the Heisenberg group $H=\mathbb{R}\times\mathbb{C}$, where $\mathbb{R}$ acts by rotation on $\mathbb{C}$. Motivated by the parametrization of $M(2)$ we can realize O_4 in the following way.

Lemma 1.2. *The oscillator group O_4 admits the following parametrization: $O_4=\mathbb{R}\times\mathbb{C}\times\mathbb{R}$ with multiplication*

$$(z,c,r)\cdot(z',c',r') = \Big(z+z'+\frac{1}{2}\,Im(e^{i(r+r')}\bar{c}c'),\, e^{ir}c'+e^{-ir'}c,\, r+r'\Big) \tag{6}$$

for $z,z',r,r'\in\mathbb{R},\ c,c'\in\mathbb{C}$.

The Lie algebra $\mathfrak{o}_4=L(O_4)=\mathbb{R}\oplus\mathbb{C}\oplus\mathbb{R}$ has Lie bracket

$$[(z,c,r),(z',c',r')] = \Big(Im(\bar{c}c'),\, 2i\cdot\det\left(\begin{smallmatrix} r & r' \\ c & c'\end{smallmatrix}\right),\, 0\Big) \tag{7}$$

for $z,z',r,r'\in\mathbb{R},\ c,c'\in\mathbb{C}$. ∎

In what follows, we explicitly compute the exponential function

$$\exp\colon \mathfrak{o}_4 \to O_4$$

of the oscillator group in this parametrization. The shape of the one-parameter-groups in the motion group $M(2)$ motivates the following definition. Set

$$\Phi(t) \overset{def}{=} \Big(\varphi(t),\, c_0\cdot\sin t,\, t\Big),$$

where $c_0\in\mathbb{C}$. By a straightforward computation using (6), we get

$$\begin{aligned}&(\varphi(t),c_0\cdot\sin t,t)\cdot(\varphi(u),c_0\cdot\sin u,u) = \\ &\qquad(\varphi(t)+\varphi(u)+\frac{|c_0|^2}{4}\Big(\sin^2 t\sin(2u)+\sin(2t)\sin^2 u\Big),\, c_0\cdot\sin(t+u),\, t+u).\end{aligned}$$

Thus we get the following functional equation for φ:

$$\varphi(t+u) = \varphi(t)+\varphi(u)+\frac{|c_0|^2}{4}(\sin^2 t\sin(2u)+\sin(2t)\sin^2 u)$$

for all $t,u\in\mathbb{R}$. If we devide both sides by u and let u tend to zero, we get the differential equation

$$\varphi'(t) = \varphi'(0)+\frac{|c_0|^2}{2}\sin^2 t$$

which integrates to

$$\varphi(t) = (\varphi'(0) + \frac{|c_0|^2}{4})\cdot t - \frac{|c_0|^2}{8}\sin(2t).$$

Thus we have

$$\Phi(t) = \Big((z_0 + \frac{|c_0|^2}{4})\cdot t - \frac{|c_0|^2}{8}\sin(2t), c_0 \sin t, t\Big), \tag{8}$$

where $z_0 \in \mathbb{R}$, $c_0 \in \mathbb{C}$, and $\Phi'(0) = (z_0, c_0, 1)$. With these preparations we get the following result.

Proposition 1.3. *The exponential function* $\exp\colon \mathfrak{o}_4 \to O_4$ *of the oscillator group is given by*

$$\exp(z, c, r) = \begin{cases} \Big(z + \frac{|c|^2}{4r} - \frac{|c|^2}{8r^2}\sin(2r), c\cdot\frac{\sin r}{r}, r\Big) & \textit{for } r \neq 0; \\ (z, c, 0) & \textit{for } r = 0. \end{cases}$$

Proof. This follows from (8) where we have only to take care to get the right parameters. It is also sufficient to note that $\frac{d}{dt}|_{t=0} \exp t\cdot(z, c, r) = (z, c, r)$. ∎

