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On the Riemannian Geometry of Finite Dimensional Mixed States

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Abstract

We consider the Riemannian geometry of the space of nonsingular density matrices $\mathcal{D}^1$ equipped with the Bures metric g^B. This space is of certain physical relevance on the background of a generalization of the Berry phase to mixed states. We determine the covariant derivative and the curvature tensor field related to the Levi-Cevita connection of $(\mathcal{D}^1, g^B)$, which allow us to calculate other curvature quantities . It turns out that $\mathcal{D}^1$ is not a space of constant curvature and not even a locally symmetric space, in contrast to what the case of two-dimensional density matrices might suggest. Moreover, we give a local description of $\mathcal{D}^1$ and explicit formulae for the Bures metric in terms of natural matrix operations containing ϱ and $d\varrho$ only.

1. Introduction

The aim of this paper is to consider the local Riemannian geometry of the space of nonsingular, normalized $n \times n$ density matrices $\mathcal{D}^1 := \{\varrho \in \mathcal{M}_{n,n}(\mathbb{C}) \mid \varrho^* = \varrho > 0,\, Tr\ \varrho = 1\}$ equipped with the Riemannian Bures metric g^B. This Riemannian metric appears on the background of a generalization of the Berry phase (see [11, 4, 16, 12, 1]) to mixed states in quantum systems proposed by Uhlmann in a series of papers ([12, 13, 14]). Moreover, this metric is just the infinitesimal version of the distance function

$$d(\varrho, \mu) = \sqrt{2 - 2Tr\left(\varrho^{\frac{1}{2}}\mu\varrho^{\frac{1}{2}}\right)^{\frac{1}{2}}} \tag{1.1}$$

given by Araki many years ago ([2], A2).

So it is natural to ask for differential geometric properties of $(\mathcal{D}^1, g^B)$ and check several ideas one gets considering the case of 2×2 density matrices. Uhlmann observed that for $n=2$ the space $\mathcal{D}^1$ is isometric to an open half shell of the 3-sphere of radius $\frac{1}{2}$ ([15]). Moreover, for $n=2$ there is an interesting relation to instantons and the Yang-Mills theory ([7]). Thus this space is an interesting geometrical object and one is led to several questions for general n, e.g., is the space $(\mathcal{D}^1, g^B)$ of constant curvature or, at least, locally symmetric?

Let us briefly explain how $(\mathcal{D}^1, g^B)$ appears. The idea of generalization of the Berry phase to mixed states proposed by Uhlmann is based on the concept of purification of mixed states, where one represents mixed states $\varrho \in \Omega$ in a Hilbert space $\mathcal{H}$ by pure states in an extended Hilbert space $\mathcal{H}^{ext}$. A standard way to do this is to take as $\mathcal{H}^{ext}$ the Hilbert space $\mathcal{H}^{HS}$ of Hilbert-Schmidt operators on $\mathcal{H}$. If

$$I\!R \ni t \mapsto \varrho(t) \in \Omega = \{\mu: \mathcal{H} \to \mathcal{H} \mid \mu \geq 0, Tr\,\mu = 1\}$$

is a sufficiently regular path of density operators then

$$I\!R \ni t \longmapsto w(t) \in \mathcal{H}^{HS}$$

is said to be a purification of ϱ iff

$$\varrho(t) = w(t)\, w(t)^*.$$

Thus, a purification of ϱ is a lift of ϱ into the fibration

$$\begin{array}{ccccc} & & S(\mathcal{H}^{HS}) & \subset & \mathcal{H}^{HS} \\ & & \downarrow \pi & & \\ \varrho : I\!R & \longrightarrow & \Omega & & \end{array} \tag{1.2}$$

$\pi(w) = ww^*$, where $S(\mathcal{H}^{HS}) = \{w \in \mathcal{H}^{HS} \mid \langle w, w\rangle = 1\}$ is the unit sphere with respect to the real part of the Hilbert-Schmidt metric;

$$\langle X, Y\rangle = \mathrm{Re}\, Tr\, XY^*; \qquad X, Y \in \mathcal{H}^{HS}. \tag{1.3}$$

Among all lifts there are distinguished ones, the so called horizontal lifts satisfying the horizontality condition

$$w^*\dot{w} = \dot{w}^* w \tag{1.4}$$

which generalizes the Berry condition in this framework. A detailed motivation and discussion of (1.4) was presented in [13, 14]. In particular, let $\mathcal{H}$ be of finite dimension n. Then the restriction of the fibration (1.2) to the (dense in Ω) manifold $\mathcal{D}^1$ of nonsingular states $\varrho > 0$ is a principal $U(n)$-bundle;

$$gl(n, \mathbb{C}) \supset S^{2n^2-1} \supset \mathcal{P} := \pi^{-1}(\mathcal{D}^1) \longrightarrow \mathcal{D}^1. \tag{1.5}$$

