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Constant Yang-Mills potentials

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The sourceless Yang-Mills equations on a manifold (E, g) for a potential $A = A_\alpha dx^\alpha$ with values in some Lie algebra L read

$$D_\alpha F^{\alpha\beta} = \partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0$$

where $[.,.]$ denotes the commutator in L, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$ are the field strength components, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ the partial derivative with respect to the local coordinates x^α of $x \in E$, D_α the gauge-covariant derivative and $g = g_{\alpha\beta} dx^\alpha dx^\beta$ a Riemannian metric. We want to discuss Yang-Mills potentials A with constant components A_α in some gauge and choice of coordinates. The Yang-Mills equations $D_\alpha F^{\alpha\beta} = 0$ then collapse to the algebraic equations

$$[F_{\alpha\beta}, A^\beta] = [[A_\alpha, A_\beta], A^\beta] = 0 \qquad \text{(YM)}$$

A general solution of (YM) is not available. It depends on the structure of L and the signature of g whether or not there exists a non-trivial solution of (YM). We can decide this problem for special types of Lie algebras - Abelian, nilpotent, compact - and for all Lie algebras L of a dimension ≤ 5. If there exists a non-trivial solution of (YM) at all, then the problem arises to find all solutions of (YM).

Let E be an n-dimensional vector space and E^* its dual. Let further $\{e_\alpha\} = \{e_1, \ldots, e_n\}$ be a base of E and $\{e^\beta\} = \{e^1, \ldots, e^n\}$ the dual base of E^*. A scalar product g on E has components $g_{\alpha\beta} := g(e_\alpha, e_\beta)$; we set $(g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}$. The pair (E, g) is interpreted as the (flat) physical space and the spacetime if g is Euclidean and Lorentzian respectively. Let L be an N-dimensional real Lie algebra and $\{X_i\} = \{X_1, \ldots, X_N\}$ a base of L, c_{ij}^k the structure constants of L with respect to $\{X_i\}$, that means $[X_i, X_j] = c_{ij}^k X_k$ for $i, j, k = 1, \ldots, N$. Physically interpreted, L is the Lie algebra of some Lie group G of gauge symmetries.

An element $A = a_\alpha^i e^\alpha \otimes X_i \in E^* \otimes L$, $a_\alpha^i \in R$, is called a one-form or a potential on E with values in L. The field strength $F \in (\wedge^2 E^*) \otimes L^1$ and the current $J \in E^* \otimes L^2$ of A are defined by

$$F(x, y) := [A(x), A(y)]$$
$$J(x) := g^{\alpha\beta}[A(e_\alpha), F(x, e_\beta)]$$

for $x, y \in E$. Note the antisymmetry: $F(x, y) = -F(y, x)$.
There is a component representation with Greek indices $\alpha, \beta, \gamma, \ldots = 1, \ldots, n$:

$A = e^\alpha \otimes A_\alpha$, where $A_\alpha \equiv A(e_\alpha) := a^i_\alpha X_i \in L$,

$F = (e^\alpha \wedge e^\beta) \otimes F_{\alpha\beta}$, where $F_{\alpha\beta} \equiv F(e_\alpha, e_\beta) := [A_\alpha, A_\beta]$,

$J = e^\alpha \otimes J_\alpha$, where $J_\alpha \equiv J(e_\alpha) := [A^\beta, F_{\alpha\beta}]$,

and another representation with Latin indices $i, j, k, \ldots = 1, \ldots, N$:

$A = a^i \otimes X_i$, where $a^i := a^i_\alpha e^\alpha \in E^*$,

$F = f^k \otimes X_k$, where $f^k := c^k_{ij} a^i \wedge a^j$,

$J = j^m \otimes X_m$, where $j^m := c^m_{kl} f^k \angle a^l$.,

The Greek indices are raised and lowered by means of $(g^{\alpha\beta})$ and $(g_{\alpha\beta})$ respectively. The inner multiplication with respect to g of a tensor by a one-form is denoted by $\angle$. The potential A is called flat if $F = 0$; it is called a Yang-Mills (abbreviated YM) potential if $J = 0$. A flat potential is trivial in the sense that it can be gauge-transformed to zero. We search for potentials A such that $F \neq 0$ and $J = 0$.

