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Geometric Aspects of SL(2)-Invariant Second Order Ordinary Differential Equations

Günter Czichowski

1. Actions of SL(2) on the Real Plane

With respect to real point transformations there are three different actions of the group SL(2) on the real $\mathbb{R}^2$-plane corresponding to certain canonical forms of the Lie algebra sl(2) (see e.g.[2,4], for the classification with respect to complex variables). If this Lie algebra is realized in the known matrix form with generators L_1, L_2, L_3,

$$\mathrm{sl}(2): \qquad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}$$

and corresponding commutator relations

$$[L_1, L_2] = L_1, \qquad [L_2, L_3] = L_3, \qquad [L_1, L_3] = L_2,$$

then the three actions correspond to three canonical forms for sl(2) given in terms of first order differential operators (vector fields) on the x, y-plane $\mathbb{R}^2$ by

$$\partial_1 = \partial_x, \qquad \partial_2 = x\partial_x + y\partial_y, \qquad \partial_3 = \frac{x^2 + \varepsilon y^2}{2}\partial_x + xy\partial_y, \qquad (\varepsilon = 0, 1, -1)$$

The values $0, 1, -1$ of the parameter ε correspond to the various SL(2)-actions as follows:

$\varepsilon = 0$: equivalence to the linear action on $\mathbb{R}^2$,
$\varepsilon = 1$: equivalence to the action by simultaneous Möbius transformations on $\mathbb{R}^2$,
$\varepsilon = -1$: action by Möbius transformations on $\mathbb{C}$.

But with respect to a geometric point of view it is more convenient to give this actions in terms of infinitesimal generators related to the known SL(2)-actions in matrix form cited above. The forms for the generators are given then as follows:

Linear action:

$$\partial_1 = y\partial_x, \qquad \partial_2 = \frac{x}{2}\partial_x - \frac{y}{2}\partial_y, \qquad \partial_3 = -\frac{x}{2}\partial_y,$$

Möbius transformations on $\mathbb{R}^2$*:*

$$\partial_1 = \partial_x + \partial_y, \qquad \partial_2 = x\partial_x + y\partial_y, \qquad \partial_3 = \frac{x^2}{2}\partial_x + \frac{y^2}{2}\partial_y,$$

Möbius transformations on $\mathbb{C}$*:*

$$\partial_1 = \partial_x, \qquad \partial_2 = x\partial_x + y\partial_y, \qquad \partial_3 = \frac{x^2 - y^2}{2}\partial_x + xy\partial_y.$$

2. SL(2)-Invariant Differential Equations and Solutions

Every action of the group SL(2) on $\mathbb{R}^2$ can be characterized by a second order ordinary differential equation which is invariant with respect to this action, and there is an interesting connection between the problem to solve this differential equation (using the symmetries) and to determine the type for the SL(2)-action in the solution space [1]:
In the following p stands for y'. If $y'' = F(x, y, p)$ is an invariant second order differential equation, then consider the differential operator

$$D = \partial_x + p\partial_y + F(x, y, p)\partial_p,$$

whose kernel is invariant under the prolonged group action and consists of all first integrals of the differential equation. Hence there exists a canonical form for the SL(2)-action in the space of first integrals (FI-form). I.e, with respect to suitable coordinates u, v in this space the following relations hold (where ∂_k' denotes the first prolongation of ∂_k):

$$\partial_1' = \partial_u, \qquad \partial_2' = u\partial_u + v\partial_v, \qquad \partial_3' = \frac{u^2 + \varepsilon v^2}{2}\partial_u + uv\partial_v, \qquad (\varepsilon = 0, 1, -1)$$

As a consequence there exist first integrals $u = u(x, y, p), \quad v = v(x, y, p)$ satisfying

$$\partial_1'(u) = 1, \;\; \partial_1'(v) = 0 \quad \partial_2'(u) = u, \;\; \partial_2'(v) = v \quad \partial_3'(u) = \frac{u^2 + \varepsilon v^2}{2}, \;\; \partial_3'(v) = uv$$

Together with the equations $D(u) = 0, D(v) = 0$ there are then 8 equations for u, v, which allow the computation of the first integrals u, v without integration. Elimination of p gives then the general solution containing u, v (constant on any solution) as parameters. The key for this procedure and the interesting fact from the geometric point of view is the change of the canonical forms in the transition from the original x, y-plane to the u, v-plane of first integrals. With respect to this effect the case of SL(2)-symmetry seems to be distinguished among the second order ordinary differential equations with three symmetries.

In the following we will give in the various cases of SL(2)-actions the invariant differential equations with corresponding first integrals and general solutions. The transition between the canonical forms respectively the various SL(2)-actions is characterized in terms of the ε-values: $(\varepsilon_1 \to \varepsilon_2)$.