The vectors $X \in T_w\mathcal{P}$ satisfying

$$w^*X = X^*w \tag{1.6}$$

are the horizontal vectors of UHLMANN's connection form A given implicitly by the equation

$$w^*wA + Aw^*w = w^*dw - (dw^*)w, \tag{1.7}$$

see [13, 14]. A certain class of connection forms including this one was considered in [6]. Since (1.6) is just the condition for X being orthogonal to the vertical vectors of (1.5) w.r. to (1.3), the horizontal subspaces are just the orthogonal complements of the fibre directions of (1.5). This connection form together with the Riemannian metric (1.3) on $\mathcal{P}$ define the Riemannian metric g^B on $\mathcal{D}^1$ we are interested in. The metric g^B is given by

$$g^B_\varrho(X, Y) = \mathrm{Re}\, Tr\, X'Y'^*; \qquad X, Y \in T_\varrho\mathcal{D}^1, \tag{1.8}$$

where X', Y' are horizontal lifts of X and Y to any point of $\pi^{-1}(\varrho)$. Using the horizontality condition (1.6) one obtains

$$g^B = \frac{1}{2} Tr\, G\, d\varrho, \tag{1.9}$$

where G (see [13, 14]) is the matrix of 1-forms on $\mathcal{D}^1$ given implicitly by

$$\varrho G + G\varrho = d\varrho; \qquad \varrho \in \mathcal{D}^1. \tag{1.10}$$

In Section 2 we make some remarks about the local structure of $\mathcal{D}^1$. It turns out, that $\mathcal{D}^1$ is locally isometric to the product of a sphere and the homogeneous space $\mathrm{U}(n)/\mathrm{T}^n$ with an invariant metric depending on the points of the sphere. Moreover, we give an explicit algebraic formula for the Bures metric tensor g^B in the case n=3 and give a method to obtain analogous formulae for general n. However, these local considerations and the formulae give not a satisfactory picture of the Riemannian space we are discussing. That is the reason for making a differential geometric approach. In Section 4 we determine the covariant derivative on $\mathcal{D}^1$. This enables us to give in Section 5 the corresponding curvature tensor field. As a consequence we obtain that $\mathcal{D}^1$ is not a space of constant curvature for $n > 2$, and not even a locally symmetric space. The physical meaning of this fact seems to be an interesting open question.

2. Notations

We denote by $\mathcal{D}$ the space of nonsingular hermitean $n \times n$ matrices and by $\mathcal{D}^1$ as above the subspace of trace one matrices for a fixed n. The spaces $\mathcal{D}$ and $\mathcal{D}^1$ carry a flat local affine structure, because they are open subsets of affine spaces (space of hermitean resp. trace one hermitean matrices). It corresponds to the flat metric

$$g^f = Tr\, d\varrho\, d\varrho \tag{2.1}$$

on $\mathcal{D}$ resp. $\mathcal{D}^1$.

We denote vector fields on $\mathcal{D}$ resp. $\mathcal{D}^1$ by X, Y, Z and W. Often we consider them as hermitean matrix valued functions due to the embedding of $\mathcal{D}$ and $\mathcal{D}^1$ into matrix spaces. In particular, let N be the vector field on $\mathcal{D}$ defined by

$$N_\varrho = \varrho; \qquad \varrho \in \mathcal{D} \quad . \tag{2.2}$$

By $[X, Y]$ we denote the commutator of the matrix valued functions X and Y in contrast to the commutator $[X, Y]_{vf}$ of vector fields considered as derivations;

$$[X, Y]_\varrho = X_\varrho Y_\varrho - Y_\varrho X_\varrho \quad , \quad [X, Y]_{vf}(f) = X(Y(f)) - Y(X(f)) \,.$$

Let L_ϱ (resp. R_ϱ) be the operator of left (resp. right) multiplication of matrices by $\varrho \in \mathcal{D}$. Note that $L_\varrho + R_\varrho$ has the spectrum $\{\lambda + \mu \mid \lambda, \mu \text{ are eigenvalues of } \varrho\}$ and, therefore, the operator $(L_\varrho + R_\varrho)^{-1}$ is well defined. Omitting the index ϱ we regard $(L+R)^{-1}$ as an operator valued function on $\mathcal{D}$ resp. $\mathcal{D}^1$. For simplicity we denote by $\overline{X}$ the matrix valued function defined by

$$\overline{X} := (L+R)^{-1}(X) \quad ; \qquad \overline{X}_\varrho = (L_\varrho + R_\varrho)^{-1}(X_\varrho) \,. \tag{2.3}$$

Formulae (1.9) and (1.10) define the Riemannian Bures metric g^B on $\mathcal{D}^1$. Of course, these formulae even define a Riemannian metric on $\mathcal{D}$ which we denote by g. The metric g^B is the pullback of g to $\mathcal{D}^1$. Since $d\varrho$ is a matrix of 1-forms we have by (1.10)

$$G = (L+R)^{-1}\,(d\varrho)$$

and in this notation the metrics read

$$g^B \ (\text{resp. } g) \ = \frac{1}{2} Tr\ (L+R)^{-1}\,(d\varrho)\,d\varrho. \tag{2.4}$$

Note that N is normal—with respect to g—to the submanifold $\mathcal{D}^1 \subset \mathcal{D}$ of codimension one.

Clearly, $L+R$ is pointwise a selfadjoint operator with respect to g and g^f. By ∇^B, ∇ and ∇^f we will denote the covariant derivatives of the Levi-Cevita connections corresponding to g^B, g and g^f. Thus, ∇^f is the ordinary covariant derivative on matrices, in particular, we have

$$\nabla^f_X N = X\ . \tag{2.5}$$

Curvatures related to the metrics above will be denoted analogously.

3. The Manifold $(\mathcal{D}^1, g^B)$

In order to form an intuition of the space $(\mathcal{D}^1, g^B)$ one would like to have a "natural" isometric embedding of $\mathcal{D}^1$ into a flat space. However, except the case n=2 ([15]) such an embedding is not known to the author.