2. Invariant wedges in the oscillator algebra

We consider on $\mathfrak{o}_4$ the symmetric, bilinear form $\mathfrak{q}\colon \mathfrak{o}_4 \times \mathfrak{o}_4 \to \mathbb{R}$ defined by

$$q((z, c, r), (z', c', r')) \stackrel{def}{=} rz' + r'z + \frac{1}{2}\, Re(\overline{c}c') \tag{9}$$

for $(z, c, r), (z', c', r') \in \mathfrak{o}_4$. The associated quadratic form is denoted by the same letter, i. e.,

$$q(z, c, r) \stackrel{def}{=} 2rz + \frac{1}{2}|c|^2. \tag{10}$$

Lemma 2.1. *The symmetric, bilinear form* $\mathfrak{q}\colon \mathfrak{o}_4 \times \mathfrak{o}_4 \to \mathbb{R}$ *defined by* (9) *is invariant, i. e., we have* $q([w_1, w_2], w_3) = q(w_1, [w_2, w_3])$ *for* $w_j \in \mathfrak{o}_4$, $j = 1, 2, 3$.

Proof. Let $w_j = (z_j, c_j, r_j) \in \mathfrak{o}_4$, $j = 1, 2, 3$. Then we have

$$\begin{aligned} q([w_1, w_2], w_3) &= q\Big((Im(\overline{c_1}c_2), 2i\cdot \det\left(\begin{smallmatrix} r_1 & r_2 \\ c_1 & c_2 \end{smallmatrix}\right), 0), (z_3, c_3, r_3)\Big) \\ &= s_3\cdot Im(\overline{c_1}c_2) + \tfrac{1}{2}\, Re(\overline{2i\cdot \det\left(\begin{smallmatrix} r_1 & r_2 \\ c_1 & c_2 \end{smallmatrix}\right)}c_3) \\ &= s_3\cdot Im(\overline{c_1}c_2) + s_1\cdot Im(\overline{c_2}c_3) - s_2\cdot Im(\overline{c_1}c_3) \end{aligned}$$

and

$$\begin{aligned} q(w_1, [w_2, w_3]) &= q\Big((z_1, c_1, r_1), (Im(\overline{c_2}c_3), 2i\cdot \det\left(\begin{smallmatrix} r_2 & r_3 \\ c_2 & c_3 \end{smallmatrix}\right), 0)\Big) \\ &= s_1\cdot Im(\overline{c_2}c_3) + \tfrac{1}{2}\, Re(2i\cdot\overline{c_1} \det\left(\begin{smallmatrix} r_2 & r_3 \\ c_2 & c_3 \end{smallmatrix}\right)) \\ &= s_1\cdot Im(\overline{c_2}c_3) + s_3\cdot Im(\overline{c_1}c_2) - s_2\cdot Im(\overline{c_1}c_3) \end{aligned}$$

These two expressions are equal, as required. ∎

Definition 2.2. *Denote by W the* invariant *Lorentz cone defined by the zero-set of the invariant quadratic form q, i. e.,*

$$W \stackrel{def}{=} \{(r,v,z) \in L \mid q(z,c,r) \leq 0, r \geq 0, z \leq 0\}. \tag{11}$$

In what follows, we denote the closed semigroup that is generated by W with

$$S \stackrel{def}{=} \overline{\langle \exp W \rangle}. \tag{12}$$

In the next paragraph we show that S is an infinitesimally generated semigroup with tangent wedge $L(S) = W$. But first, we determine the exponential image $\exp W$ of W. We have

$$(z_0, c_0, 1) \in W \iff z_0 + \frac{1}{4}|c_0|^2 \leq 0. \tag{13}$$

We denote by $\Phi_{z_0,c_0}\colon \mathbb{R} \to O_4$ the one-parameter group generated by $(z_0, c_0, 1)$. Hence we have by Proposition 1.3

$$\begin{aligned} \Phi_{z_0,c_0} &\stackrel{def}{=} \exp t\cdot(z_0, c_0, 1) \\ &= \Big((z_0 + \frac{|c_0|^2}{4})\cdot t - \frac{|c_0|^2}{8}\sin(2t), c_0 \sin t, t\Big). \end{aligned}$$

