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On pairs of self-adjoint operators

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This survey is based on the works of Kiev mathematicians and others in the structure theory of pairs of bounded and unbounded self-adjoint operators that satisfy an algebraic relation.

It was a part of a series of lectures given by the authors at the University of Leipzig in the Fall of 1992. For earlier publications on the subject by the authors see [27, 29, 30].

0. Introduction

I. Let H be a separable complex (finite or infinite-dimensional) Hilbert space. We consider pairs $A = A^*$ and $B = B^*$ of self-adjoint (bounded or unbounded) operators which are solutions of the equation

$$P_2(A, B) = \alpha A^2 + \beta_1 AB + \beta_2 BA + \gamma B^2 + \delta A + \epsilon B + \chi I = 0,$$

where α, β_1, β_2, γ, δ, ϵ, $\chi \in \mathbb{C}$. (Of course, if the operators are unbounded, then it is necessary to specify in what sense we understand this equation). Suppose that

$$P_2^*(A, B) = \overline{\alpha} A^2 + \overline{\beta_1} BA + \overline{\beta_2} AB + \overline{\gamma} B^2 + \overline{\delta} A + \overline{\epsilon} B + \overline{\chi} I = P_2(A, B),$$

so we can write this equation as

$$P_2(A, B) = \alpha A^2 + \beta\{A, B\} + i\hbar[A, B] + \gamma B^2 + \delta A + \epsilon B + \chi I = 0, \qquad (1)$$

where α, β, $\hbar$, γ, δ, ϵ, $\chi \in \mathbb{R}$, $[A, B] = AB - BA$ is the commutator, and $\{A, B\} = AB + BA$ is the anticommutator. We also have that $\beta = \frac{1}{2}(\beta_1 + \beta_2)$ and $\hbar = \frac{1}{2i}(\beta_1 - \beta_2)$. In Section 1, we first study pairs $A = A^*$, $B = B^*$ which satisfy (1).

1. If we consider a commuting pair of operators which act in a one-dimensional Hilbert space ($\dim H = 1$), $A = a$ and $B = b$ ($a, b \in \mathbb{R}$), then the solution of (1) is a conic—a second order curve which corresponds to a set of points (a, b) in the real plane $\mathbb{R}^2$, whose coordinates satisfy the equation:

$$P_2(a, b) = \alpha a^2 + 2\beta ab + \gamma b^2 + \delta a + \epsilon b + \chi = 0 \quad (\alpha, \beta, \gamma, \delta, \epsilon, \chi \in \mathbb{R})$$

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Conversely, for any curve (2) in the plane, one can consider a class of non-commutative deformations of it which depend on a parameter $\hbar$. These deformations are given by (1) which can be written as

$$\alpha A^2 + \beta\{A, B\} + \gamma B^2 + \delta A + \epsilon B + \chi I = \frac{\hbar}{i}[A, B] \tag{2}$$

In Section 1.1, we apply a non-degenerate real affine change of variables to $A = A^*$ and $B = B^*$ (so that the new variables are formally self-adjoint) to reduce equation (1) to 19 different kinds of operator equations, and 4 series of equations which depend on a parameter.

It should be noted that to every equation $P_2(A, B) = 0$ one can associate an algebra with involution ($*$-algebra) $\mathfrak{A}_{P_2(\cdot,\cdot)}$ with two self-adjoint generators $A = A^*$, $B = B^*$. This algebra is a factor-algebra of the free $*$-algebra with two generators, $\mathbb{C}\langle A, B\rangle$, with respect to the two-sided ideal generated by relation (1), i.e., it is a $*$-algebra with two generators which satisfy quadratic relation (1).

2. In what follows, we study the structure of the pairs A, B which satisfy these equations. We take *the structure theorem for a single self-adjoint operator A in H*, that gives a decomposition of any self-adjoint operator A into a direct sum (or a direct integral) of the simplest self-adjoint operators which are operators of multiplication on a real constant to be a model for structure theorems given in the article. We consider two cases:

a) $A = A^* \in L(H)$ is a bounded operator in H. Then by applying the spectral theorem to a bounded self-adjoint operator we can represent A as a spectral integral, i.e. there is a resolution of the identity, $E_A(\cdot)$, which is an orthogonal operator-valued measure on the Borel σ-algebra $\mathfrak{B}(\mathbb{R}^1)$ with the support in the compact set $[-\|A\|, \|A\|]$ such that

$$A = \int_{-\|A\|}^{\|A\|} \lambda \, dE_A(\lambda).$$

b) $A = A^*$ is an unbounded operator. Then the support of the resolution of the identity $E_A(\cdot)$ lies in $\mathbb{R}^1$ and

$$A = \int_{\mathbb{R}^1} \lambda \, dE_A(\lambda), \quad D(A) = \{f \in H \mid \int_{\mathbb{R}^1} \lambda^2 \, d(E_A(\lambda)f, f) < \infty\}.$$

As in representation theory, we consider a collection of self-adjoint operators $A_1, A_2, \dots, A_n$ as a simple basic building block if it is irreducible.

Definition 0.1. We say that a collection of self-adjoint operators $A_k = \int_{\mathbb{R}} \lambda_k \, dE_k(\lambda_k)$, $k = 1, \dots, n$ is irreducible if there is no subspace of H (different from H and $\{0\}$), invariant with respect to all the operators $E_k(\Delta)$ ($k = 1, \dots, n$; $\Delta \in \mathfrak{B}(\mathbb{R}^1)$).

If the operators in the collection $(A_k)_{k=1}^n$ are bounded, then the collection will be irreducible if there is no non-trivial subspace of H, which is invariant with respect to all the operators of the collection $(A_k)_{k=1}^n$.

A collection of self-adjoint operators, $(A_k)_{k=1}^n$ is irreducible if and only if any bounded operator C that commutes with all A_k, $k = 1, \dots, n$ (i.e. that commutes with all their spectral projections) is a multiple of the identity.

As in representation theory, we also study pairs (as well as collections) of operators in H up to unitary equivalence.

Definition 0.2. Collections $(A_k)_{k=1}^n$ of operators in a space H and $(\tilde{A}_k)_{k=1}^n$ in a space $\tilde{H}$ are called unitarily equivalent if there exists a unitary operator $U\colon H \to \tilde{H}$ such that the diagrams

$$\begin{array}{ccc} H & \xrightarrow{A_k} & H \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \tilde{H} & \xrightarrow{\tilde{A}_k} & \tilde{H} \end{array}$$

are commutative for all $k = 1, \dots, n$, i.e. $UA_k = \tilde{A}_k U$.

The purpose of Section 1 of the article is for every quadratic relation
1) to describe up to unitary equivalence all irreducible
 a) pairs of bounded self-adjoint operators,
 b) pairs of unbounded selfadjoint operators
that satisfy the relation and
2) to prove a structure theorem on decomposition of any such pair into irreducible ones.

The use of representation theory language is not accidental. To describe pairs of bounded operators that satisfy the relation $P_2(A,B) = 0$ means to describe representations of generators in the corresponding $*$-algebra $\mathfrak{A}_{P_2(\cdot,\cdot)}$, which satisfy the required quadratic relation, by bounded operators.

It is clear that if the operators are unbounded, then it is necessary to define the meaning of the operator equality (1). This can be done in different ways. Consider, for example, the property of commutativity of a pair of self-adjoint operators A and B on an invariant set Φ, which is dense in H, such that the operators are essentially self-adjoint on Φ,

$$(\forall f \in \Phi) \quad (AB - BA)f = 0.$$

This property is, generally speaking, not equivalent to the property of commutativity of their spectral projections. Moreover, there exist *irreducible* pairs of self-adjoint operators, which commute on a dense set in an *infite-dimensional* space H (see, e.g. [39]).