Note that every $\varrho \in \mathcal{D}^1$ can be uniquely decomposed as $\varrho = ss^*$, where s is a triangular complex matrix with real positive entries on the diagonal;

$$s = \sum_{i>j} z_{ij} E_{ij} + \sum_i \mu_i E_{ii} \qquad .$$

Since ϱ has trace one the coefficients satisfy

$$\sum_{i>j} \mid z_{ij} \mid^2 + \sum_i \mu_i^2 = 1 \qquad .$$

Thus $\mathcal{D}^1$ is diffeomorphic to a segment $(\mu_i > 0)$ of a (n^2-1)-sphere. But this diffeomorphism is not an isometry as we will see later. However, let us consider the submanifold Λ of diagonal matrices $\lambda = \mu^2$, where

$$\mu = \sum_i \mu_i E_{ii}\,; \quad \mu_i > 0\,, \quad \sum_i \mu_i^2 = 1 \quad .$$

Denoting the pullback of the 1-form G to Λ by $\iota^* G$, we get explicitly

$$\iota^* G = \frac{1}{2}\mu^{-2} d\mu^2$$

and, therefore, we obtain for the pullback of g^B to Λ

$$\iota^* g^B = \frac{1}{4} Tr\ \mu^{-2} d(\mu^2) d(\mu^2) = Tr\ d\mu d\mu = \sum_i d\mu_i d\mu_i \quad .$$

Thus Λ is isometric to the segment

$$\mathcal{S} = \left\{(\mu_1, \ldots, \mu_n) \in I\!R^n_+ \mid \mu_1^2 + \ldots + \mu_n^2 = 1\right\}$$

of the $(n-1)$-sphere of radius 1 ([15]). A little bit more of the local structure of $\mathcal{D}^1$ one can see as follows.
Using the decomposition $\varrho = u\mu^2 u^*$ $, u \in U(n)/T^n$, for a generic $\varrho \in \mathcal{D}^1$ a simple calculation shows that

$$(L_\varrho + R_\varrho)^{-1}(d\varrho) = Adu\left(\frac{1}{2}\lambda^{-1}d\lambda + (L_\lambda + R_\lambda)^{-1}\left([u^*du, \lambda]\right)\right) \quad , \tag{3.1}$$

where $\lambda = \mu^2$ is diagonal with different eigenvalues only. Here we used the same symbol u resp. du for classes and their representatives. This makes sense, because the right hand side of (3.1) does not depend on the the choice of representatives. Indeed, elements of T^n commute with λ. Inserting (3.1) into (2.4) we get

$$g^B_\varrho = Tr\ d\mu d\mu \ + \ \frac{1}{2}Tr\left\{\ (L_\lambda + R_\lambda)^{-1}\left([u^*du, \lambda]\right)[u^*du, \lambda]\ \right\}. \tag{3.2}$$

Thus, in a generic point ϱ the manifold $\mathcal{D}^1$ is locally isometric to $S^{n-1} \times U(n)/T^n$, where the metric on the homogeneous space $U(n)/T^n$ given by the second term of (3.2) depends on the parameter $\lambda \in S^{n-1}$. Moreover, this metric is invariant under the natural left $U(n)$-action.

Formula (2.4) giving the metric g^B is rather implicit because the operator $(L+R)^{-1}$ is not given in natural matrix operations. In [9] an explicit formula for the Bures metric was given for n=2, only. Here we show, how one can obtain for every fixed n an explicit formula for the inverse of the operator $L+R$ and, therefore, for g^B, too. We first consider the cases n=2 and n=3. Note, that for $\varrho \in \mathcal{D}^1$ holds

$$\frac{L-R}{L+R} \ = \ L - R \qquad \text{for } n = 2 \tag{3.3}$$

$$\frac{L-R}{L+R} \ = \ \frac{3}{1 - Tr\ \varrho^3}\,(Id - L)\circ(Id - R)\circ(L - R) \quad \text{for } n = 3\,. \tag{3.4}$$

Inserting these formulae into

$$\frac{2}{L+R} \ = \ L^{-1}\left(Id + \frac{L-R}{L+R}\right)$$

we get for the inverse of $L+R$

$$\frac{2}{L+R} \ = \ Id \ + \ \frac{1}{|\varrho|}(Id - L)\circ(Id - R) \qquad \text{for } n = 2 \tag{3.5}$$

$$\begin{aligned} \frac{2}{L+R} \ &= \ Id \ + \ \frac{3}{1 - Tr\ \varrho^3}(Id - L)\circ(Id - R) \\ &\quad + \ \frac{3|\varrho|}{1 - Tr\ \varrho^3}(Id - L^{-1})\circ(Id - R^{-1}) \quad \text{for } n = 3\,, \end{aligned} \tag{3.6}$$

where $|\varrho| := \mathrm{Det}\,\varrho$. Here we used that from the Hamilton-Cayley theorem we know

$$\begin{aligned} L^2 - L + |\varrho|\, Id &= 0 \qquad \text{for } n=2 \\ -R^3 + R^2 - \frac{1 - Tr\,\varrho^3 + 3|\varrho|}{3} R + |\varrho|\, Id &= 0 \qquad \text{for } n=3\,. \end{aligned}$$

Thus we obtain by (2.4) for the Bures metric in these two cases

$$g^B_\varrho = \frac{1}{4} Tr\, \left\{ d\varrho d\varrho + \frac{1}{|\varrho|}(d\varrho - \varrho d\varrho)(d\varrho - \varrho d\varrho) \right\} \qquad \text{for } n=2 \tag{3.7}$$

$$\begin{aligned} g^B_\varrho = \frac{1}{4} Tr\, &\left\{ d\varrho d\varrho + \frac{3}{1 - Tr\,\varrho^3}(d\varrho - \varrho d\varrho)(d\varrho - \varrho d\varrho) \right. \\ &\left. + \frac{3|\varrho|}{1 - Tr\,\varrho^3}(d\varrho - \varrho^{-1} d\varrho)(d\varrho - \varrho^{-1} d\varrho) \right\} \qquad \text{for } n=3\,. \end{aligned} \tag{3.8}$$

