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An example of a solvable Lie algebra

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We shall consider a solvable real Lie algebra $\mathfrak{g}$ which is linearly spanned by the set $comp\,\mathfrak{g}$ of its compact elements and use the notation of [1]. In particular, the diagram in [1] below Proposition 2.10 gives a good orientation on the subalgebras involved in our discussion. It was shown in [1] that a Cartan algebra in a real Lie algebra spanned by its compact elements has nilpotent class ≤ 2. It was left open whether there are algebras in which the Cartan algebras are nonabelian. In the following we present an example of dimension 13 showing that this can happen.

Let $\mathfrak{g}$ be a real vector space with a basis

$$\{u, x, y, z, t, e_1, e_2, f_1, f_2, g_1, g_2, h_1, h_2\}$$

and define the following relations on the basis elements:

$$\begin{array}{llll}
[u, a_1] = a_2, & [u, a_2] = -a_1 & \text{for } a = e, f, g, h, & \\
[e_1, e_2] = x, & [f_1, f_2] = y, & & \\
[e_1, f_2] = -t, & [e_2, f_1] = t, & & \\
[x, f_i] = h_i, & [y, e_i] = g_i & \text{for } i = 1, 2, & [x, y] = z, \\
[t, e_i] = -h_i, & [t, f_i] = -g_i & \text{for } i = 1, 2, & \\
[e_1, g_2] = -\frac{1}{2}z, & [e_2, g_1] = \frac{1}{2}z, & [f_1, h_2] = \frac{1}{2}z, & [f_2, h_1] = -\frac{1}{2}z.
\end{array}$$

All other brackets between basis elements are set equal to 0. Bilinear extension yield a bilinear product $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

We define on $\mathfrak{g}$ the structure of a graded vector space by setting $\mathfrak{g}^0 = \mathbb{R}\cdot u$, $\mathfrak{g}^1 = span\{e_1, e_2, f_1, f_2\}$, $\mathfrak{g}^2 = span\{x, y, t\}$, $\mathfrak{g}^3 = span\{g_1, g_2, h_1, h_2\}$, $\mathfrak{g}^4 = \mathbb{R}\cdot z$. It is readily verified that $[\mathfrak{g}^m, \mathfrak{g}^n] \subseteq \mathfrak{g}^{m+n}$. Thus $\mathfrak{g}$ is a graded algebra with respect to the bracket.

Lemma. $\mathfrak{g}$ *is a solvable Lie algebra.*

Proof. We verify the validity of the Jacobi identity by straightforward, but tedious calculation. I have written a program which executes the necessary verifications. The gradation permits us a quick inspection of the solvability of $\mathfrak{g}$. ∎

We verify quickly that the subspace $\mathfrak{h} = span\{u, x, y, z, t\}$ is a Cartan subalgebra and $\mathfrak{t} \overset{def}{=} \mathfrak{h} \cap comp\,\mathfrak{g} = span\{u, z\}$. Also, the sum of the weight spaces with respect to $\mathfrak{t}$ is $\mathfrak{t}^+ = span\{e_1, e_2, f_1, f_2, g_1, g_2, h_1, h_2\}$ (cf. [1]), and $\mathfrak{g}' = \mathcal{C}^\infty = \langle \mathfrak{t}^+ \rangle = span\{e_1, e_2, f_1, f_2, g_1, g_2, h_1, h_2, t, x, y, z\}$. It follows that $\mathfrak{e} \overset{def}{=} \mathfrak{h} \cap \mathfrak{g}' = span\{x, y, t, z\}$. Further, in the notation of [1], we have $\mathfrak{z} = \mathbb{R}\cdot z = \mathfrak{j}$ and may take $\mathfrak{e}_1 = span\{x, y, t\}$ as a complement of $\mathfrak{j}$ in $\mathfrak{e}$, and $\mathfrak{t}_1 = \mathbb{R}\cdot u$ as a complement of $\mathfrak{z}$ in $\mathfrak{t}$. With these choices we obtain $\mathfrak{g}' = \mathfrak{e}_1 \oplus \mathfrak{t}^+$. Therefore, we have $\dim \mathfrak{t} = 2$, $\dim \mathfrak{h} = 5$, $\dim \mathfrak{e} = 4$, $\dim \mathfrak{e}_1 = 3$, $\dim \mathfrak{t}^+ = 8$, $\dim \mathfrak{g}' = \dim\langle \mathfrak{t}^+ \rangle = 12$.

The following diagram records the gradation and the relevant subalgebras:

degree	$\mathfrak{h}$		$\mathfrak{t}^+$
	$\mathfrak{t}$	$\mathfrak{e}_1$	
0	u		
1			$e_1 \quad e_2 \quad f_1 \quad f_2$
2		$x \quad y \quad t$	
3			$g_1 \quad g_2 \quad h_1 \quad h_2$
4	z		

The following equations will show that $\mathfrak{g} = span(comp(\mathfrak{g}))$. From [1], Lemma 2.2(iii) we know that $comp = \Gamma{\cdot}\mathfrak{t}$. We will be finished if we can show that $\mathfrak{g}$ is spanned by elements of the form $e^{ad\, w}(u)$ with $w \in \mathcal{C}^\infty$. Now we compute:

$$
\begin{aligned}
e^{ad(e_1+e_2)}(u) &= u - e_2 + e_1 - 2x,\\
e^{ad(e_1-e_2)}(u) &= u - e_2 - e_1 - 2x,\\
e^{ad(e_2-e_1)}(u) &= u + e_1 + e_2 - 2x,\\
e^{ad(f_1+f_2)}(u) &= u - f_2 + f_1 - 2y,\\
e^{ad(f_1-f_2)}(u) &= u - f_2 - f_1 - 2y,\\
e^{ad(f_2-f_1)}(u) &= u + f_1 + f_2 - 2y,\\
e^{ad(e_1+f_1)}(u) &= u - e_2 - f_2 - x - y + 3g_1 + 3h_1 + 2t,\\
e^{ad(a_1)}(u) &= u - a_2,\\
e^{ad(a_2)}(u) &= u + a_1 \qquad\qquad \text{for } a = g,\, h.
\end{aligned}
$$

We see that the span of these elements is $\mathfrak{g}$ and this shows that $\mathfrak{g} = span\ comp$.

References

[1] Hofmann, K.,H., *Compact elements in solvable real Lie algebras*, Seminar Sophus Lie **2** (1992), this volume.

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