So the meaning of what is a representation of the $*$-algebra $\mathfrak{A}_{P_2(\cdot,\cdot)}$ by unbounded operators (as well as what is a pair of unbounded operators which satisfy a relation) needs to be additionally defined.

3. Even for a pair of bounded operators, the problem of describing, up to unitary equivalence, irreducible pairs of bounded self-adjoint operators without any relations is very difficult.

If there are such pairs that the weakly closed $*$-algebra $\mathfrak{A}$ (equivalently: W^*-algebra, von Neumann algebra) generated by these operators is not of a type I

factor, then this fact serves as an indication that the unitary classification problem for pairs of operators in H is a difficult one.

Following the general ideology of $*$-representation theory, we accept the definition below:

Definition 0.3. We will say that a quadratic relation is *wild* if there exist pairs of bounded operators, that satisfy this relation and that generate a factor which is not of type I. Otherwise we will say that the relation is *tame*.

A pair of self-adjoint operators (without relation) is wild. In Section 1.2, we will show that the only canonical relations $0 = 0$ and $A^2 = I$ are wild. The system of two wild relations $A^2 = I$, $B^2 = I$ is tame and has only one-dimensional and two-dimensional irreducible representations (see Section 1.3).

In Sections 1.3–1.5 we will describe up to unitary equivalence all irreducible representations by bounded and unbounded operators and give the corresponding structure theorems for the tame relations.

The description problem for pairs A, B of bounded self-adjoint operators that satisfy the relation $P_2(A,B) = 0$ is equivalent to the description problem for $*$-representations of $*$-algebra $\mathfrak{A}_{P_2(\cdot,\cdot)}$ by bounded operators. In its turn, the description problem for representations of $*$-algebra $\mathfrak{A}_{P_2(\cdot,\cdot)}$ by bounded operators can be reduced to the representation problem for the corresponding C^*-algebra $\mathcal{C}_{P_2(\cdot,\cdot)}$. There exists a construction of this C^*-algebra for a number of tame relations.

If the operators which satisfy relation (1) are unbounded, whether or not the relation is wild depends on the meaning we attach to the statement saying that a pair of unbounded operators should satisfy the relation. Von Neumann algebras generated by the spectral projections of the solutions may have a factor representation which is not of type I if we use one definition and may not have such if we use another (see in [39] an example of unbounded self-adjoint operators A and B that commute on a dense set, and that generate a factor which is not of type I). Hence, when we consider a pair of unbounded operators, we will say that the relation is tame or wild with respect to a definition analogous to the one formulated above for bounded operators.

II. 1. Classification problems for polynomial relations $P_n(A,B)$, $A = A^*$, $B = B^*$, are much more complicated if $n \geq 3$.

For example, the I. Newton's affine classification of third order relations of the form

$$\begin{aligned} P_3(A,B) = aA^3 + bB^3 + c\{A^2, B\} + d\{A, B^2\} + \\ + eA^2 + fB^2 + g\{A,B\} + kA + lB + mI = 0 \end{aligned}$$

(that do not contain commutators) or, which is the same thing, cubics on the real plane comprises already 72 types of relations [25, 43]. This is why the structure of pairs of self-adjoint operators that satisfy a polynomial relations (or several polynomial relations) of degree greater than two was studied for particular important examples or classes of examples ([4, 5, 46] etc.).

In Section 2.1 of this article, following [1, 2, 26, 27, 46], we give structure theorems for pairs of bounded or unbounded operators such that a study of the relation that they satisfy (polynomial or not) can be reduced to a study of the dynamical system

$$CU = CF(C)$$

$(C = C^*,\ U^* = U^{-1},\ F\colon [a,b] \to \mathbb{R}^1)$.

This class of relations contains (see [44]) the relations of the form $XX^* = f(X^*X) \quad (X = A + iB)$ (which is satisfied, for example, by generators of the algebra of polynomials on a quantum disk [18]).

2. In Section 2.2 of this article, following [5, 38], we study the structure of *bounded* self-adjoint operators that satisfy a (not only polynomial) relation or several relations which are linear with respect to B (semi-linear). Such relations appear in the study of exactly solvable problems of quantum physics (see, for example, [15]).

It should be noted that both of these relation classes admit a generalization to a system containing several self-adjoint generators [5, 27, 46]. Note also that families of (unbounded) operators that are connected by relations of such form (and more complex) arise in the theory of q-oscillator systems ([6, 9, 35, 36] etc.), representation theory of quantum groups ([13, 49, 50] etc.), quantum homogeneous spaces ([40, 41] etc.), Sklyanin algebras ([3, 19, 45, 48] etc.).

1. Pairs of self-adjoint operators satisfying quadratic relations

In this section, we study pairs of self-adjoint bounded and unbounded operators A, B, which satisfy the following relation

$$P_2(A,B) = \alpha A^2 + \beta\{A,B\} + i\hbar[A,B] + \gamma B^2 + \delta A + \epsilon B + \chi I = 0 \tag{3}$$
$$(\alpha, \beta, \hbar, \gamma, \delta, \epsilon, \chi \in \mathbb{R})$$

1.1. Classification

Let us start with the homogeneous quadratic relation

$$\tfrac{\hbar}{i}[A,B] = \alpha A^2 + \beta\{A,B\} + \gamma B^2 \qquad (\alpha, \beta, \gamma, \hbar \in \mathbb{R}) \tag{4}$$

Proposition 1.1. *By using a non-degenerate linear transformation, relation* (4) *can be reduced to one of the following forms:*

(0_0) $\quad 0 = 0$	(IV_0) $\quad [A,B] = 0$
(I_0) $\quad A^2 = 0$	(V_0) $\quad \frac{1}{i}[A,B] = B^2$
(II_0) $\quad A^2 + B^2 = 0$	(VI_0) $\quad \frac{1}{i}[A,B] = q(A^2 + B^2)$ $\quad (q > 0)$
(III_0) $\quad A^2 - B^2 = 0$	(VII_0) $\quad \frac{1}{i}[A,B] = q(A^2 - B^2)$ $\quad (q > 0)$

Proof. By using a non-degenerate linear transformation, we can reduce

$$\alpha A^2 + \beta\{A, B\} + \gamma B^2$$

to diagonal form. If $\hbar = 0$, then equation (4) will take one of the forms (0_0)–(III_0). If $\hbar \neq 0$, then by using the same transformation, we reduce the right hand side of identity (4) to the corresponding form and then if $\frac{\hbar}{i}[A, B] = 0$, we get (IV_0), if $\frac{\hbar}{i}[A, B] = A^2$, replace B by $\hbar B$ to get (V_0), if $\frac{\hbar}{i}[A, B] = A^2 \pm B^2$, we get (VI_0) and (VII_0) with $q = \frac{1}{\hbar} > 0$ by substituting A with $\frac{\hbar}{|\hbar|}A$. ∎

By applying a similar arguments we can prove the following statement.