In (3.8) we could additionally substitute ϱ^{-1} by powers of ϱ and its invariants as in the first case. The formula given in [9] for the case $n{=}2$ can be reproduced from (3.7) using the following two identities, which are easily derived from the equation $\varrho^2 - \varrho + |\varrho|\, \mathbb{1} = 0$.

$$\begin{aligned} Tr\,(\varrho\, d\varrho\, d\varrho + d\varrho\, \varrho\, d\varrho) &= Tr\; d\varrho\, d\varrho\;, \\ Tr\; \varrho\, d\varrho\, \varrho\, d\varrho &= |\varrho|\, Tr\; d\varrho\, d\varrho + d|\varrho|\; d|\varrho|\;. \end{aligned}$$

Using this we get from (3.7)

$$\begin{aligned} g^B_\varrho &= \frac{1}{2}\, Tr\; d\varrho\, d\varrho \;+\; \frac{1}{4|\varrho|}\; d|\varrho|\; d|\varrho| \\ &= \frac{1}{2}\, Tr\; d\varrho\, d\varrho \;+\; d\sqrt{|\varrho|}\; d\sqrt{|\varrho|} \qquad \text{(see [9])}. \end{aligned} \tag{3.9}$$

To get similar formulae for arbitrary n one can proceed as follows:
Denote by

$$\chi_\varphi(x) = \sum_i \sigma_{m-i}(\varphi)(-x)^i$$

the characteristic polynomial of a linear mapping φ, where $\sigma_k(\varphi) := Tr\ \wedge^k \varphi$ is the elementary invariant of degree k. By the Hamilton-Cayley theorem we have

$$\chi_{L+R}(L{+}R) = 0$$

and, therefore, multiplying this equation by $(L{+}R)^{-1}$

$$(L+R)^{-1} = -\frac{1}{\mathrm{Det}(L{+}R)} \sum_{i=1}^{n^2} \sigma_{n^2-i}(L{+}R)\,(-L{-}R)^{i-1} \quad . \tag{3.10}$$

The invariants $\sigma_k(L+R)$ may be expressed by polynomials of invariants of the matrix ϱ using the following identities:

$$\sigma_k(L+R) = \sum_{i=0}^{k} \sigma_i(L)\sigma_{k-i}(R) \tag{3.11}$$

$$\chi_L(x) = \chi_R(x) = (\chi_\varrho(x))^n \quad . \tag{3.12}$$

The invariant $\sigma_i(\varrho)$ is a polynomial in $Tr\ \varrho^j$, $j = 2, \ldots, i$. Formulae (3.11) and (3.12) can be easily verified assuming ϱ to be diagonal. Finally, using $\chi_\varrho(\varrho) = 0$ we get the Bures metric g^B for every fixed n in the form

$$g^B = \frac{1}{\sigma_{n^2}(L+R)}\ Tr \sum_{i,j=0}^{n-1} C_{ij}\varrho^i d\varrho \varrho^j d\varrho \ , \tag{3.13}$$

where the $C_{ij} \in I\!R$ are certain polynomials of invariants of ϱ. Thus, we have the Riemannian Bures metric g^B in terms of ϱ, $d\varrho$ and natural matrix operations. However, it seems that such algebraic formulae do not contribute to much to a better understanding of the Riemannian metric g^B and are rather unsuitable for the calculation of curvatures. A general approach to this problem will be presented in the next sections.

4. The Covariant Derivative

In order to determine the covariant derivatives ∇ and ∇^B related to g and g^B we use the underlying flat local affine structure on $\mathcal{D}$ and $\mathcal{D}^1$. Note, that by (2.4) and (2.1) we have the relation

$$2g(X,Y) = g^f(\overline{X}, Y) = g^f(X, \overline{Y}) \quad . \tag{4.1}$$

The flat covariant derivative of the (1,1)-tensor field $L+R$ is simply calculated, namely,

$$\left(\nabla^f_X(L+R)\right)(Y) = XY + YX \quad , \tag{4.2}$$

where the fields X and Y are considered on the right hand side as matrix valued functions and XY denotes their product. This implies

$$\left(\nabla^f_X(L+R)^{-1}\right)(Y) = -(L+R)^{-1}(X\overline{Y} + \overline{Y}X) \tag{4.3}$$

and, therefore, using $\nabla^f g^f = 0$

$$\begin{aligned} 2\left(\nabla^f_X g\right)(Y,Z) &= g^f\left(\left(\nabla^f_X(L+R)^{-1}\right)(Y), Z\right) \\ &= -\ g^f(X\overline{Y} + \overline{Y}X, \overline{Z}) \ . \end{aligned} \tag{4.4}$$

Any two covariant derivatives on a manifold differ on vector fields by a certain (1,2) tensor field. Relating ∇ and ∇^B to ∇^f we prove the following

Theorem 4.1. *The covariant derivatives ∇ and ∇^B corresponding to the Levi-Cevita connections on $(\mathcal{D}, g)$ and $(\mathcal{D}^1, g^B)$ are given by*

$$\nabla_X Y = \nabla^f_X Y + S(X,Y) \tag{4.5}$$
$$\nabla^B{}_X Y = \nabla^f_X Y + S(X,Y) + 2g^B(X,Y)\cdot N \tag{4.6}$$

with

$$S(X,Y) := -\overline{X}\,N\,\overline{Y} - \overline{Y}\,N\,\overline{X} \quad . \tag{4.7}$$

For notations see (2.2) and (2.3).