Now we specialize the type of the Lie algebra L in order to make the problem tractable. The following three structural theorems are essential.

Theorem 1. *Let $L = L_I \oplus L_{II}$ be the direct sum of Lie algebras L_I, L_{II} and let $A = A_I + A_{II}$ be the corresponding decomposition of an L-valued potential into an L_I-valued potential A_I and an L_{II}-valued potential A_{II}. The YM equations for A are equivalent to both the YM equations for A_I and A_{II}.*

Theorem 2. *Let $L = L_I \niplus L_{II}$ be the semidirect sum of an ideal L_I and a subalgebra L_{II} and let $A = A_I + A_{II}$ be the corresponding decomposition of a potential. The YM equations for A imply the YM equations for A_{II}.*

Theorem 3. *If a Lie subalgebra M of L admits a non-flat YM potential then so does L.*

The proofs are easy and omitted here.[1]

Theorem 4. *Every YM potential A with values in a 3-nilpotent Lie algebra L is a constant YM potential.*

Proof. (YM) holds identically: $[[L, L], L] = 0 \Rightarrow [[A_\alpha, A_\beta], A^\beta] = 0$. ■

Example . The Heisenberg algebra $H(2m + 1)$ has the only non-vanishing structure relations $[X_i, Y_i] = Z$ for $i = 1, \ldots, m$ with respect to some base $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z\}$. It is 3-nilpotent.

Corollary . *Every non-Abelian nilpotent Lie algebra L admits a non-flat YM potential A.*

Proof. It is known that the 3-dimensional Heisenberg algebra appears as a subalgebra of every nilpotent Lie algebra L: $H(3) \subseteq L$. ∎

Theorem 5. *If $L' = [L, L]$ is Abelian and of codimension 1 in L then L admits only flat YM potentials.*

Proof. Let a base $\{X_1, \ldots, X_{N-1}\}$ of L' be completed to a base $\{X_0, X_1, \ldots, X_{N-1}\}$ of L. Since L' is Abelian, the only non-trivial commutator relations are $[X_p, X_0] = c^q_{p0} X_q$, where $p, q = 1, \ldots, N-1$. Since L' has dimension $N-1$, the matrix $c = (c^q_{p0})$ is regular and so is its square $b = (b^q_p) := c^2$. Now F and J reduce to the components $f^r = 2c^r_{q0}(a^q \wedge a^0)$ and $j^p = b^p_q(a^q \wedge a^0)\angle a^0$ respectively. The YM equations become equivalent to $(a^q \wedge a^0)\angle a^0 = 0$. Hence each pair (a^q, a^0), $p = 1, \ldots, N-1$, is linearly dependent and $F = 0$. ∎

Theorem 6. *A constant YM potential on a Euclidean or Minkowski space with values in a compact Lie algebra is flat.[1]*

Corollary . *Let $L = L_I \niplus L_{II}$ be the semidirect sum of an ideal L_I and a compact subalgebra L_{II} and let $A = A_I + A_{II}$ be the corresponding decomposition of a potential. There exists a gauge in which the YM equations for A reduce to the YM equations for A_I and to $A_{II} = 0$.*

Concisely expressed: Compact right summands in a semidirect decomposition of a Lie algebra can be ignored. The situation may appear as a special case to the Levi-Malcev theorem which states that every Lie algebra is the semidirect sum $L = I \niplus S$ of a solvable ideal I and a semisimple subalgebra S.

Now we want to consider low values of $\dim L \leq 5$ and Euclidean (E, g). We make use of the classification of the isomorphy types of Lie algebras. The Levi-Malcev theorem allows to reduce the problem of classification of all Lie algebras to the following subproblems:

Classification of the solvable Lie algebras. Solvable Lie algebras are completely classified in the literature up to dimension 6. ([2], [3], [4], [5], [6], [7])

Classification of all semisimple, or rather, simple Lie algebras. Simple Lie algebras are completely classified nowadays.