Proposition 1.2. *By using an affine change of variables, equation* (3) *can be reduced to one of the following forms:*

(0_0) $0 = 0$	(0_1) $\chi I = 0$ $(\chi \in \mathbb{R}, \chi \neq 0)$	(0_2) $A = 0$
(I_0) $A^2 = 0$	(I_1) $A^2 = I$ (I_1') $A^2 = -I$	(I_2) $A^2 = B$
(II_0) $A^2 + B^2 = 0$	(II_1) $A^2 + B^2 = I$ (II_1') $A^2 + B^2 = -I$	
(III_0) $A^2 - B^2 = 0$ *or* $\{A, B\} = 0$	(III_1) $A^2 - B^2 = I$ *or* $\{A, B\} = I$	
(IV_0) $[A, B] = 0$	(IV_1) $\frac{1}{i}[A, B] = I$	(IV_2) $\frac{1}{i}[A, B] = A$
(V_0) $\frac{1}{i}[A, B] = A^2$	(V_1) $\frac{1}{i}[A, B] = A^2 + I$ (V_1') $\frac{1}{i}[A, B] = A^2 - I$	(V_2) $\frac{1}{i}[A, B] = A^2 + B$
(VI_0) $\frac{1}{i}[A, B] = q(A^2 + B^2)$ $(q > 0)$	(VI_1) $\frac{1}{i}[A, B] = q(A^2 + B^2) + I$ $(q \in \mathbb{R}, q \neq 0)$	
(VII_0) $\frac{1}{i}[A, B] = q(A^2 - B^2)$ $(q > 0)$	(VII_1) $\frac{1}{i}[A, B] = q(A^2 - B^2) + I$ $(q \in \mathbb{R}, q \neq 0)$	

Our next aim is to find out for each relation (0_0)–(VII_0)whether it is wild or tame and if it is tame, to describe pairs of (bounded or unbounded) self-adjoint operators, which satisfy this relation.

Considering solutions of the equations we have:

(0_1) $\chi I = 0$ $(\chi \neq 0)$. There are no pairs A, B which satisfy (0_1);

(0_2) $A = 0$. Since $B = B^*$ it is an arbitrary (bounded or unbounded) self-adjoint operator, the only irreducible representations are one-dimensional, $A = 0$,

$B = b$ and their structure is given by the structure theorem for a single operator B;

(I_0) $A^2 = 0$. Because $A = A^*$, $A^2 = A = 0$ and case (I_0) is similar to (0_2). The structure of any solution of equation (I_0) is the following: $A = 0$, $B = \int_{\mathbb{R}^1} \lambda \, dE_B(\lambda)$, where $E_B(\cdot)$ is an identity decomposition for the operator B;

(I_1') $A^2 = -I$. This equation doesn't have solutions.

(I_2) $A^2 = B$. The structure of any (bounded or unbounded) solution of equation (I_2) has the form $A = \int_{\mathbb{R}^1} \lambda \, dE_A(\lambda)$, $B = \int_{\mathbb{R}^1} \lambda^2 \, dE_A(\lambda)$, where $E_A(\cdot)$ is an identity decomposition for the operator A;

(II_0) $A^2 + B^2 = 0$. Here, $A = B = 0$;

(II_1') $A^2 + B^2 = -I$. There are no solutions.

The rest of the relations can be divided into four groups:

wild relations		
(0_0) $0 = 0$	(I_1) $A^2 = I$	
binormal relations		
	(II_1) $A^2 + B^2 = I$	
(III_0) $\{A, B\} = 0$	(III_1) $\{A, B\} = I$	
Lie algebras and their non-linear transformations		
(IV_0) $[A, B] = 0$	(IV_1) $\frac{1}{i}[A, B] = I$	(IV_2) $\frac{1}{i}[A, B] = A$
(V_0) $\frac{1}{i}[A, B] = A^2$	(V_1) $\frac{1}{i}[A, B] = A^2 + I$ (V_1') $\frac{1}{i}[A, B] = A^2 - I$	(V_2) $\frac{1}{i}[A, B] = A^2 + B$
q-relations		
(VI_0) $\frac{1}{i}[A, B] = q(A^2 + B^2)$ $(q > 0)$	(VI_1) $\frac{1}{i}[A, B] = q(A^2 + B^2) + I$ $(q \in \mathbb{R}, q \neq 0)$	
(VII_0) $\frac{1}{i}[A, B] = q(A^2 - B^2)$ $(q > 0)$	(VII_1) $\frac{1}{i}[A, B] = q(A^2 - B^2) + I$ $(q \in \mathbb{R}, q \neq 0)$	

In what follows, we study structure problems for each of these groups of relations.

1.2. Wild relations

1. First of all we show that the relation (0_0) $0 = 0$ is wild, i.e. there exist pairs of bounded self-adjoint operators A, B (that satisfy relation (0_0) or, which is the same, that do not satisfy any relation) and such that the W^*-algebra generated by these operators is a factor which is not of type I.

Example 1.3. (See [23]). Let G be a countable discrete group such that for each $g \neq e$ the class of the conjugate elements $G_g = \{g_0^{-1} g g_0 \mid g_0 \in G\}$ is infinite. In the space $H = L_2(G) = \{f\colon G \to \mathbb{C} \mid \sum_{g \in G} |f(g)|^2 < \infty\}$, consider unitary operators

$$(R_{g_0} f)(g) = f(g g_0), \qquad (L_{g_0} f)(g) = f(g_0^{-1} g).$$

Let $\mathfrak{A}_r$ be a von Neumann algebra of all bounded operators A that commute with all the operators L_{g_0}, $g_0 \in G$:

$$\mathfrak{A}_r = \{A \mid (\forall g_0 \in G) \quad A L_{g_0} = L_{g_0} A\} = \{L_{g_0}, g_0 \in G\}'.$$

Correspondingly write

$$\mathfrak{A}_l = \{B \mid (\forall g_0 \in G) \quad B R_{g_0} = R_{g_0} B\} = \{R_{g_0}, g_0 \in G\}'.$$

The von Neumann algebras $\mathfrak{A}_r$ and $\mathfrak{A}_l$ are type II_1 factors. ■

Proposition 1.4. *Relation* (0_0) $0 = 0$ *is wild.*

Proof. Consider a free group with two generators u, v. The unitary operators of its right regular representation $R_u = U = \int_{[0,2\pi)} e^{i\phi}\, dE_U(\phi)$ and $R_v = V = \int_{[0,2\pi)} e^{i\phi}\, dE_V(\phi)$ generate a factor which is not of type I. Then the pair of bounded self-adjoint operators $A = \int_{[0,2\pi)} \phi\, dE_V(\phi)$ and $B = \int_{[0,2\pi)} \phi\, dE_V(\phi)$ also generate a factor which is not of type I. ■

We now give two proofs showing that relation (I_1) is wild, i.e., the problem to describe, up to unitary equivalence, pairs of self-adjoint operators A, B such that $A^2 = I$ is wild.