Proof. First we show (4.5). Let X and Y be vector fields on $\mathcal{D}$ and define a covariant derivative by $\widetilde{\nabla}_X Y := \nabla^f_X Y + S(X,Y)$, where S is given by (4.7). Since S is a symmetric (1,2) tensor field on $\mathcal{D}$ the torsion of $\widetilde{\nabla}$ vanishes. Thus, in order to show that $\widetilde{\nabla} = \nabla$ we have to show $\widetilde{\nabla} g = 0$. It is sufficient to show $\left(\widetilde{\nabla} g\right)(Y,Y) = 0$. By the Leibniz rule, (4.7), (4.4), (4.1) and (2.1) we get

$$\begin{aligned}
\left(\widetilde{\nabla}_X g\right)(Y,Y) &= \widetilde{\nabla}_X(g(Y,Y)) - 2g\left(\widetilde{\nabla}_X Y, Y\right) \\
&= \nabla^f_X(g(Y,Y)) - 2g\left(\nabla^f_X Y, Y\right) + 2g(\overline{X}\,N\,\overline{Y} + \overline{Y}\,N\,\overline{X}, Y) \\
&= \left(\nabla^f_X g\right)(Y,Y)) + 2g(\overline{X}\,N\,\overline{Y} + \overline{Y}\,N\,\overline{X}, Y) \\
&= -\frac{1}{2} g^f(X\,\overline{Y} + \overline{Y}\,X, \overline{Y}) + g^f(\overline{X}\,N\,\overline{Y} + \overline{Y}\,N\,\overline{X}, \overline{Y}) \\
&= Tr\,(-X\,\overline{Y}\,\overline{Y} + (\overline{X}\,N + N\,\overline{X})\,\overline{Y}\,\overline{Y}) \\
&= Tr\,(-X\,\overline{Y}\,\overline{Y} + X\,\overline{Y}\,\overline{Y}) = 0\,.
\end{aligned} \tag{4.8}$$

Now, let X and Y be vector fields tangent to $\mathcal{D}^1$; $Tr\,X_\varrho = 0$, $Tr\,Y_\varrho = 0$ for $\varrho \in \mathcal{D}^1$. Then $\nabla^f_X Y$ is tangent to $\mathcal{D}^1$, too. Because $\mathcal{D}^1$ is a Riemannian submanifold of $\mathcal{D}$, $\nabla^B{}_X Y$ is just the component tangent to $\mathcal{D}^1$ of $\nabla_X Y$. Using that N is normal to $\mathcal{D}^1$ and $g_\varrho(N,N) = \frac{1}{4}$ for $\varrho \in \mathcal{D}^1$ we get

$$\begin{aligned}
\nabla^B{}_X Y &= \nabla_X Y - 4g\,(\nabla_X Y, N)\cdot N \\
&= \nabla_X Y + 4g\left(\overline{X}\,N\,\overline{Y} + \overline{Y}\,N\,\overline{X}, N\right)\cdot N \\
&= \nabla_X Y + Tr\,((\overline{X}\,N\,\overline{Y} + \overline{Y}\,N\,\overline{X})\mathbb{1})\cdot N \\
&= \nabla^f_X Y - \overline{X}\,N\,\overline{Y} - \overline{Y}\,N\,\overline{X} + 2g^B(X,Y)\cdot N \quad .
\end{aligned} \tag{4.9}$$

This finishes the proof. ■

The geodesics on $\mathcal{D}^1$ connecting two states were given explicitly in [15]. By the construction of the Bures metric they are the projection under π of shortest circles connecting representatives $\pi^{-1}(\varrho)$ and $\pi^{-1}(\mu)$ in the principal bundle (1.5). This allows one to determine the exponential mapping related to the covariant derivative obtained above. Here we give the final result, only. Let $\varrho \in \mathcal{D}^1$ and $h \in T_\varrho \mathcal{D}^1$ be a traceless hermitean matrix. Then the exponential mapping on $(\mathcal{D}^1, g^B)$ is given by

$$\begin{aligned}
Exp_\varrho(h) = \cos^2(\|h\|)\varrho \;&+\; \frac{1}{\|h\|^2}\sin^2(\|h\|)\,\overline{h}\,\varrho\,\overline{h} \\
&+\; \frac{1}{\|h\|}\sin(\|h\|)\cos(\|h\|)\,h\,,
\end{aligned} \tag{4.10}$$

where

$$\|h\|^2 = g^B(h,h) \quad \text{and} \quad \varrho\,\overline{h} + \overline{h}\,\varrho = h\ .$$

A direct verification that $\varrho(t) := Exp_{\varrho}(th)$ satisfies the geodesic equation

$$\nabla^B_{\dot{\varrho}}\dot{\varrho} = 0$$

is not straightforward.