Since $(z_0, c_0, 1) + \mathbb{R}^-\cdot(1,0,0) \subseteq W$ whenever $(z_0, c_0, 1) \in W$, the boundary of $\exp W$ is just the image of ∂W. Hence we have as "boundary curves" Φ_{c_0} of $\exp W$

$$\Phi_{c_0}(t) = \Big(-\frac{|c_0|^2}{8}\cdot \sin(2t), c_0 \sin t, t\Big), \quad t \geq 0, \tag{14}$$

i. e., the one-parameter semigroups Φ_{z_0,c_0} with $z_0 = -\frac{|c_0|^2}{4}$.

In Figure 1 we see the projection of Φ_{c_0} into $\mathbb{R} \times 0 \times \mathbb{R}$, resp., $0 \times \mathbb{R}\cdot c_0 \times \mathbb{R}$.

Figure 1: *Projection of Φ_{c_0} into* $\mathbb{R} \times 0 \times \mathbb{R}$, *resp.,* $0 \times \mathbb{R}\cdot c_0 \times \mathbb{R}$.

Lemma 2.3. *For the intersection of the exponential image of W with the hyperplane $H_t \overset{def}{=} \mathbb{R} \times \mathbb{C} \times \{t\}$ the following holds.*

(i) For $t = k{\cdot}\pi$, $k \in \mathbb{N}$:

$$(\exp W) \cap H_t = \mathbb{R}^- \times \{0\} \times \{k{\cdot}\pi\}.$$

(ii) For $t \neq k{\cdot}\pi$, $k \in \mathbb{N}$:

$$(\exp W) \cap H_t = \Big\{\Big(-\frac{\sin(2t)}{4(1-\cos(2t))}{\cdot}|\tilde{c}|^2, \tilde{c}, t\Big) \;\Big|\; \tilde{c} \in \mathbb{C}\Big\} + \mathbb{R}^- \times \{0\} \times \{0\}.$$

Proof. The case $t = 0$ is obvious, since $\exp(z, c, 0) = (z, c, 0)$, and $(z, c, 0) \in W$ if and only if $z \leq 0$ and $c = 0$.

By Proposition 1.3, we have for $t = k{\cdot}\pi$, $k \in \mathbb{N} \setminus \{0\}$,

$$\exp(z, c, k{\cdot}\pi) = (z + \frac{|c|^2}{4k{\cdot}\pi}, 0, k{\cdot}\pi).$$

Again, the fact, that $(z, c, k{\cdot}\pi) \in W$ if and only if $z + \frac{|c|^2}{4k\cdot\pi} \leq 0$, proves the assertion in the case $t = k{\cdot}\pi$.

For $(z, c, t) \in W$ we have

$$\exp(z, c, t) = \Big(z + \frac{|c|^2}{4t} - \frac{|c|^2}{8t}\sin(2t), c{\cdot}\frac{\sin t}{t}, t\Big),$$

where $z + \frac{|c|^2}{4t} \leq 0$. Thus we have

$$(\exp W) \cap H_t = \Big\{\Big(-\frac{|c|^2}{8t}\sin(2t), c{\cdot}\frac{\sin t}{t}, t\Big) \;\Big|\; c \in \mathbb{C}\Big\} + \mathbb{R}^- \times \{0\} \times \{0\}.$$

Set $\tilde{c} = c{\cdot}\frac{\sin t}{t}$. Then a straightforward calculation using $\sin^2 t = \frac{1}{2}(1 - \cos(2t))$ proves the assertion. ■

For a fixed $\tilde{c} \in \mathbb{C}$ we identify the two-dimensional subspace $H_{t,\tilde{c}} \overset{def}{=} \mathbb{R} \times \mathbb{R}{\cdot}\tilde{c} \times \{t\} \cap H_t$ with $E \overset{def}{=} \mathbb{R} \times \mathbb{R}{\cdot}\tilde{c}$. Then $(\exp W) \cap H_{t,\tilde{c}}$ for $t \neq k{\cdot}\pi$ is just the region of E below te graph of the function $\{(f(x), x) \mid f(x) = -\frac{\sin(2t)|\tilde{c}|^2}{4(1-\cos(2t))}{\cdot}x^2\}$. For $t = k{\cdot}\pi$ it is the half-ray $\mathbb{R}^1 \times \{0\}$ (see Figure 2).