Proposition 1.5. *Relation $A^2 = I$ is wild, i.e., there exist pairs A, B of bounded self-adjoint operators such that $A^2 = I$ and the von Neumann algebra $\mathfrak{A} = \{A, B\}''$ generated by these operators is not a type I factor.*

Proof. a) Consider the countable discrete group $G = \mathbb{Z}_2 * \mathbb{Z}$ ($*$ is the free product of groups). Its generators u and v satisfy the generating relation $u^2 = e$. The group $\mathbb{Z}_2 * \mathbb{Z}$ satisfies the condition: $\forall g \neq e$, the class of the conjugate elements $G_g = \{g_0^{-1} g g_0 \mid g_0 \in G\}$ is infinite. It follows then from Example 1.3 that the unitary operators of its right regular representation $u \overset{R}{\to} R_u$ and $v \overset{R}{\to} R_v = \int_{[0,2\pi)} e^{i\phi}\, dE(\phi)$ generate in $L(l_2(G))$ a von Neumann algebra which is a type II_1 factor and $R_u^2 = R_{u^2} = I$. So, the bounded self-adjoint operators $A = R_u$ and $B = \int_{[0,2\pi)} \phi\, dE(\phi)$ are such that $A^2 = I$ and the von Neumann algebra generated by these operators is not a type I factor.

b) The operator A is self-adjoint and unitary at the same time because $A^2 = I$ and so it can be written as $A = P_{H_1} - P_{H_1^\perp}$ (H_1 is a subspace in H). Let us choose $B = P_{\mathfrak{h}_1} + \frac{1}{2} P_{\mathfrak{h}_2} + \frac{1}{3} P_{\mathfrak{h}_3}$, where $\mathfrak{h}_1 \perp \mathfrak{h}_2$ is a pair of mutually orthogonal subspaces in H and $\mathfrak{h}_3 = H \ominus (\mathfrak{h}_1 \oplus \mathfrak{h}_2)$. We will show that even for such pairs A, B, the subspaces H_1, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ in H can be chosen in such a way that the W^*-algebra $\{A, B\}'' = \mathfrak{A}(A, B)$ generated by the operators A and B is not a type I factor. The algebra $\{A, B\}''$ and the algebra $\{P_{H_1}, P_{\mathfrak{h}_1}, P_{\mathfrak{h}_2}\}'' = \mathfrak{A}(P_{H_1}, P_{\mathfrak{h}_1}, P_{\mathfrak{h}_2})$, generated by three projections $P_{H_1}^* = P_{H_1}$, $P_{\mathfrak{h}_1}^* = P_{\mathfrak{h}_1}$, $P_{\mathfrak{h}_2}^* = P_{\mathfrak{h}_2}$, two of which are mutually orthogonal, $P_{\mathfrak{h}_1} \cdot P_{\mathfrak{h}_2} = 0$, coinside.

Lemma 1.6. *The unitary classification problem for three projections, two of which are mutually orthogonal, is wild.*

Proof. The von Neumann algebra $\{P_{H_1}, P_{\mathfrak{h}_1}, P_{\mathfrak{h}_2}\}''$ is generated by three unitary and at the same time self-adjoint operators $A = P_{H_1} - P_{H_1^\perp}$, $U_1 = P_{\mathfrak{h}_1} - P_{\mathfrak{h}_1^\perp}$ and $U_2 = P_{\mathfrak{h}_2} - P_{\mathfrak{h}_2^\perp}$ such that $[U_1, U_2] = 0$. These three unitary operators define a unitary representation of the generators a, u_1 and u_2 of the group $G = \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$ with the generators satisfying the relations $a^2 = u_1^2 = u_2^2 = e$, $u_1u_2 = u_2u_1$. Conversely, any unitary representation of the group $G \ni g \mapsto \pi(g)$ is given by three unitary self-adjoint operators $A = \pi(A)$, $U_1 = \pi(u_1)$ and $U_2 = \pi(u_2)$ such that $[U_1, U_2] = 0$. But the group G has the property that the class G_g of any element $g \neq e$ is infinite and the operators of the right regular representation, $A = R_a$, $U_1 = R_{u_1}$ and $U_2 = R_{u_2}$, generate, by virtue of Example 1.3, a factor which is of type II. ∎

According to this lemma, there exist pairs of self-adjoint operators A, B such that $A^2 = I$ and $\{A, B\}'' = \{P_{H_1}, P_{\mathfrak{h}_1}, P_{\mathfrak{h}_2}\}''$ is not a type I factor, i.e., the relation $A^2 = I$ is wild. ∎

We gave the second proof of Proposition 1.5 in order to draw the reader's attention to the fact that the unitary classification problem for three orthogonal projections P_{H_1}, P_{H_2}, P_{H_3} in H (or, which is the same thing, unitary classification problem for three subspaces H_1, H_2, H_3 in H) is wild even under the additional condition that H_2 and H_3 are mutually orthogonal.

2. These classification problems are wild not only in the sense of Definition 0.3, i.e., not only do there exist pairs of operators or collections of projections that generate a factor, which is not of type I, but they also contain a standard wild problem "as a subproblem" (See [34]).

For a standard of complexity (a model wild problem), following [20] we choose the unitary classification problem for a pair of bounded self-adjoint (or unitary) operators without any relations[2]. This problem contains as a subproblem the description problem for irreducible collections of n $(n = 2, 3, \ldots, \infty)$ self-adjoint operators (see [20, 37]).

Here, to make it easier for the reader, we will not try to formalize the words: the description problem for the collection $A_1, \ldots, A_n$ of operators that satisfy the relations $F_k(A_1, \ldots, A_n) = 0$, $k = 1 \ldots, m$ contains as a subproblem a standard wild problem of describing pairs U, V of unitary (or self-adjoint) operators which do not satisfy any relations (see [34]). All these attempts actually mean that we describe the properties of a procedure which allows for a pair U, V in H, to construct a collection $A_1^{(U,V)}, \ldots, A_n^{(U,V)}$ in $\mathfrak{h}$ such that $F_k(A_1^{(U,V)}, \ldots, A_n^{(U,V)}) = 0$ and the W^*-algebras $\{U, V\}'$ in H and $\{A_1^{(U,V)}, \ldots, A_n^{(U,V)}\}'$ in $\mathfrak{h}$ are isomorphic.

Example 1.7. Let us show that *the problem of unitary classification of four orthogonal projections* (or, which is the same thing, four subspaces in a Hibert space) contains a standard wild problem as a subproblem, and that there exist four

[2] For linear algebra problems, one chooses another well known unsolved problem for a standard of complexity (model wild problem), which is to classify non-similar pairs of matrices (see [14])

orthogonal projections P_1, P_2, P_3, P_4 such that the W^*-algebra $\{P_1, P_2, P_3, P_4\}''$ generated by these projections is not a type I factor. To do that, for every pair of unitary operators u, v in H, we construct four orthogonal projections in $\mathfrak{h} = H \oplus H$:

$$P_1^{(U,V)} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2^{(U,V)} = \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & \frac{1}{2}I \end{pmatrix},$$
$$P_3^{(U,V)} = \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}U^* \\ \frac{1}{2}U & \frac{1}{2}I \end{pmatrix}, \quad P_4^{(U,V)} = \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}V^* \\ \frac{1}{2}V & \frac{1}{2}I \end{pmatrix}.$$

The point of this construction is that these four orthogonal projections $\{P_k^{(U,V)}\}_{k=1}^4$ in $\mathfrak{H}$ generate a factor, the W^*-algebra $\{P_1, P_2, P_3, P_4\}''$, if and only if the pair U, V generates the factor $\{U, V\}''$ in H. Moreover, the following statement holds.

Proposition 1.8. *The von Neumann algebras* $\{U, V\}'$ *and* $\{P_k^{(U,V)}, k = \overline{1,4}\}'$ *are isomorphic.*

Proof. The operator $\mathcal{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ in $\mathfrak{h}$ commutes with these four projections if and only if it has the form $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$, where $C \in L(H)$ and $[C, U] = [C, V] = 0$. ■

Continuing the discussion of the example we note that for any von Neumann algebra $\mathfrak{A}$ the two von Neumann algebras $\mathfrak{A}$ and $\mathfrak{A}'$ are both factors and simultaneously are, or fail to be, of type I. Also, there exist pairs U, V in H such that the von Neumann algebra $\{U, V\}''$ generated by this pair is not a type I factor. Hence it follows that $\{U, V\}' = \{P_k^{(U,V)}, k = 1, 2, 3, 4\}'$ is not a type I factor and, consequently, the von Neumann algebra $\{P_k^{(U,V)}, k = 1, 2, 3, 4\}''$ generated by the four orthogonal projections is not a type I factor. ■

We gave this example because it is easy to write four orthogonal projections corresponding to a pair of unitary operators.

