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The rule of Littlewood/Richardson and its treatment by a computer

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1. Introduction

The irreducible summands of the sum decomposition of the tensor product of two irreducible representations of a classical group or algebra are computable by aid of the rule of Littlewood/Richardson (shortly LR-rule). The LR-rule is effective also for other decomposition problems. It originated from the task to decompose the product of two Schur functions into a sum of Schur functions. The idea connecting all these problems is the characterization of the objects by a partition $\mathbf{l} = (l_1, l_2, \ldots, l_p)$, $l_1 \geq l_2 \geq \ldots \geq l_p \geq 0$, $l_i \in \mathbb{N}$ or by a Young frame with p rows and l_i boxes in the i-th row $(i = 1, \ldots, n)$. We get the summands of the product of two such objects adding the boxes of the second frame in a various way to the first frame. With the LR-rule we have to expect certain restrictions. It is possible that two frames with only a few rows give a great number of summands. So the use of a computer for the calculation appears to be indicated; for this purpose the LR-rule will be translated in a formula.

2. The rule of Littlewood/Richardson

We consider two Young frames $\mathbf{l} = (l_1, l_2, ..., l_p)$ and $\mathbf{m} = (m_1, m_2, ..., m_q)$. In the j-th row of $\mathbf{m}$ we write in all the boxes the number j and denote these boxes by j-boxes. Now we add the boxes of $\mathbf{m}$ to $\mathbf{l}$ beginning with the 1-boxes, next the 2-boxes etc. In the built frame the following conditions must be satisfied:

I. The frame gotten from $\mathbf{l}$ after the addition of the 1-boxes (2-boxes,...) must be admissible, i.e., the length of successive rows are non-increasing.

II. The frame does not contain boxes with equal labels appearing in a column.

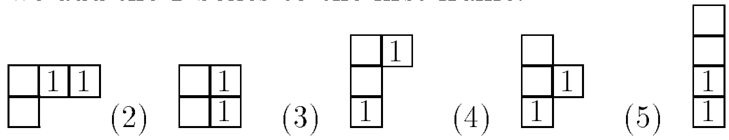
III. The number of $(j+1)$-boxes is never greater than the number of j-boxes if we count up the boxes in the rows from the right to the left and from top to bottom $(j = 1, ..., q-1)$.

We sum symbolically all Young frames built in this way and denote this sum as the result of the multiplication of $\mathbf{l}$ and $\mathbf{m}$.

Example: The two factors are $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$ ((1,1) and (2,1) resp.).

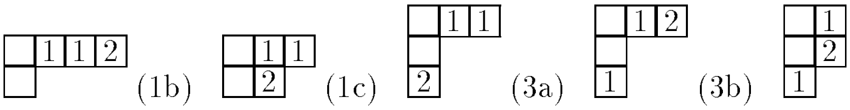
We fill the second frame with numbers: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$.

Now we add the 1-boxes to the first frame:

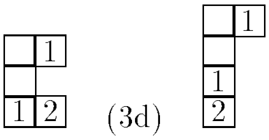


(1) (2) (3) (4) (5)

By I. is forbidden: (4); by II. are forbidden: (2),(5). Now we add the 2-box to (1) and (3):



(1a) (1b) (1c) (3a) (3b)



(3c) (3d)

By I. are forbidden: (3c); by III. are forbidden: (1a), (3a). The result is the following:(1b),(1c),(3b) and (3d) or (3,2),(3,1,1),(2,2,1) and (2,1,1,1).

3. A closed formula for the LR-rule

A general formula for the LR-rule is given in [S91]. There we can find also formulas for some Clebsch-Gordan-series of classical Lie algebras and superalgebras. Here we will prove only a special case, but the same idea leads to the general proof. We look to $\mathbf{l} = (l_1, \ldots, l_p)$ und $\mathbf{m} = (m_1, m_2)$ and complete $\mathbf{l}$ by $l_{p+1} = l_{p+2} = 0$. We consider an admissible distribution of the 1-boxes and the 2-boxes of $\mathbf{m}$ at $\mathbf{l}$ and denote the number of i-boxes in the h-th row by k_{hi}. If we put first $k_{p+1,1}$ 1-boxes to the $(p+1)$-th row of $\mathbf{l}$, then k_{p1} 1-boxes to the p-th row etc., we have the following restrictions by I. and II.:

$$k_{h1} \leq \min(l_{h-1} - l_h, m_1 - \sum_{j=h+1}^{p+1} k_{j1}) = Min(l, m; h, 1)$$

$$k_{11} = m_1 - \sum_{j=2}^{p+1} k_{j1}$$

for $h = 2, \ldots p+1$. For the parameters k_{h2} we get from I. and II.:

$$k_{h2} \leq \min((l_{h-1} + k_{h-1,1}) - (l_h + k_{h1}),\ m_2 - \sum_{j=h+1}^{p+2} k_{j2}) = Min(l, m; h, 2)$$

$$k_{11} = m_1 - \sum_{j=2}^{p+1} k_{j1}$$

for $h = 2, \ldots p+2$.

Now we have to consider also condition III. :

$$\sum_{j=2}^{h} k_{j2} \leq \sum_{j=1}^{h-1} k_{j1}$$

or

$$\sum_{j=3}^{h} k_{j2} + m_2 - \sum_{j=3}^{p+2} k_{j2} \leq \sum_{j=2}^{h-1} k_{j1} + m_1 - \sum_{j=2}^{p+1} k_{j1}$$

or

$$m_2 - m_1 - \sum_{j=h+2}^{p+2} k_{j2} + \sum_{j=h}^{p+1} k_{j1} \leq k_{h+1,2}$$

for $h = 2, ..., p+1$.

So we get

$$Max(m; h, 2) = \max(0, m_2 - m_1 + \sum_{j=h-1}^{p+1} k_{j1} - \sum_{j=h+1}^{p+2} k_{j2}) \leq k_{h2}$$

$(h = 3, ...p+1)$ and can press the LR-rule in the following form:

$$l * m =$$

$$\sum_{k_{p+1,1}} \dots \sum_{k_{21}} \sum_{k_{p+2,2}} \dots \sum_{k_{32}} (l_1 + k_{11}, l_2 + k_{21} + k_{22}, \dots, l_{p+1} + k_{p+1,1} + k_{p+2,2}, l_{p+2} + k_{p+2,2})$$

$$0 \leq k_{h1} \leq Min(l, m; h, 1)$$

$$Max(m; h, 2) \leq k_{h2} \leq Min(l, m; h, 2)$$

$$k_{11} = m_1 - \sum_{j=2}^{p+1} k_{j1}, k_{22} = m_2 - \sum_{j=3}^{p+2} k_{j2}$$

4. Treatment by a computer

This formula and the general formula also can be treated by a computer. We have written in Greifswald a C-program which give the result of the LR-rule using the proved formula. For example, we have computed the following cases: (1) (5,3,1)$*$(2,2,1) (2) (6,5,3,3,1)$*$(3,2,1) (3) (8,6,5,4,2,1)$*$(4,2,2,1) The sums consists of (without consideration of multiplicities) 37 summands (1), 436 summands (2), 10483 summands (3). These results show that it is necessary to use a computer. It is possible to compute the summands of the LR-rule by an other procedure. This procedure is realized in Bayreuth using the special computer-algebra- system SYMCHAR. The theoretical basis of this procedure is given in the paper [LS85] of Lascoux/Schützenberger. The first comparisons show that this procedure should be a little quicker than our computation. But it seems that an advantage of the formulas of [S91] consist in the possibility of computing the Clebsch-Gordan-series of representations of classical Lie algebras and superalgebras without change of the program: We only have to replace the definition of $Max(m;h,i)$ and $Min(l,m;h,i)$ by a similar definition.

References

[LS85] Lascoux, A. and Schützenberger, M.-P., *Schubert polynomials and the Littlewood-Richardson rule*, Letters in Math. Phys. **10**(1985), 111–124.

[S91] Schlosser, H., *A closed formula for the rule of Littlewood/Richardson with applications in the theory of representations of $gl(V)$ and the superalgebra $pl(V)$*, Math. Nachr. **151** (1991), 315–326.

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