Since some of the differential equations contain a real parameter c, it should be noted that this transition depends on the c-value too. The exceptional c-values are given by the zeros of certain differential invariants I_1, I_2 [3]. The computations are made with REDUCE. They are, after the formulation of the corresponding equations with the suitable ε-value, in some sense straightforward, but not simple for technical reasons (appearence of radicals, factorizations). As a consequence of the solution procedure described above one gets a good model for the SL(2)-action in the first integral space. We will demonstrate this in a more detailed fashion in the first case ($\varepsilon = 0$).

2.1. Linear Action ($\varepsilon = 0$)

The general form of the corresponding differential equation, which is invariant under the linear action, is given by

$$y'' = c(px - y)^3.$$

After a simple scale transformations one can assume $c = 1$ ore $c = -1$.
2.1.1. Differential equation:

$$y'' = (px - y)^3, \qquad (\varepsilon = 0 \to \varepsilon = -1)$$

First Integrals:

$$u = \frac{p^2x^3y - 2px^2y^2 + p + xy^3}{p^2x^2y^2 + p^2 - 2pxy^3 + y^4}, \qquad v = \frac{p^2x^2 - 2pxy + y^2}{p^2x^2y^2 + p^2 - 2pxy^3 + y^4}.$$

General solution:

$$u^2y^2 - 2uxy + x^2 + v^2y^2 - v = 0.$$

This equation defines for $v > 0$ a two-parametric set of ellipses in central position and with area equal to π, which is of course invariant under the linear action of SL(2). For the limit $v \to 0$ one gets the straight lines $y = \frac{1}{u}x$, which form an invariant one-parametric set of solutions. These are the "boundary" elements in the upper halfplane ($v > 0$) of the u, v-space, which is the solution space as SL(2)-orbit. There is also a geometric argument to construct the general solution without any integration: Take the unit circle $x^2+y^2 = 1, (u = 0, v = 1)$ as a special solution, which is invariant with respect to the rotations in SL(2). Application of the elements of SL(2) leads then to the two-parametric family of ellipses defined above. Hence one gets in the solution space clearly the situation of the SL(2)-action by Möbius transformations on the upper half-plane of $\mathbb{C}$ ($\varepsilon = -1$) : A half-plane as a two-parametric orbit is bounded by a one-parameter orbit. Analogous considerations hold for the following cases, and we give only the results in term of first integrals, solutions and additional remarks in some cases.

2.1.2. Differential equation:

$$y'' = -(px - y)^3, \qquad (\varepsilon = 0 \to \varepsilon = 1)$$

First Integrals:

$$u = \frac{p^2x^3y - 2px^2y^2 - p + xy^3}{p^2x^2y^2 - p^2 - 2pxy^3 + y^4}, \qquad v = \frac{p^2x^2 - 2pxy + y^2}{p^2x^2y^2 - p^2 - 2pxy^3 + y^4}.$$

General Solution: $u^2y^2 - 2uxy + x^2 - v^2y^2 + v = 0$.
The solutions are hyperbolae and lines, the interpretation is analogous to that in the previous case.

2.2. Action by Möbius Transformations on $\mathbb{R}^2$ ($\varepsilon = 1$)

In the following four cases the solutions are hyperbolae and lines, the general solution can be contained by application of the SL(2)-action to a hyperbola $(x+r)(y-r) = s$ (with suitable r, s).

2.2.1. Differential equation:

$$y'' = \frac{cp^{\frac{3}{2}} - 2p^2 - 2p}{x-y}, \quad (\varepsilon = 1 \to \varepsilon = 1, \quad |c| > 4)$$

First Integrals:

$$u = \frac{c\sqrt{p}(x+y) - 4px - 4y}{2(c\sqrt{p} - 2p - 2)}, \qquad v = \frac{\sqrt{c^2-16}\sqrt{p}(x-y)}{2(c\sqrt{p} - 2p - 2)}$$

General Solution:
$-\sqrt{c^2-16}cxv + \sqrt{c^2-16}cyv - c^2u^2 + c^2ux + c^2uy - c^2xy + c^2v^2 + 16u^2 - 16ux - 16uy + 16xy - 16v^2 = 0$

2.2.2. Differential equation:

$$y'' = \frac{cp^{\frac{3}{2}} - 2p^2 - 2p}{x-y}, \quad (\varepsilon = 1 \to \varepsilon = -1, \quad |c| < 4).$$

First Integrals:

$$u = \frac{c\sqrt{p}(x+y) - 4px - 4y}{2(c\sqrt{p} - 2p - 2)}, \qquad v = \frac{\sqrt{16-c^2}\sqrt{p}(x-y)}{2(c\sqrt{p} - 2p - 2)}.$$

General Solution:
$-cv\sqrt{16-c^2}x + cv\sqrt{16-c^2}y + c^2u^2 - c^2ux - c^2uy + c^2v^2 + c^2xy - 16u^2 + 16ux + 16uy - 16v^2 - 16xy = 0.$