Example 1.9. The description problem for three orthogonal projections, two of which are mutually orthogonal, contains a standard wild problem as a subproblem.

Following the article [20], it is easy to give a construction method for three orthogonal projections $\{P_k^{(U,V)}, k = 1, 2, 3\}$, subject to the condition $P_2^{(U,V)} \perp P_3^{(U,V)}$ in $\mathfrak{h} = \bigoplus_{k=1}^{20} H_k$ $(H_k = H)$, which correspond to a pair of unitary operators in H, such that the following statement holds.

Proposition 1.10. *The von Neumann algebras* $\{U, V\}'$ *in* H *and* $\{P_k^{(U,V)}, k = 1, 2, 3\}'$ *in* $\mathfrak{h}$ *are isomorphic.*

If a problem contains as a subproblem the classification problem for a pair of operators, then by reasoning as in Examples 1.7 and 1.9, it is easy to show that it is wild, i.e. it has solutions that generate a factor, which is not of type I (if a problem is wild in the sense of a pair of operators, it is wild in the sense of a factor). But the converse is by no means true, i.e., if a relation (or several relations) is wild, it does not mean that their solution contains a pair as its fragment.

Example 1.11. The Cuntz algebra [7] is a wild $*$-algebra. This algebra is generated by S_1 and S_2 which satisfy the relations:

$$(S_1^*S_1)^2 = S_1^*S_1,\ (S_2^*S_2)^2 = S_2^*S_2,\ S_1^*S_1 + S_2^*S_2 = I.$$

But since the Cuntz algebra is nuclear, the unitary classification problem for non-selfadjoint operators satisfying these relation does not contain as a subproblem the same problem for a pair of operators.

1.3. Binormal operators

1. The problem to describe the pairs of bounded operators $A = A^*$, $B = B^*$ such that $A^2 = I$ is wild. Nevertheless, if we add to the wild relation $A^2 = I$ the additional wild relation $B^2 = I$, then the arising problem to describe the structure of pairs A, B of self-adjoint unitary operators is the description problem for unitary representations of the tame group $\mathbb{Z}_2 * \mathbb{Z}_2$.

Proposition 1.12. *The irreducible unitary non-equivalent pairs of self-adjoint unitary operators are:*

1) *four pairs of operators in* $\mathbb{C}^1$ *:* $A = \pm 1$, $B = \pm 1$*;*

2) *pairs of operators in* $\mathbb{C}^2$ *:*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \quad (\phi \in (0,\pi))$$

The corresponding structure theorem also holds (see [17, 32] etc.). We formulate it in convenient for us form.

Theorem 1.13. (The structure theorem for a pair of self-adjoint unitary operators in the resolution of identity form). *To any pair of self-adjoint unitary operators,* A*,* B *in* H *there uniquely correspond the orthogonal decomposition,* $H = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ *of* H *into invariant with respect to* A *and* B *subspaces* $\mathfrak{h}_1 = \oplus_{j,k=\pm 1} H_{j,k}$ *and* $\mathfrak{h}_2 = \mathbb{C}^2 \otimes H_+$*, and the resolution of identity* $dE(\cdot)$ *on* $(0,\pi)$ *with values in the projections onto subspaces of* H_+ *(here* $E((0,\pi)) = I_+$ *is the identity operator in* H_+*), such that the following representation holds:*

$$\begin{aligned} A &= \sum_{j,k=\pm 1} jP_{H_{j,k}} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_+ \\ B &= \sum_{j,k=\pm 1} kP_{H_{j,k}} + \\ &\quad + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \int_0^\pi \cos\phi\, dE(\phi) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \int_0^\pi \sin\phi\, dE(\phi) \end{aligned}$$

Proof. The proof of this theorem follows from the decomposition of H with respect to the spectrum of selfadjoint operator $\frac{1}{2}\{A, B\}$ that commutes with both A and B. ∎

2. Now we can easily describe the *bounded* self-adjoint solutions of relations (II_1) $A^2 + B^2 = I$, (III_0) $A^2 = B^2$ (or, which is the same, $\{\tilde{A}, \tilde{B}\} = 0$) and (III_1) $A^2 - B^2 = I$ (or, which is the same, $\{\tilde{A}, \tilde{B}\} = I$). These relations we call *binormal*, since the corresponding C^*-algebra, which is generated by single non-selfadjoint operator $X = A + iB$ has only one- and two-dimensional irreducible representations (i.e., the operator X is binormal, see, e.g. [10]).

Proposition 1.14. *The irreducible self-adjoint solutions of relations (II_1), (III_0) and (III_1) are one- and two-dimensional. Each solution is determined by a point of a circle (II_1), or of a pair of intersecting lines (III_0), or of a hyperbola (III_1) and by an irreducible pair of projections.*

The corresponding structure theorems also hold.

3. For "the circle" (II_1) $A^2 + B^2 = I$ only bounded solutions exist, since $\|A\| \le 1$, $\|B\| \le 1$. In constrast, for relations (III_0) and (III_1) a class of "integrable" representations by unbounded operators was defined and investigated in [27, 29, 30, 37].

There is no need in using unbounded operators for studying *irreducible* representations of relations (III_0), (III_1) either. Irreducibility here implies that the operators A^2 and B^2 commute with both A and B, hence are scalar. Thus the operators A, B are bounded, and all the irreducible representations are given by Proposition 1.14 above.

If we consider reducible representations of (III_0) and (III_1) in unbounded operators, new representations appear. The class of "integrable" anticommuting pairs (relation (III_1)) was introduced by F.-H. Vasilescu [47] (see also [33, 37]). Following [37], we define the class of "integrable" ones and give the structure theorem.

Definition 1.15. (See [37]). A pair of unbounded self-adjoint operators anticommute if

$$\{A_m, B_m\} = A_m B_m + B_m A_m = 0$$

for any $m = 1, 2, \dots$, where $A_m = \int_{-m}^{m} \lambda \, dE_A(\lambda)$ and $B_m = \int_{-m}^{m} \lambda \, dE_B(\lambda)$ are bounded operators.

Note that in the definition of integrable anticommuting pair of operators (relation (III_0)) one could require the operators $f_{odd}(A)$ and $g_{odd}(B)$ to anticommute for any measurable bounded odd functions $f_{odd}(\cdot)$, $g_{odd}(\cdot)$. Moreover, the following proposition holds:

Proposition 1.16. (See [37]). *For a pair of anticommuting operators to be integrable, it is necessary and sufficient to have*

$$\{\sin tA, \sin sB\} = 0 \qquad \forall\ t, s \in \mathbb{R}^1.$$

The corresponding structure theorem holds for unbounded anticommuting operators [37].

To consider relation (III_1), we introduce $\tilde{A} = (1/\sqrt{2})(A + B)$ and $\tilde{B} = (1/\sqrt{2})(A - B)$, then (III_1) is equivalent to $\{\tilde{A}, \tilde{B}\} = I$.

Definition 1.17. Representation of (III_1) is called integrable, if the operators $\tilde{A}$ and $\tilde{B} - \frac{1}{2}\tilde{A}^{-1}$ anticommute in the sense of the previous definition (since $\ker A$ is invariant, the relation implies $\ker A = 0$).

