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# Moufang loops and Malcev algebras

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**Abstract**

The paper contains the proof of the uniqueness of a connected and simply connected analytical Moufang loop having a given tangent Malcev algebra. This result completes the extension of the global Lie theory to analytical Moufang loops investigated by E. N. Kuzmin and F. S. Kerdman.

## 1. Preliminaries

One of the central questions in differentiable loop theory is the generalization of the Lie algebra - Lie group correspondence to the non-associative case. For local analytical Moufang loops this question is solved by E. N. Kuzmin [5] using Campbell-Hausdorff formulas. The tangent algebras of these loops are called Malcev algebras. The possibility of realizing a given Malcev algebra as tangent algebra of a unique global Moufang loop has been investigated by F. S. Kerdman [3]. He proved the following:

**Theorem.** *For any real Malcev algebra there exists an analytical Moufang loop whose tangent algebra is the given Malcev algebra. If the loop is solvable or semisimple and simply connected, then it is uniquely determined by its tangent Malcev algebra up to isomorphism.*

The uniqueness question is open in the general case. This is closely related to the extension problem of local homomorphisms and isomorphisms of analytical loops (cf. K. H. Hofmann and K. Strambach [2], Problem IX.6.33). The aim of our paper is the proof of the second assertion of Kerdman's theorem without the assumption of solvability or semisimplicity by the investigation of the existence and uniqueness up to isomorphism of global extensions of analytical local loop isomorphisms.

## 2. Loops, 3-nets and Chern connection

***Definition 2.1.*** *We say that the manifold $\mathcal{N}$ has a 3-net structure, if there are given three direct product decompositions $\mathcal{X}_i \times \mathcal{X}_{i+1}$ of $\mathcal{N}$ such that the vertical leaves of the $i$-th decomposition $\mathcal{X}_i \times \mathcal{X}_{i+1}$ coincide with the horizontal leaves of the $i+1$-th decomposition $\pmod 3, i = 1,2,3$. In the following we choose one of these product decompositions and its horizontal and vertical leaves we call horizontal and vertical lines, respectively. The leaves of the third foliation in the given product decompositions are called transversal lines of the 3-net $\mathcal{N}$.*

It is clear that the lines of a 3-net are of the same dimension which is equal to the half of the dimension of the net manifold. We denote by $\mathcal{H}, V$ and $\mathcal{T}$ the family of horizontal, vertical and transversal lines, respectively, they are called sometimes the directions of the 3-net. Each direction $\mathcal{H}, V, T$ has a natural differentiable manifold structure. The dimension of the manifolds of parallel lines (i.e., of the directions $\mathcal{H}, V$ and $\mathcal{T}$) is the same as the dimension of lines. It follows from the above definitions that the function assigning to every point the horizontal (vertical or transversal) line incident with this point and the function defined by the intersection of two non-parallel lines are differentiable. We call the tangent spaces of the horizontal, vertical and transversal lines through a point $x \in \mathcal{N}$ the horizontal, vertical and transversal tangent spaces at $x$ and we denote them $T^h\mathcal{N}$, $T^v\mathcal{N}$ and $T^t\mathcal{N}$, respectively. The projection operators of the tangent bundle to the horizontal and vertical distributions with respect to the direct sum decomposition $T\mathcal{N} = T^h\mathcal{N} \oplus T^v\mathcal{N}$ we denote by $H\colon T\mathcal{N} \to T^h\mathcal{N}$, $V\colon T\mathcal{N} \to T^v\mathcal{N}$. The projection operators $H$ and $V$ are $(1,1)$-tensor fields on $\mathcal{N}$ satisfying $H^2 = H$, $V^2 = V$ and $H + V = I$, where $I$ is the identity operator. We denote by $J$ the $(1,1)$-tensor field on $T\mathcal{N}$ defined by the following properties: $J^2 = I$, $JH = HJ$ and $\ker(J + I) = T^t\mathcal{N}$.

***Proposition 2.2.*** *There exists a unique covariant derivation $\nabla$ on the 3-net manifold $\mathcal{N}$ which satisfies*

$$\nabla H = \nabla V = \nabla J = 0,$$

*and*

$$T(HX, VY) = 0 \text{ for any vectorfields } X,\ Y \text{ on } \mathcal{N},$$

*where $T(X,Y)$ denotes the torsion tensorfield of the covariant derivation $\nabla$.*

**Proof.** **Cf. Theorem 3.2 in [7].** □

***Definition 2.3. The covariant derivation described in the preceding Proposition is called the Chern connection of the $3$-net $\mathcal{N}$.***

**It is well known that if we fix an origin $O$ in the 3-net manifold $\mathcal{N}$, then a loop multiplication $x \circ y$ can be introduced on the horizontal line $L$ through $O$. This operation is defined by the projection maps onto the lines, thus it is differentiable. Moreover $\mathcal{N}$ is diffeomorphic to the direct product $L \times L$, where the lines of the 3-net $\mathcal{N}$ have the following form:**