5. The Curvature Tensor Field

We denote by $\mathcal{R}$ and $\mathcal{R}^B$ the curvature tensor fields of ∇ and ∇^B obtained in the last section. As usual we use the same symbol for the (0,4) and (1,3) tensor fields, e.g.

$$\mathcal{R}(W,Z,X,Y) = g(\mathcal{R}(X,Y)Z,W).$$

The basic result for what follows is

Theorem 5.1. *The curvature tensor fields of the Levi-Cevita connections of* $(\mathcal{D},g)$ *and* $(\mathcal{D}^1,g^B)$ *are given by*

$$\begin{aligned}\mathcal{R}\,(W,Z,X,Y) \;=\;& 2g\left(iN[\,\overline{X}\,,\,\overline{Y}\,]N, i[\,\overline{W}\,,\,\overline{Z}\,]\right)\\ &+g\left(iN[\,\overline{Z}\,,\,\overline{Y}\,]N, i[\,\overline{W}\,,\,\overline{X}\,]\right)\\ &-g\left(iN[\,\overline{Z}\,,\,\overline{X}\,]N, i[\,\overline{W}\,,\,\overline{Y}\,]\right)\end{aligned} \tag{5.1}$$

$$\begin{aligned}\mathcal{R}^B\,(W,Z,X,Y) \;=\;& \mathcal{R}\,(W,Z,X,Y)\\ &+g^B\,(Y,Z)\,g^B\,(X,W) - g^B\,(X,Z)\,g^B\,(Y,W)\end{aligned} \tag{5.2}$$

For notations compare Section 2. The imaginary units were introduced to make the arguments of g hermitean.

Proof. First we show (5.1), then (5.2) follows from the Gauss theorem. Equation (5.1) we could get by a straightforward but quite lengthy computation after inserting (4.5) into the definition of the curvature. The proof is simplified if we use algebraic properties of quadrilinear mappings of the curvature type (comp. [10]). For this purpose denote by $\widetilde{\mathcal{R}}(W,Z,X,Y)$ the right hand side of (5.1). $\widetilde{\mathcal{R}}$ is antisymmetric in the first two and in the last two arguments. Moreover, it satisfies

$$\widetilde{\mathcal{R}}(W,Z,X,Y) + \widetilde{\mathcal{R}}(W,X,Y,Z) + \widetilde{\mathcal{R}}(W,Y,Z,X) = 0\ .$$

Thus (comp. [10]), $\mathcal{R} = \widetilde{\mathcal{R}}$ follows if we prove

$$\mathcal{R}(X,Y,X,Y) = \widetilde{\mathcal{R}}(X,Y,X,Y)\ ,$$

where we have from the right hand side of (5.1)

$$\widetilde{\mathcal{R}}(X,Y,X,Y) = 3g\left(iN[\,\overline{X}\,,\,\overline{Y}\,]N, i[\,\overline{X}\,,\,\overline{Y}\,]\right)\ .$$

The essential technical problem is solved by the following lemma, whose proof is given in the Appendix.

Lemma 5.2. *Let X and Y be vector fields on $\mathcal{D}$. Then*

$$\mathcal{R}(X,Y)Y := \left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]_{vf}}\right)Y = 3\left[\frac{L\circ R}{L+R}\left([\overline{X}\,,\overline{Y}]\right),\overline{Y}\right]\ . \tag{5.3}$$

This lemma and definition (2.4) of g yield

$$\begin{aligned}\mathcal{R}(X,Y,X,Y) &= g(\mathcal{R}(X,Y)Y,X)\\ &= \frac{3}{2}Tr\,\left[\frac{L\circ R}{L+R}\left([\overline{X}\,,\overline{Y}]\right),\overline{Y}\right]\overline{X}\\ &= \frac{3}{2}Tr\,\frac{1}{L+R}\left(N[\overline{X}\,,\overline{Y}]N\right)[\overline{Y}\,,\overline{X}]\\ &= 3g\left(iN[\overline{X}\,,\overline{Y}]N, i[\overline{X}\,,\overline{Y}]\right) = \widetilde{\mathcal{R}}(X,Y,X,Y)\ .\end{aligned} \tag{5.4}$$

This proves the assertion (5.1). Now, let X, Y, Z and W be vector fields tangent to the Riemannian submanifold $\mathcal{D}^1 \subset \mathcal{D}$. Then by the Gauss formula (comp. [10])

$$\begin{aligned}\mathcal{R}(W,Z,X,Y) = \mathcal{R}^B(W,Z,X,Y) &+ g(\alpha(X,Z),\alpha(Y,W))\\ &- g(\alpha(Y,Z),\alpha(X,W))\ ,\end{aligned} \tag{5.5}$$

where $\alpha(X,Z)$ is the component normal to $\mathcal{D}^1$ of $\nabla_X Z$. Since ${\nabla^B}_X Z$ is the corresponding tangent component, we get

$$\nabla_X Z = {\nabla^B}_X Z + \alpha(X,Z)$$

and, therefore, by Theorem 4.1

$$\alpha(X,Z) = -2g^B(X,Z)N\ .$$

We inserte this and related formulae for the other pairs of fields into (5.5). Using $g^B(N,N) = \frac{1}{4}$ we get the assertion (5.2). ∎

6. The Sectional Curvature

Theorem 5.1 giving the curvature tensor fields on $\mathcal{D}$ and $\mathcal{D}^1$ allows us to determine other curvature quantities. We restrict ourselves to the sectional curvature $\mathcal{K}^B$ of $\mathcal{D}^1$. Let X and Y be vector fields on $\mathcal{D}^1$. Then the sectional curvature of the planes ρ generated by X and Y is given by

$$\mathcal{K}^B(\rho) = \frac{1}{\|\,X\wedge Y\,\|^2}\,\mathcal{R}^B(X,Y,X,Y)\ ,$$

where

$$\|\,X\wedge Y\,\|^2 = g^B(X,X)g^B(Y,Y) - g^B(X,Y)g^B(X,Y)$$

is the square of the area of the parallelograms generated by X and Y. Thus we obtain from Theorem 5.1