Classification of the derivations of solvable Lie algebras. This problem is solved up to dimension 9. [8]

The number of isomorphy types of N-dimensional Lie algebras rapidly increases with N:

$\dim L = N$	Number of isomorphy types	Number of indecomposable types
1	1	1
2	2	1
3	8	6
4	19	10
5	–	40
6	–	> 100

Let us follow MUBARAKZYANOV's notation for the isomorphy types of indecomposable Lie algebras: $L^{h,\dots}_{N,k}$ means the k-th type of the Lie algebra of dimension N. Eventual superscripts $h, \dots$ stand for the continous parameters on which L depends. Mubarakzyanov further abbreviates the direct sum of m copies of a Lie algebra L by mL.

We use the Levi-Malcev theorem and, moreover, the following structure results to reduce the big number of isomorphy types of low-dimensional Lie algebras:

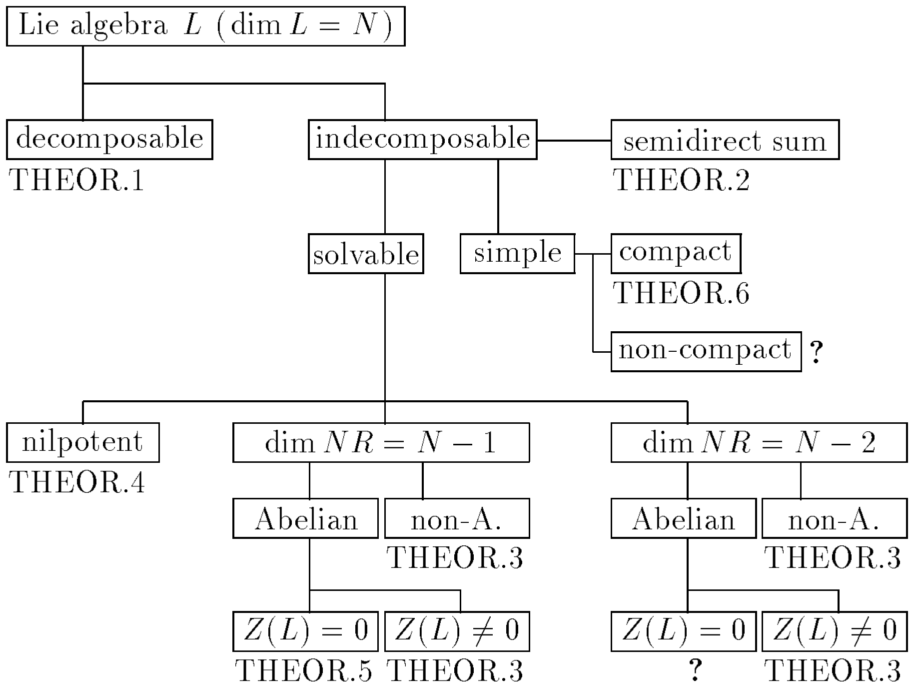
– MUBARAKZYANOV classifies solvable Lie algebras with respect to their maximal nilradical NR, where $\dim NR \geq \frac{N}{2}$. Furthermore he classifies with respect to the dimension of the centre $Z(L)$ of L, where $\dim Z(L) \leq 2\dim NR - N$. Only the fact whether $Z(L) = 0$ or $Z(L) \neq 0$ matters here.

– Every indecomposable solvable Lie algebra with a non-vanishing centre $Z(L) \neq 0$ contains the 3-dimensional Heisenberg algebra as a subalgebra. According to Theorem 3 the corresponding (YM) has a non-flat solution.

In view of the above theorems we should work inductively with respect to N.

The existence of non-flat YM potentials remaines open only for

- non-compact simple Lie algebras and
- indecomposable solvable Lie algebras with Abelian nilradical of $\dim NR \leq N - 2$ and with $Z(L) = 0$.