Figure 2: $(\exp W) \cap H_{t,\tilde{c}}$.

We have choosen the parametrization of O_4 in such a way that the one-parameter group $\exp(z, c, t)$ is entirely contained in the three-dimensional subspace $\mathbb{R} \times \mathbb{R}{\cdot}c \times \mathbb{R}$. Since the image of W under the exponential function is invariant under the action of $Ad(\exp(0, 0, r))$, which induces a rotation in $\{0\} \times \mathbb{C} \times \{0\}$, we can visualize $\exp W$ in $\mathbb{R} \times \mathbb{R}{\cdot}c \times \mathbb{R}$ as in Figure 3.

Figure 3: *Intersection of* $\exp W$ *with* $\mathbb{R} \times \mathbb{R}{\cdot}c \times \mathbb{R}$.

3. The semigroup $S = \overline{\langle \exp W \rangle}$

In this section we consider the closed semigroup $S = \overline{\langle \exp W \rangle}$ generated by W. We show that it is infinitesimally generated with tangent wedge $L(S) = W$. It turns out that $\exp W$ is not dense in S. Indeed, there are interior points in $S \setminus \exp W$. This makes this example so important for the discussion of *divisible* semigroups and Lie semialgebras in [HR].

Lemma 3.1. *Let* $B \subseteq O_4$ *defined by* $B \overset{def}{=} \mathbb{R} \times \mathbb{C} \times]-\pi, \pi[$. *Then the restriction of the exponential map*

$$\exp|_B \colon B \to B$$

is a diffeomorphism.

Proof. This follows from the explicit formula for the exponential map in Proposition 1.3. ∎

Lemma 3.2. *Let* W *be the invariant cone of Definition 2.2. Then the infinitesimally generated semigroup* $S = \overline{\langle \exp W \rangle}$ *satisfies* *(i)* $L(S) = W$.

(ii) With B *of Lemma 3.1 we have*

$$S \cap B = (\exp W) \cap B.$$

Proof. Let $C = \{((z,c,v),(z',c',v')) \in B \times B \mid -\pi \leq r, r', r + r' \leq \pi\}$. Then $C = \{(x,y) \in B \times B \mid \exp x \cdot \exp y \in B\}$. By Proposition 3.1, we have

$$\exp sx \cdot \exp ty \in \exp B \text{ für alle } s,t \in [0,1]$$

for all $(x,y) \in C$. Now [HHL] II.2.41 implies

$$\exp x \cdot \exp y \in \exp W \text{ for all } (x,y) \in C \cap (W \times W).$$

Since $[\pi, +\infty[\times \mathbb{C} \times \mathbb{R}$ is a semigroup ideal in $\mathbb{R}^+ \times \mathbb{C} \times \mathbb{R}$, this shows (ii). But then (i) is obvious. ∎

With these preparations we can explicitly describe the semigroup $S = \overline{\langle \exp W \rangle}$.

Proposition 3.3. *Let* $O_4 = \mathbb{R} \times \mathbb{C} \times \mathbb{R}$ *be the oscillator group in the parametrization Lemma 1.2, and* W *the invariant Lorentz cone of Definition 2.2. Set* $S = S_1 \cup S_2$ *with*

$$S_1 \; = \; \left\{(z,c,r) \in O_4 \;\middle|\; z \leq -\frac{\sin(2r)}{4(1-\cos(2r))}|c|^2\right\} \cup \mathbb{R}^- \times \{0\} \times \{0\}$$

and

$$S_2 \; = \; [\pi, +\infty[\times \mathbb{C} \times \mathbb{R}.$$

Then

(i) $S = \overline{\langle \exp W \rangle}$,

(ii) $L(S) = W$.