2.2.3. Differential equation:

$$y'' = \frac{4p^{\frac{3}{2}} - 2p^2 - 2p}{x-y}, \qquad (\varepsilon = 1 \to \varepsilon = 0, c = 4)$$

First Integrals:

$$u = \frac{\sqrt{p}x - y}{\sqrt{p} - 1}, \qquad v = \frac{\sqrt{p}(x-y)}{p - 2\sqrt{p} + 1}$$

General Solution: $u^2 - ux - uy - vx + vy + xy = 0.$

2.2.4. Differential equation:

$$y'' = \frac{-4p^{\frac{3}{2}} - 2p^2 - 2p}{x-y}, \quad (\varepsilon = 1 \to \varepsilon = 0, \quad c = -4).$$

First Integrals:

$$u = \frac{\sqrt{p}x + y}{\sqrt{p} + 1}, \qquad v = \frac{\sqrt{p}(x-y)}{2\sqrt{p} + p + 1}.$$

General Solution: $u^2 - ux - uy + vx - vy + xy = 0.$

2.3. Action by Möbius transformations on $\mathbb{C}$, $(\varepsilon = -1)$

In the following four cases the solutions are circles and lines, the general solution is contained by application of the Möbius transformations given by the elements of SL(2) in the complex plane to a special circle or line as a special solution.

2.3.1. Differential equation:

$$y'' = \frac{c(p^2+1)^{\frac{3}{2}} - p^2 - 1}{y}, \quad (\varepsilon = -1 \to \varepsilon = 1, \quad |c| < 1).$$

First Integrals:

$$u = \frac{c\sqrt{p^2+1}x - py - x}{c\sqrt{p^2+1} - 1}, \qquad v = \frac{\sqrt{1-c^2}\sqrt{p^2+1}y}{\sqrt{p^2+1}c - 1}.$$

General Solution:
$-2\sqrt{1-c^2}cvy - c^2u^2 + 2c^2ux + c^2v^2 - c^2x^2 - c^2y^2 + u^2 - 2ux - v^2 + x^2 + y^2 = 0.$

The solutions are circles and lines which intersect the x-axis under a fixed angle.The two-parametric family of this curves can be constructed geometrically by the application of SL(2) on the solution $y = rx$ (r suitable). The case $c = 0$, $y'' = \frac{-p^2-1}{y}$ is of special interest: The solutions are circles perpendicular to the x-axis, the group SL(2) is then acting by Möbius transformations in the Poincaré model of the non-euclidean plane.

2.3.2. Differential Equation:

$$y'' = \frac{c(p^2+1)^{\frac{3}{2}} - p^2 - 1}{y}, \quad (\varepsilon = -1 \to \varepsilon = -1, \quad |c| > 1).$$

First Integrals:

$$u = \frac{c\sqrt{p^2+1}x - py - x}{c\sqrt{p^2+1} - 1}, \qquad v = \frac{\sqrt{c^2-1}\sqrt{p^2+1}y}{c\sqrt{p^2+1} - 1}.$$

General Solution:
$-2\sqrt{c^2-1}cvy + c^2u^2 - 2c^2ux + c^2v^2 + c^2x^2 + c^2y^2 - u^2 + 2ux - v^2 - x^2 - y^2 = 0.$
The solutions are circles $(x-a)^2 + (y-b)^2 = \mathbb{R}^2$ with $\frac{b}{r} = c$.

2.3.3. Differential Equation:

$$y'' = \frac{(p^2+1)^{\frac{3}{2}} - p^2 - 1}{y}, \quad (\varepsilon = -1 \to \varepsilon = 0, \quad c = 1).$$

First Integrals:

$$u = \frac{\sqrt{p^2+1}x - py - x}{\sqrt{p^2+1} - 1}, \qquad v = \frac{-y(\sqrt{p^2+1} + p^2 + 1)}{p^2}.$$

General Solution: $u^2 - 2ux + 2vy + x^2 + y^2 = 0.$

The solutions are circles tangent to the x-axis, the solution space is then a two-parametric orbit with the invariant solution $y = 0$ as boundary point.

2.3.4. Differential Equation:

$$y'' = -\frac{(p^2+1)^{\frac{3}{2}} + p^2 + 1}{y}, \quad (\varepsilon = -1 \to \varepsilon = 0, \quad c = -1).$$

First Integrals:

$$u = \frac{\sqrt{p^2+1}x + py + x}{\sqrt{p^2+1}+1}, \quad v = \frac{-y(\sqrt{p^2+1} + p^2 + 1)}{p^2}.$$

General Solution: $u^2 - 2ux - 2vy + x^2 + y^2 = 0$. The solution space can be interpreted as in the previous case.

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Fachrichtungen Mathematik/Informatik
Ernst Moritz Arndt Universität Greifswald
Friedrich Ludwig Jahn-Strasse 15a
D-17487 Greifswald, Germany

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