Let us now go through all isomorphy types of Lie algebras with $N \leq 4$ and the open cases for $N = 5$ in the following tables.

$\dim L \leq 3$

$L_{N,k}^h$	Non-vanishing commutator relations	Remarks
L_1		Abelian $\Rightarrow$ Every A is flat.
$2L_1$		decomposable $\Rightarrow$ THEOR.1
L_2	$[X_1, X_2] = X_1$	(YM) $\Rightarrow (a^1 \wedge a^2) \angle a^2 = 0$ $\Rightarrow a^1, a^2$ linearly dependent $\Rightarrow (a^1 \wedge a^2) = 0$, hence $F = 0$, i. e. A is flat
$3L_1$		decomposable $\Rightarrow$ THEOR.1
$L_2 \oplus L_1$	$[X_1, X_2] = X_1$	decomposable $\Rightarrow$ THEOR.1
$L_{3,1}$	$[X_2, X_3] = X_1$	$L_{3,1} \cong H(3)$ nilpotent $\Rightarrow$ THEOR.4
$L_{3,2}$	$[X_1, X_3] = X_1$ $[X_2, X_3] = X_2$	$NR = 2L_1$, $Z(L) = 0$ $\Rightarrow$ THEOR.5
$L_{3,3}^h$	$[X_1, X_3] = X_1$ $[X_2, X_3] = hX_2$ $h \leq \|1\|$, $h \neq 0$	$NR = 2L_1$, $Z(L) = 0$ $\Rightarrow$ THEOR.5
$L_{3,4}^p$	$[X_1, X_3] = pX_1 - X_2$ $[X_2, X_3] = X_1 + pX_2$ $p \geq 0$	$NR = 2L_1$, $Z(L) = 0$ $\Rightarrow$ THEOR.5
$L_{3,5}$	$[X_1, X_2] = X_3$ $[X_2, X_3] = X_1$ $[X_3, X_1] = X_2$	$L_{3,5} \cong so(3)$ compact $\Rightarrow$ THEOR.6
$L_{3,6}$	$[X_1, X_2] = -X_3$ $[X_2, X_3] = X_1$	$L_{3,6} \cong sl(2, R)$ (YM) $\Rightarrow f^3 \angle a^2 = f^2 \angle a^3$, $f^3 \angle a^1 = -f^1 \angle a^3$, $f^2 \angle a^1 = f^1 \angle a^2$ $\Rightarrow a^1, a^2, a^3$ linearly dependent, hence $F = 0$, i. e. A is flat

If L has dimension $N \leq 3$ and admites a non-flat YM potential then $L \cong H(3)$.

$\dim L = 4$ (indecomposable)