Theorem 1.18. *To the integrable representation of relation (III_1) there uniquely correspond the orthogonal decomposition*

$$H = H_0 \oplus H_1 = H_0 \oplus (\mathbb{C}^2 \otimes H_+)$$

together with the resolutions of identity $E_0(\cdot)$ on $K_0 = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1\lambda_2 = 1/2\}$ and $E_+(\cdot)$ on $K_+ = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 > 0, \lambda_2 > 0\}$ with values in the projections onto subspaces of H_0, H_+ respectively, such that

$$\begin{aligned} A &= \int_{K_0} \lambda_1 \, dE_0(\lambda_1, \lambda_2) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \int_{K_+} \lambda_1 \, dE_+(\lambda_1, \lambda_2), \\ B &= \int_{K_0} \lambda_2 \, dE_0(\lambda_1, \lambda_2) + \\ &\quad + \int_{K_+} \left(\frac{1}{2\lambda_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \otimes dE_+(\lambda_1, \lambda_2). \end{aligned}$$

1.4. Lie algebras and their non-linear trasformations

Lie relations (IV_0), (IV_1), (IV_2), as like as relations (V_0), (V_1), (V_1') (about (V_2) see below) can be treated from the common point of view. Namely, all these relations are partial cases of the relation

$$[A, B] = iP_2(A) \tag{5}$$

where $P_2(A)$ is a real quadratic polynomial.

To begin with, consider bounded pairs, A, B which satisfy (5).

Proposition 1.19. *The irreducible pairs, A, B of bounded self-adjoint operators which satisfy the relation (5) are one-dimensional, and they are: $A = \lambda$, $B = \mu$, where $P_2(\lambda) = 0$, $\mu \in \mathbb{R}^1$. An arbitrary bounded pair is of the form:*

$$A = \int_M \lambda \, dE(\lambda, \mu), \qquad B = \int_M \mu \, dE(\lambda, \mu),$$

where $E(\cdot, \cdot)$ is a resolution of the identity on $M = \{(\lambda, \mu) \in \mathbb{R}^2 \mid P_2(\lambda) = 0\}$.

Proof. Indeed, (5) implies $[A, [A, B]] = 0$ and due to Kleineke-Shirokov theorem (see, e.g. [16]) the operator $[A, B]$ is quasi-nilpotent. But since $[A, B]$ is skew-adjoint, it means $[A, B] = 0$. Then $P_2(A) = 0$ and the assertion follows from the spectral theorem for a pair of commuting self-adjoint operators. ∎

Remark 1.20. Proposition 1.19 implies that if the polynomial, $P_2(\cdot)$ has no real roots, then there are no bounded self-adjoint pairs, which satisfy (5). In particular, there are no bounded pairs which satisfy CCR (relation (IV_1)) or $[A,B] = i(A^2 + I)$ (relation (V_1)).

Now we pass to the investigation of pairs of unbounded self-adjoint operators which satisfy (5) (see also [8]). As usually, speaking about unbounded operators, which satisfy a relation, it is necessary to specify its operator sense. One of the possible ways here is to consider the relation (5) as a nonlinear deformation of CCR.

First, we make some formal transformations which, in general, may be out of sense for operators. Let Φ be an invariant with respect to A and B set, $\Phi \subset D(A) \cap D(B)$.

Proposition 1.21. *For any polynomial* $P(\cdot)$

$$[P(A), B]f = iP_2(A)P'(A)f, \quad f \in \Phi \tag{6}$$

Proof. Indeed, for monomials we have by induction:

$$\begin{aligned}
[A^n, B]f &= A^n Bf - BA^n f = A(A^{n-1}B)f - BA^n f = \\
&= A(BA^{n-1} + [A^{n-1}, B])f - BA^n f = \\
&= BA^n f + [A, B]A^{n-1} f + A[A^{n-1}, B]f - BA^n f = \\
&= niP_2(A)A^{n-1} f,
\end{aligned}$$

and so we have (6). ∎

Suppose now that the relation (6) holds not for polynomials only, but for some other functions also, say, for such a function $f(\cdot)$, that $f'(\lambda) = 1/P_2(\lambda)$. Then we have

$$[f(A), B] = iP_2(A)f'(A) = iI.$$

As a condition of "integrability" for a pair A, B satisfying (5) one could take the following: the pair, A, B is *integrable*, if the pair $f(A)$, B satisfies CCR in the Weyl form. However, this approach has certain disadvantages.

Example 1.22. Let $P_2(\lambda) = \lambda^2 + 1$ (relation (V_1)). In this case $f(\lambda) = \operatorname{arctg}\lambda$ and we have

$$[\operatorname{arctg} A, B] = iI.$$

But since $\operatorname{arctg}(\cdot)$ is a bounded function, then the operator $\operatorname{arctg} A$ is also bounded. However, no one of the operators satisfying CCR in the Weyl form can be bounded. So, for such a definition of integrability the relation $[A,B] = i(A^2+I)$ has no solution.

Nevertheless, the pair of operators in $H = L_2([-\frac{\pi}{2}, \frac{\pi}{2}])$

$$(Af)(\lambda) = \operatorname{tg}\lambda f(\lambda), \quad (Bf)(\lambda) = \frac{1}{i}\frac{d}{d\lambda} f(\lambda),$$

defined on the set $\Phi = C_0^\infty([-\frac{\pi}{2}, \frac{\pi}{2}])$ satisfies the relation $[A,B]f = i(A^2 + I)$f for $f \in \Phi$. ∎

To define the class of integrable pairs of unbounded self-adjoint operators which satisfy a relation of the form (5), using certain formal manipulations we will obtain the relations which contain only bounded functions of self-adjoint operators A and B, similarly to that is in classical Lie algebras representation theory.

Introduce a sequence of polynomials,

$$D_0(A) = A,\ D_1(A) = P_2(A),\ D_2(A) = P_2(A)P_2'(A), \dots$$
$$D_k(A) = P_2(A)D_{k-1}'(A).$$

Then the following formal equalities follow:

$$\begin{aligned} AB^n &= \sum_{k=0}^{n} C_n^k i^k D_k(A), \\ Ae^{itB} &= e^{itB} S_t(A), \qquad (7) \\ e^{itA}e^{isB} &= e^{isB}e^{itS_s(A)}, \qquad (8) \end{aligned}$$

where we denote

$$S_t(A) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} D_k(A). \qquad (9)$$

Note that the relation (8) contains only bounded operators and so can be taken as definition of integrability for pairs, satisfying (5).

Here, we will not concern ourselves with the convergence of the series in (8), (9). We will find the form of the mapping $S_t(\cdot)$ using other methods.

First of all, (7) implies the following properties of $S_t(\cdot)$:

1. $S_0(\lambda) = \lambda$ for almost all λ with respect to the spectral measure of A;
2. $S_{t_1+t_2}(\lambda) = S_{t_1}(S_{t_2}(\lambda))$ for fixed t_1, t_2 and almost all λ;
3. $S_t(A)$ is differentiable with respect to t family of operators. Indeed, $S_t(A)f = e^{-itB}Ae^{itB}f$ and the family $S_t(A)$ satisfies the Lax equation

$$\frac{d}{dt}S_t(A)f = -i[S_t(A), B]f.$$

In particular, since $S_0(A) = A$, we have for $t = 0$:

$$\frac{d}{dt}S_t(A)\Big|_{t=0} f = -P_2(A)f.$$

One can check directly that for $P_2(\lambda) \not\equiv 0$ all these conditions are satisfied by the function

$$S_t(\lambda) = \phi^{-1}(\phi(\lambda) - t), \quad \phi(\lambda) = -\int \frac{d\lambda}{P_2(\lambda)}, \qquad (10)$$

where the branch serving as ϕ^{-1} is selected in such a fashion that $S_0(\lambda) = \lambda$ and $S_t(\lambda)$ is a group of bijections of an extended line $\mathbb{R} \cup \{\infty\}$.