Proposition 6.1. *The sectional curvature of the planes ρ generated by the fields X and Y on $\mathcal{D}^1$ is given by*

$$\mathcal{K}^B(\rho) = 1 + \frac{3}{\|\, X \wedge Y \,\|^2}\; g\left(iN[\overline{X}\,,\, \overline{Y}]N, i[\overline{X}\,,\, \overline{Y}]\right) \tag{6.1}$$

■

Corollary 6.2. *The sectional curvature on $\mathcal{D}^1$ fulfils*

$$\mathcal{K}^B \geq 1\,. \tag{6.2}$$

Proof. Note that $Z \longmapsto g_\varrho(\varrho Z \varrho, Z)$ is a positive definite quadratic form on hermitean matrices, because $\varrho > 0$. Thus (6.2) is obvious by (6.1). ■

Corollary 6.3. *The space $\mathcal{D}^1$ is not a space of constant curvature for $n \geq 3$.*

Proof. Let $\mu = \frac{1}{n}\,\mathbb{1}$. We calculate $\mathcal{K}^B$ at μ and show that it is not independent of the plane ρ. First, note that

$$\frac{1}{L_\mu + R_\mu} = \frac{n}{2}\, Id\,.$$

Thus we get by (2.4) and (6.1)

$$\mathcal{K}^B_\mu(\rho) = 1 - \frac{3}{4}\, n\, \frac{Tr\,([X,Y]^2)}{Tr\,(XX) Tr\,(YY) - (Tr\,XY)^2}\quad .$$

Inserting $X = E_{11} - E_{22}$, $Y = E_{11} - E_{33}$ and $X' = E_{12} + E_{21}$, $Y' = E_{23} + E_{32}$ we get $\mathcal{K}^B_\mu(\rho) = 1$ and $\mathcal{K}^B_\mu(\rho') = 1 + \frac{3}{8}n$. ■

Since $\mathcal{D}^1$ is not a space of constant curvature we ask whether the sectional curvature is invariant under parallel displacement or not. A space with this property is called a locally symmetric space ([8]). An equivalent property is that the local geodesic reflections $exp_\varrho(X_\varrho) \mapsto exp_\varrho(-X_\varrho)$ are local isometries. This would mean that locally there is no essential difference between going along a geodesic in a direction X_ϱ and the opposite direction $-X_\varrho$. From this point of view the following theorem should not be a surprise, although we do not give a physical interpretation of this fact.

Theorem 6.4. *The space $\mathcal{D}^1$ is not locally symmetric for $n \geq 3$.*

Proof. It is a well known that a space is locally symmetric iff the covariant derivative of its curvature tensor field vanishes ([8]). We show that $\nabla^B \mathcal{R}^B$ does not vanish at $\mu = \frac{1}{n}\mathbb{1}$. Since $\nabla^B g^B = 0$ we have by Theorem 5.1 $\nabla^B \mathcal{R}^B = \nabla^B \mathcal{R}$, where we regard $\mathcal{R}$ as a (0,4)-tensor field on $\mathcal{D}^1$ given by (5.1). Since $\overline{N} = \frac{1}{2}\mathbb{1}$ we see from (5.1) that N annihilates $\mathcal{R}$. Thus we have

$$\begin{aligned}
\left(\nabla^B \mathcal{R}\right)(X,Y,X,Y) &= \nabla^B(\mathcal{R}(X,Y,X,Y)) &&- 2\,\mathcal{R}(\nabla^B X, Y, X, Y) \\
& &&- 2\,\mathcal{R}(X, \nabla^B Y, X, Y) \\
&= \nabla(\mathcal{R}(X,Y,X,Y)) &&- 2\,\mathcal{R}(\nabla X, Y, X, Y) \\
& &&- 2\,\mathcal{R}(X, \nabla Y, X, Y) \\
&= (\nabla \mathcal{R})(X,Y,X,Y) &&,
\end{aligned}$$

where we used that by Theorem 4.1. $\nabla^B - \nabla = 2\, g^B\, N$. Hence, $\nabla^B \mathcal{R}^B = \nabla \mathcal{R}$ on $\mathcal{D}^1$. Define the (1,2)-tensor field C on $\mathcal{D}$ by

$$C(X,Y) := i\,[\,\overline{X}\,,\,\overline{Y}\,]\,. \tag{6.3}$$

Then we have by (5.1)

$$\mathcal{R}(X,Y,X,Y) = 3\,g(L\circ R\circ C(X,Y), C(X,Y))$$

and, therefore,

$$\begin{aligned}(\nabla\mathcal{R})\,(X,Y,X,Y) &= 3\,g((\nabla\; L\circ R)\circ C(X,Y), C(X,Y)) \\ &+ \; 6\,g(L\circ R\circ (\nabla C)(X,Y), C(X,Y))\,. \end{aligned}\tag{6.4}$$

The covariant derivatives of $L\circ R$ and C we calculate using the Leibniz rule and $\nabla = \nabla^f + S$, where S is given by (4.7), e.g.

$$\begin{aligned}(\nabla_Z\; L\circ R)\,(X) &= \nabla_Z(L\circ R(X)) - L\circ R\,(\nabla_Z X) \\ &= ZXN + NXZ + S(Z,NXN) - NS(Z,X)N\,.\end{aligned}$$