$L^h_{N,k}$	Non-vanishing commutator relations	Remarks
$L_{4,1}$	$[X_2, X_4] = X_1, [X_3, X_4] = X_2$	nilpotent $\Rightarrow$ THEOR.4
$L^{\alpha}_{4,2}$	$[X_1, X_4] = \alpha X_1, [X_2, X_4] = X_2$ $[X_3, X_4] = X_2 + X_3,\ \alpha \neq 0$	$NR = 3L_1,\ Z(L) = 0$ $\Rightarrow$ THEOR.5
$L_{4,3}$	$[X_1, X_4] = X_1$ $[X_2, X_4] = X_1 + X_2$ $[X_3, X_4] = X_2 + X_3$	$NR = 3L_1,\ Z(L) = 0$ $\Rightarrow$ THEOR.5
$L^{\beta,\gamma}_{4,4}$	$[X_1, X_4] = X_1, [X_2, X_4] = \beta X_2$ $[X_3, X_4] = X_2 + X_3,\ \beta, \gamma \neq 0$	$NR = 3L_1,\ Z(L) = 0$ $\Rightarrow$ THEOR.5
$L^{\alpha,p}_{4,5}$	$[X_1, X_4] = \alpha X_1$ $[X_2, X_4] = pX_2 - X_3$ $[X_3, X_4] = X_2 + pX_3$ $\alpha \neq 0, p \geq 0$	$NR = 3L_1,\ Z(L) = 0$ $\Rightarrow$ THEOR.5
$L_{4,6}$	$[X_1, X_4] = X_1, [X_3, X_4] = X_2$	$NR = 3L_1,\ Z(L) \neq 0$ $\Rightarrow$ THEOR.3
$L_{4,7}$	$[X_2, X_3] = X_1, [X_1, X_4] = 2X_1$ $[X_2, X_4] = X_2$ $[X_3, X_4] = X_2 + X_3$	$NR = H(3)$ $\Rightarrow$ THEOR.3
$L^h_{4,8}$	$[X_2, X_3] = X_1$ $[X_1, X_4] = (1 + h)X_1$ $[X_2, X_4] = X_2, [X_3, X_4] = hX_3$ $h \leq \|1\|, h \neq 0$	$NR = H(3)$ $\Rightarrow$ THEOR.3
$L^p_{4,9}$	$[X_2, X_3] = X_1, [X_1, X_4] = 2pX_1$ $[X_2, X_4] = pX_2 - X_3$ $[X_3, X_4] = X_2 + pX_3,\ p \geq 0$	$NR = H(3)$ $\Rightarrow$ THEOR.3
$L_{4,10}$	$[X_1, X_3] = X_1, [X_2, X_3] = X_2$ $[X_1, X_4] = -X_2, [X_2, X_4] = X_1$	$NR = 2L_1,\ Z(L) = 0$ $\exists$ a non-flat A^*

* $A = (e^1 + e^2) \otimes (X_1 + X_2) + e^3 \otimes X_3 + e^4 \otimes X_4$ on a Euclidean space (E, g) with orthonormal base $\{e^1, \dots, e^4\}$ of E^* is a non-flat YM potential:
$F_{13} = F_{23} = X_1 + X_2,\ F_{14} = F_{24} = X_1 - X_2$, but $J = 0$.

If L has dimension $N \geq 4$ then the number of Lie algebras which admits a non-flat YM potential are predominant.

$\dim L = 5$

with L indecomposable, solvable, $NR = 3L_1$, $Z(L) = 0$

$L_{5,33}^{\alpha\beta}$: $[X_1, X_4] = X_1$, $[X_3, X_4] = \alpha X_3$, $[X_2, X_5] = X_2$, $[X_3, X_5] = \beta X_3$, $(\alpha, \beta) \neq (0, 0)$

$$\begin{aligned}
\text{(YM)} \quad &\Rightarrow \quad (a^1 \wedge a^4)\angle a^4 = 0,\ (a^2 \wedge a^5)\angle a^5 = 0, \\
&\quad\quad (a^3 \wedge b)\angle b = 0,\ b := \alpha a^4 + \beta a^5 \\
&\Rightarrow \quad (a^1, a^4),\ (a^2, a^5),\ (a^3, b) \text{ linearly dependent} \\
&\Rightarrow \quad F = 2(a^1 \wedge a^4) \otimes X_1 + 2(a^2 \wedge a^5) \otimes X_2 + 2(a^3 \wedge b) \otimes X_3 = 0, \\
&\quad\quad \text{i. e., } A \text{ is flat}
\end{aligned}$$

$L_{5,34}^{\alpha}$: $[X_1, X_4] = \alpha X_1$, $[X_2, X_4] = X_2$, $[X_3, X_4] = X_3$, $[X_1, X_5] = X_1$, $[X_3, X_5] = X_2$

$H(3)$ is a subalgebra $\Rightarrow$ THEOR.3

$L_{5,35}^{\alpha\beta}$: $[X_1, X_4] = \alpha X_1$, $[X_2, X_4] = X_2$, $[X_3, X_4] = X_3$, $[X_1, X_5] = \beta X_1$, $[X_2, X_5] = -X_3$, $[X_3, X_5] = X_2$, $(\alpha, \beta) \neq (0, 0)$

$L_{4,10}$ is a subalgebra $\Rightarrow$ THEOR.3

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