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The solution of the epimorphism problem for Hausdorff topological groups

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Consider the category $\mathcal{G}$ of Hausdorff topological groups. Then a morphism $e: G \to H$ in $\mathcal{G}$ is an *epimorphism* if for any pair of morphisms $f, g: H \to K$ satisfying $fe = ge$ we must have $f = g$. Clearly every morphism e with $\overline{e(G)} = H$ is an epimorphism in $\mathcal{G}$. It was an open problem since the late sixties whether the converse is true. This problem is equivalent to the following problem in [3]:

Problem 512. [K. H. Hofmann] *If G is a Hausdorff topological group and H a proper closed subgroup of G, must there exist distinct morphisms of Hausdorff topological groups $f, g: G \to K$ with $f|H = g|H$?*

There is a considerable literature on the problem with many partial results (see the references below). On the power of these partial results it was conjectured that the answer to the problem is yes. However, the answer is negative.

Theorem 1. *Let $\mathbb{T}$ be a circle (a one-dimensional torus). Let $G = \mathrm{Aut}\,\mathbb{T}$ be the group of all self-homeomorphisms of $\mathbb{T}$ with the topology of uniform convergence, and let H be the stability subgroup at some point in $\mathbb{T}$. Then for any pair of morphisms $f, g: G \to K$ into a Hausdorff topological group with $f|H = g|H$ we must have $f = g$.*

Proof. (Sketch) Define $j: G \times G \to K$ by $j(x,y) = f(x)g(y)^{-1}$. Let $\mathcal{U}$ be the coarsest uniformity on $G \times G$ for which j is uniformly continuous w.r.t. $\inf\{\mathcal{L}, \mathcal{R}\}$ (where $\mathcal{L}$ and $\mathcal{R}$ denote the left- and right- uniform structures on K, respectively). Let H act on the right on $G \times G$ via $(x,y)\cdot h = (xh, yh)$, and denote the orbit space of this action by X. Equip X with the quotient uniformity of $(G \times G, \mathcal{U})$. Since $j(xh, yh) = j(x,y)$ for all $h \in H$, the map j induces a map $j': X \to K$. It turns out that the image Δ of $\mathrm{diag}(G \times G)$ in X considered as a uniform subspace of X has the trivial (coarsest) uniformity. Since K is Hausdorff, j' is constant on Δ which means that j is constant on $\mathrm{diag}(G \times G)$. This says $f = g$. ∎

A detailed proof will appear elsewhere.

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