Definition 1.23. We say that a pair of unbounded selfadjoint operators satisfying (5) is integrable, if the relations

$$e^{itA}e^{-isB} = e^{-isB}e^{itS_s(A)} \quad \forall\, t, s \in \mathbb{R},$$

hold, where $S_t(\cdot)$ is defined by (10).

Example 1.24. In the Lie situation, the definition reduces to the traditional one for a representation of a Lie algebra to be integrable to a unitary representation of the corresponding Lie group. The relation $S_t(\cdot)$ has the form:

$$(IV_0)\ S_t(\lambda)=\lambda, \quad (IV_1)\ S_t(\lambda)=\lambda+t, \quad (IV_2)\ S_t(\lambda)=e^t\lambda.$$

The relation (7) here is:

(IV_0) $(\forall t, s \in \mathbb{R})$ $e^{itA}e^{isB} = e^{isB}e^{itA}$; the condition is that the one-parameter groups commute, and this is equivalent to commuting of spectral projections for A and B;

(IV_1) $(\forall t, s \in \mathbb{R})$ $e^{itA}e^{isB} = e^{-its}e^{isB}e^{itA}$; we have CCR in the Weyl form;

(IV_2) $(\forall t, s \in \mathbb{R})$ $e^{itA}e^{isB} = e^{isB}e^{ite^{-s}A}$; this is one of the forms of defining relations in the group of affine transformations of a line.

Example 1.25. In cases (V_0), (V_1), (V_1') we have correspondingly:

$$(V_0)\ S_t(\lambda)=\frac{\lambda}{1-t\lambda}, \qquad (V_1)\ S_t(\lambda)=\frac{\lambda+\operatorname{tg} t}{1-\lambda\operatorname{tg} t},$$
$$(V_1')\ S_t(\lambda)=\frac{\lambda(e^{-2t}+1)+e^{-2t}-1}{\lambda(e^{-2t}-1)+e^{-2t}+1}.$$

We also give an equivalent form of the definition.

Proposition 1.26. *The pair, A, B satisfying* (5) *is integrable if and only if*

$$(\forall\ t\in\mathbb{R},\ \Delta\in\mathfrak{B}(\mathbb{R}))\quad E_A(\Delta)e^{-itB}=e^{-itB}E_A(S_t^{-1}(\Delta)). \tag{11}$$

Proof. (See [27]). Indeed,

$$(\forall u, v\in H)\quad (e^{itA}e^{-isB}u,v)=\Big(\int_{\mathbb{R}} e^{it}\,dE_A(\lambda)e^{-isB}u,v\Big)=$$
$$=\Big(\int_{\mathbb{R}} e^{itS_s(\lambda)}\,dE_A(\lambda)u,e^{isB}v\Big)=\int_{\mathbb{R}} e^{it\lambda}\,d(E_A(S_t^{-1}(\lambda))u,e^{isB}v),$$

and due to uniqueness property of Fourier transform of a measure we have

$$(E_A(\Delta)e^{-isB}u,v)=(e^{-isB}E_A(S_t^{-1}(\Delta))u,v).$$

Conversely, using a functional calculus for the self-adjoint operator A we get

$$e^{itA}e^{-isB}=\int_{\mathbb{R}} e^{it}\lambda\,dE_A(\lambda)e^{-isB} \;=\; e^{-isB}\int_{\mathbb{R}} e^{it\lambda}\,dE_A(S_s^{-1}(\lambda))=$$
$$=e^{-isB}\int_{\mathbb{R}} e^{itS_s(\lambda)}\,dE_A(\lambda) \;=\; e^{-isB}e^{itS_s(A)}.$$

This completes the proof. ∎

Now we pass to an investigation of irreducible integrable pairs, A, B of self-adjoint operators satisfying the relation (5). First of all, note that the spectral measure of operator A is quasi-invariant and ergodic with respect to action of the mapping $S_t(\cdot)$, $t \in \mathbb{R}^1$. Indeed, the quasi-invariance of the measure follows directly from the relation

$$e^{isB} E_A(\Delta) e^{-isB} = E_A(S_s^{-1}(\Delta))$$

and from the fact that the operators e^{isB}, $s \in \mathbb{R}$ are invertible. Moreover, each measurable $S_t(\cdot)$-invariant set $\Delta \in \mathfrak{B}(\mathbb{R})$ defines an invariant with respect to A and B subspace $H_0 = E_A(\Delta)\, H$. So, for an irreducible pair we have $E_A(\Delta) = I$ or $E_A(\Delta) = 0$. In all the cases considered the action of $S_t(\cdot)$ satisfies the conditions of the Glimm Theorem, so the class of equivalent ergodic measures (or, which is the same, the spectral measure of A) is uniquely determined by some orbit, $O_\lambda = \{S_t(\lambda) \mid t \in \mathbb{R}\}$ which is a support of the spectral measure.

We fix an orbit and find out all irreducible pairs, A, B which correspond to the orbit. Depending on the nature of the action $S_t(\cdot)$ of $\mathbb{R}^1$ on the orbit O_{λ_0} of λ_0, different situations occur. The action may be trivial ($S_t(\lambda_0) = \lambda_0$ for all $t \in \mathbb{R}$), free ($S_t(\lambda_0) = S_{t'}(\lambda_0)$ if and only if $t = t'$), or periodic ($S_{t+\gamma}(\lambda_0) = S_t(\lambda_0)$, γ is the period).

First consider the trivial action.

Proposition 1.27. *$O_{\lambda_0} = \{\lambda_0\}$ if and only if λ_0 is a root of $P_2(\lambda)$. To the orbit there corresponds the family of one-dimensional representations: $H = \mathbb{C}^1$, $A = \lambda_0$, $B = \mu$ with $\mu \in \mathbb{R}^1$.*

Proof. Indeed, since the spectral measure of the operator is concentrated on the single point, λ_0, we have $A = \lambda_0 I$. Therefore, $[A, B] = 0$, $P_2(A) = P_2(\lambda_0) = 0$, and B is an arbitrary self-adjoint operator. Irreducibility implies $\dim H = 1$. ∎

Now consider a free action of $\mathbb{R}^1$ on the orbit.

Proposition 1.28. *If $\mathbb{R}^1$ acts freely on O_{λ_0}, then a unique integrable pair satisfying (5) corresponds to the orbit.*

Proof. Define the resolution of identity, $E(\cdot)$ on $\mathbb{R}^1$, putting for $\delta \in \mathfrak{B}(\mathbb{R})$

$$E(\delta) = E_A(\Delta), \quad \text{where } \Delta = \{S_\alpha(\lambda) \mid \alpha \in \delta\} \subset O_{\lambda_0}.$$

Since $S_t(\Delta) = \{S_\alpha(\lambda_0) \mid \alpha \in \delta + t\}$, the following relation holds

$$E(\delta) e^{-itB} = e^{-itB} E(\delta - t).$$

write $\tilde{A} = \int \lambda \, dE(\lambda)$. Then the operators $\tilde{A}$ and B satisfy CCR in the Weyl form, which implies uniqueness of the irreducible pair corresponding to O_{λ_0}. The choice of another initial point λ_0 leads to a unitary equivalent pair. ∎

Consider, at last, a periodic action of $\mathbb{R}^1$ on O_{λ_0}.