We are interested in the value of this quantity at μ only. Thus, for hermitian traceless matrices X,Y,Z we get

$$(\nabla_Z\; L\circ R)_\mu\,(X) = \frac{1}{n}\,(ZX + XZ) \tag{6.5}$$

and, analogously,

$$(\nabla_Z\; C)_\mu\,(X,Y) = -i\,\frac{n^3}{8}\;\left(\,Z[X,Y] + [X,Y]Z + XZY - YZX\,\right)\,. \tag{6.6}$$

Moreover, we have at μ

$$g_\mu(X,Y) \;=\; \frac{n}{4}\,Tr\;XY \tag{6.7}$$

$$L\circ R\;_\mu \;=\; \frac{1}{n^2}\;Id \tag{6.8}$$

$$C_\mu(X,Y) \;=\; i\,\frac{n^2}{4}\,[X,Y]\,. \tag{6.9}$$

Finally, we insert (6.5)–(6.9) into (6.4) and obtain

$$(\nabla_Z\mathcal{R})_\mu\,(X,Y,X,Y) = \frac{3}{64}\,n^4\;Tr\;Z\left\{(XY)^2 + (YX)^2 - X^2Y^2 - Y^2X^2\right\}. \tag{6.10}$$

But, in general the right hand side does not vanish for $n \geq 3$, e.g. $X = E_{12} + E_{21}$, $Y = E_{23} + E_{32}$, $Z = E_{11} - E_{22}$ and, therefore, $\nabla^B\mathcal{R}^B \neq 0$. ∎

7. Appendix

Proof of Lemma 5.2. By Theorem 4.1 we have $\nabla = \nabla^f + S$, where S is the symmetric (1,2)-tensor field on $\mathcal{D}$ given by (4.7). Using the Leibniz rule and the vanishing of curvature and torsion of ∇^f we get

$$\begin{aligned} & \left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]_{vf}}\right) Y \\ = & \left(\nabla^f_X S\right)(Y,Y) - \left(\nabla^f_Y S\right)(X,Y) + S(S(Y,Y),X) - S(S(X,Y),Y) \,. \end{aligned} \tag{7.1}$$

To determine $\nabla^f S$ one replaces—using the Leibniz rule—N and $(L{+}R)^{-1}$ involved in S by their flat covariant derivatives (2.5) and (4.3). To simplify the notation we write X , Y instead of $\overline{X}$, $\overline{Y}$. This yields

$$\left(\nabla^f_X S\right)(Y,Y) = 2\left\{(\overline{X\,Y{+}Y\,X})N\,Y \;+\; Y\,N(\overline{X\,Y{+}Y\,X}) - \;Y\,X\,Y\right\} .$$

Overline means as defined in Section 2 the application of $(L+R)^{-1}$. Now, we substitute X by $N\,X{+}X\,N$ and obtain

$$\begin{aligned} \left(\nabla^f_X S\right)(Y,Y) \;=\; & 2\Big\{(N\overline{X\,Y} + \overline{X\,N\,Y} + \overline{Y\,N\,X} + \overline{Y\,X}N)N\,Y \\ & +Y\,N(N\overline{X\,Y} + \overline{X\,N\,Y} + \overline{Y\,N\,X} + \overline{Y\,X}N) \\ & -Y\,(N\,X \,+\, X\,N)\,Y\,\Big\} . \end{aligned}$$

Next, we substitute in the last two terms $X\,Y$ by $N\overline{X\,Y} + \overline{X\,Y}N$ and similary for $Y\,X$. This yields

$$\begin{aligned} \left(\nabla^f_X S\right)(Y,Y) \;=\; & 2\left[N\overline{[X\,,Y\,]}N,\, Y\right] \\ + \;& 2(\overline{X\,N\,Y} + \overline{Y\,N\,X})N\,Y \\ + \;& 2\,Y\,N(\overline{X\,N\,Y} + \overline{Y\,N\,X}) \,. \end{aligned} \tag{7.2}$$

Analogously we find for the second term of (7.1)

$$\begin{aligned} -\left(\nabla^f_Y S\right)(X,Y) \;=\; & \left[N\overline{[X\,,Y\,]}N,\, Y\right] \\ - \;& (\overline{X\,N\,Y} + \overline{Y\,N\,X})N\,Y \\ - \;& Y\,N(\overline{X\,N\,Y} + \overline{Y\,N\,X}) \\ - \;& 2\overline{Y\,N\,Y}N\,X \,-2\,X\,N\overline{Y\,N\,Y} \,. \end{aligned} \tag{7.3}$$

The remaining terms of (7.1) yield

$$S(S(Y,Y),X) \;=\; 2\overline{Y\,N\,Y}N\,X \,+2\,X\,N\overline{Y\,N\,Y} \tag{7.4}$$

$$\begin{aligned} -S(S(X,Y),Y) \;=\; & -\overline{X\,N\,Y}N\,Y \,-\, Y\,N\overline{X\,N\,Y} \\ & -\overline{Y\,N\,X}N\,Y \,-\, Y\,N\overline{Y\,N\,X} \,. \end{aligned} \tag{7.5}$$

Finally, we add equations (7.2)–(7.5) and get by (7.1) the assertion of Lemma 5.2

$$\begin{aligned} \mathcal{R}(X,Y)Y \;=\; & 3\left[N\overline{[X\,,Y\,]}N,\, Y\right] \\ \;=\; & 3\left[\frac{L\circ R}{L{+}R}\left([\overline{X}\,,\overline{Y}]\right), \overline{Y}\right] . \end{aligned}$$

∎

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