***horizontal lines:*** $L \times \{y_0\}$, $y_0 \in L$,

***vertical lines:*** $\{x_0\} \times L$, $x_0 \in L$,

***transversal lines:*** $\{(x,y) : x \circ y = z_0,\ z_0 \in L\}$.

**From the other hand if there is given a differentiable loop $L$ then the lines of the preceding form in the product manifold $L \times L$ determine a 3-net structure, which is called the *canonical $3$-net of the loop* $L$ and is denoted by $\mathcal{N}(L)$.**

## 3. Extension of local loop isomorphisms

***Proposition 3.1.*** *Let $L$ be a connected and simply connected analytical loop and $U$ a connected neighbourhood of the identity $e \in L$. Suppose that $M$ is a connected analytical loop and $f\colon U \to V \subset M$ is a bijective analytical map such that $x, y, xy \in U$ implies $f(xy) = f(x)f(y)$. Then the map $f$ can be extended uniquely to a global analytical loop homomorhism $F\colon L \to M$.*

**Proof.** The mapping $f \times f\colon U \times U \to V \times V$ is a local affine map with respect to the Chern connections of the 3-nets $\mathcal{N}(L)$ and $\mathcal{N}(M)$. Since $L \times L$ is connected and simply connected, the map $f \times f$ can be extended to a unique global affine map $\Phi\colon L \times L \to M \times M$ (cf. Kobayashi-Nomizu I. Theorem 6.1 in Chapter VI [4]). Since $\Phi$ preserves the parallel translation with respect to the Chern connection and the tensor fields $H$, $V$, and $J$ associated with the 3-nets are parallel, they are preserved by the map $\Phi$. Since the tangent distributions of the foliations of a 3-net can be described uniquely by the tensor fields $H$, $V$, and $J$, it follows that the map $\Phi$ preserves the foliations of the 3-nets $\mathcal{N}(L)$ and $\mathcal{N}(M)$. The restriction of $\Phi$ to the horizontal line $L \times \{e\}$, $e \in L$ preserves the loop multiplication, hence it determines the unique extending loop homomorphism $F\colon L \to M$ of the map $f\colon U \to V \in M$. $\square$

**Remark.** Since the local map $f$ is bijective and the loop $L$ is simply connected, it is isomorphic to the universal covering loop of $M$ and the extended map $F : L \to M$ corresponds to the universal covering loopmorphism of $L$.

## 4. Moufang loops and Malcev algebras

***Definition 4.1.*** *A loop $L$ is called Moufang loop, if it satisfies the identity $x(y(zy)) = ((xy)z)y$.*

***Definition 4.2.*** *A vector space $\mathfrak{m}$ with a distributive multiplication $[\cdot,\cdot]$ is called a Malcev algebra provided the following conditions are satisfied:*

$$\begin{aligned} [x,y] &= -[y,x], \\ [[x,y],[x,z]] &= [[[x,y],z],x] + [[[y,z],x],x] + [[[z,x],x],y] \end{aligned}$$

*for all $x, y, z \in \mathfrak{m}$ (cf.[9]).*

***Definition 4.3.*** *The tangent algebra $\mathfrak{l}$ of an analytical loop $L$ is the tangent space $T_eL$ of $L$ at the identity $e$ equipped with the commutator bracket operation defined by*

$$[X,Y] := [T_e\lambda_x X, T_e\lambda_x Y], \qquad x \in L,\ X, Y \in T_e$$

*where $\lambda_x = (y \mapsto xy)$ is the right multiplication map $L \to L$ and the operation $[\cdot,\cdot]$ on the right hand side denotes the Lie bracket of vector fields.*

Let $L$ be an analytical Moufang loop. Then its tangent algebra $\mathfrak{m}=T_eL$ is a Malcev algebra (cf. I. A. Malcev [6], E. N. Kuzmin [5]).

*Theorem 4.4. Let* $\mathfrak{m}$ *be a Malcev algebra. Then there exists an up to isomorphism uniquely determined connected and simply connected analytical Moufang loop* $L$*, whose tangent algebra is the given Malcev algebra.*

Proof. The existence of a Moufang loop to a given tangent Malcev algebra has been proved by F. S. Kerdman in [3]. A simply connected one can be found by the universal covering loop construction [2]. If two connected and simply connected Moufang loops $L$ and $M$ have the same tangent Malcev algebra, then they have to coincide in a neighbourhood $U \subseteq L$, $U \subseteq M$ of the identity element $e \in U$, since the local analytical Moufang loop associated with a Malcev algebra is uniquely determined up to local isomorphism (cf.[5]). It follows from Proposition 3.1 that the identity map $\iota: U \to U$ can be extended to a unique analytical loop isomorphism. □

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