Proof. (i) First, we show $S \subseteq S_* \overset{def}{=} \overline{\langle \exp W \rangle}$. By Proposition 3.2,

$$S_1 = S_* \cap B = (\exp W) \cap B \tag{15}$$

holds. This implies $S_1 \subseteq S_*$. We have $\Phi_c(\frac{\pi}{2}) = \exp \frac{\pi}{2} \cdot (\frac{|c|^2}{4}, c, 1) = (0, c, \frac{\pi}{2})$, and therefore $\{0\} \times \mathbb{C} \times \{\frac{\pi}{2}\} \subseteq S_*$. For $r \in \mathbb{R}^+$, $z \in \mathbb{R}^-$ the relations $(0,0,r) = \exp(0,0,r) \in S_*$ and $(z,0,0) = \exp(z,0,0) \in S^*$ hold. Because of

$$(\pi, v, 0) \cdot (r, 0, 0) = (\pi + r, v, 0) \tag{16}$$

and

$$(r, v, 0) \cdot (0, 0, z) = (r, v, z) \tag{17}$$

this implies $\mathbb{R}^- \times \mathbb{C} \times [\frac{\pi}{2}, +\infty[\subseteq S_*$. By definition of S_1 and Lemma 2.3, equation (16) implies $S \setminus (\mathbb{R}^+ \setminus \{0\} \times \{0\} \times [\pi, +\infty[) \subseteq S_*$.

It remains to show, that all points of the form $(z, 0, \pi)$ with $z \in \mathbb{R}^+ \setminus \{0\}$ are in S_*. Consider the one-parametergroups Φ_{c_0} and $\Phi_{e^{-2is}c_0}$. Then we have

$$\Phi_{c_0}(t)\cdot\Phi_{e^{-2is}c_0}(t) = \Big(-\frac{|c_0|^2}{4}\cdot\sin(2t) + \frac{1}{2}\sin 2(t-s)\cdot|c_0|^2, (e^{i(t-2s)} + e^{-it})\cdot c_0, 2t\Big).$$

In particular, for $s - \frac{\pi}{2} = t = t_0$ we get

$$\Phi_{c_0}(t_0)\cdot\Phi_{e^{-2it_0}c_0}(t_0) = (-\frac{|c_0|^2}{4}\cdot\sin(2t_0), 0, 2t_0) \in S_*.$$

Thus every element of the form $(z, 0, \pi)$ with $z \in \mathbb{R}^+ \setminus \{0\}$ is approximable within S_* by a suitable choose of $t_0 > \frac{\pi}{2}$ and $c_0 \in \mathbb{C}$. Since S_* is closed, this shows $S \subseteq S_*$.

For the proof of (i) it remains to show that S is a semigroup. Obviously, $SS_2 \cup S_2S \subseteq S$ holds. By (15) and Proposition 3.2, we also have $S_1S_1 \subseteq S$. This proves the assertion. Claim (ii) now follows from Lemma 3.2. ∎

Proposition 3.3 shows, that the entire half-space $[\pi, +\infty[\times\mathbb{C}\times\mathbb{R}$ is contained in $S = \overline{\langle \exp W\rangle}$, whereas $\exp W$ misses open sets in $[\pi, +\infty[\times\mathbb{C}\times\mathbb{R}$ by Lemma 2.3.

References

[HHL] Hilgert, J., Hofmann, K. H., and Lawson, J. D., "Lie Groups, Convex Cones, and Semigroups", Oxford Univ. Press, 1989.

[HR] Hofmann, K. H., and Ruppert, W. A. F., *The structure of Lie groups which support closed divisible subsemigroups*, Proc. Oberwolfach Meeting on Semigroups, July 1991.

Technische Hochschule Darmstadt
Schloßgartenstraße 7
D-6100 Darmstadt
Germany