Proposition 1.29. *In the case of a periodic action of $\mathbb{R}^1$ on O_{λ_0}, the family of unitarily non-equivalent integrable irreducible pairs parametrized by the points of a circle corresponds to the orbit.*

Proof. Let $\lambda \in O_{\lambda_0}$. The mapping $S_t(\cdot)$ is one-to-one on the period $[t_0, t_0+\gamma)$ and is measurable. Define the resolution of the identity $E(\cdot)$ on S^1 by taking $\forall \delta \in \mathfrak{B}(\mathbb{R})$

$$E(\delta) = E_A(\Delta), \quad \text{where } \Delta = \{S_\alpha(\lambda_0) \mid \alpha \in [t_0, t_0+\gamma),\ e^{2\pi i\alpha/\gamma} \in \delta\}.$$

We have

$$\begin{aligned} E(\delta)e^{-itB} = E_A(\Delta)e^{-itB} = e^{-itB}E_A(S_t^{-1}(\Delta)) = \\ = e^{-itB}E(e^{it}\delta). \end{aligned} \tag{12}$$

On the other hand, consider the group G of matrices of the form

$$G \ni g = \begin{pmatrix} 1 & t & \alpha \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \qquad t, \alpha \in \mathbb{R}^1,\ n \in \mathbb{Z},$$

which is the semidirect product $\mathbb{R} \rtimes (\mathbb{Z} \times \mathbb{R}) = \{(t, n, \alpha) \mid t, \alpha \in \mathbb{R}^1,\ n \in \mathbb{Z}\}$. Consider a unitary representation of this group such that $(0, 0, \alpha) \mapsto e^{i\alpha}$. Let $(t, 0, 0) \mapsto U_t$, $(0, 1, 0) \mapsto V$ be the representation operators and

$$V = \int_{S^1} \lambda \, dE_V(\lambda).$$

Then

$$\begin{aligned} \int_{S^1} \lambda \, dE_V(\lambda)U_{-t} = VU_{-t} = e^{-it}U_{-t}V = \\ = U_{-t}\int_{S^1} e^{-it}\lambda \, dE_V(\lambda) = U_{-t}\int_{S^1} \lambda \, dE_V(e^{it}\lambda), \end{aligned}$$

which implies that for all $\delta \in \mathfrak{B}(S^1)$ and all $t \in \mathbb{R}$ we have

$$E_V(\delta)U_{-t} = U_{-t}E_V(e^{it}\delta),$$

i.e., the relation (12) determines an unitary representation of the group G, such that $(0, 0, \alpha) \mapsto e^{i\alpha}$. Due to Mackey's representation theory for semi-direct products [21], the representations are parametrized by the points of a circle and are realized in the space $L_2([0, 2\pi), d\lambda)$ by the formulae

$$\begin{aligned} (Vf)(\lambda) &= e^{i\lambda}f(\lambda), \\ (Uf)(\lambda) &= e^{i\alpha[\frac{t+\lambda}{2\pi}]}f(t+\lambda \pmod{2\pi}), \qquad \alpha \in [0, 2\pi) \approx S^1. \end{aligned}$$

This accomplishes the proof. ∎

Now we give a list of all irreducible integrable representations of relations (V_0), (V_1), (V_1').

(V_0). The dynamical system generated by

$$S_t(\lambda) = \frac{\lambda}{1 - t\lambda}, \quad t \in \mathbb{R}^1,$$

has two orbits: one-point orbit $O_0 = \{0\}$ and $O_\infty = \mathbb{R}^1 \setminus \{0\}$. To the one-point orbit there corresponds the family of one-dimensional pairs, $A = 0$, $B = \mu \in \mathbb{R}$. On the orbit O_∞ the group $\mathbb{R}^1$ acts freely and to this orbit there corresponds a unique irreducible integrable pair in $L_2(\mathbb{R}^1, d\lambda)$:

$$(Af)(\lambda) = -\frac{1}{\lambda} f(\lambda), \quad (Bf)(\lambda) = \frac{1}{i}\frac{d}{d\lambda} f(\lambda) \tag{13}$$

(the operators are defined on the natural domains).

(V_1). The dynamical system generated by

$$S_t(\lambda) = \frac{\lambda + \operatorname{tg} t}{1 - \lambda \operatorname{tg} t} \quad t \in \mathbb{R}^1,$$

has no stationary points ($\lambda^2 + 1$ has no real roots). To a unique periodical orbit $\mathbb{R}^1$ there corresponds a family of irreducible integrable pairs of self-adjoint operators in $L_2([-\frac{\pi}{2}, \frac{\pi}{2}])$, which correspond to a parameter $\alpha = e^{i\phi} \in S^1$:

$$(Af)(\lambda) = \operatorname{tg} \lambda f(\lambda), \quad (Bf)(\lambda) = \frac{1}{i}\frac{d}{d\lambda} f(\lambda). \tag{14}$$

Here the operator A is defined on the natural domain, and $D(B)$ contains absolutely continuous functions satisfying the boundary condition $f(-\frac{\pi}{2}) = \alpha f(\frac{\pi}{2})$. To the different values of $\alpha \in S^1$ there correspond different self-adjoint extension of the symmetric operator $B_0 = \frac{1}{i}\frac{d}{d\lambda}$ with $D(B_0) = C_0^1([-\frac{\pi}{2}, \frac{\pi}{2}])$.

(V_1'). The dynamical system

$$S_t(\lambda) = \frac{\lambda(e^{-2t} + 1) + e^{-2t} - 1}{\lambda(e^{-2t} - 1) + e^{-2t} + 1}, \quad t \in \mathbb{R}^1,$$

has two fixed points, $\lambda_1 = -1$ and $\lambda_2 = 1$ (which are the roots of the polynomial $\lambda^2 - 1$) to which one-dimensional irreducible pairs correspond, $A = \pm 1$, $B = \mu \in \mathbb{R}^1$, and two other orbits, $O_0 = (-1, 1)$ and $O_\infty = \mathbb{R}^1 \setminus [-1, 1]$ with free action of $\mathbb{R}^1$. To these orbits there correspond the following pairs of unbounded self-adjoint operators in $L_2(\mathbb{R}^1)$:

$$(Af)(\lambda) = \frac{e^{-2\lambda} - 1}{e^{-2\lambda} + 1}, \quad (Bf)(\lambda) = \frac{1}{i}\frac{d}{d\lambda} f(\lambda),$$

(corresponds to O_0) and

$$(Af)(\lambda) = \frac{e^{-2\lambda} + 1}{e^{-2\lambda} - 1}, \quad (Bf)(\lambda) = \frac{1}{i}\frac{d}{d\lambda} f(\lambda)$$

(correponds to O_∞). The operators are defined on the natural domains.

Remark 1.30. In the Lie situation (for unitary representations of arbitrary real Lie groups) there always exists a Gårding domain, which is dense in H invariant with respect to operators of infinitesimal representation of the Lie algebra and is their *essential domain.* In particular, such a domain exists for an integrable pair of operators satisfying one of the relations (IV_0)–(IV_2). Moreover, in the Lie situation Φ may be chosen to lie in the set of *analytic* vectors for the infinitesimal representation. However, this is not true if one considers the relations (V_0)–(V_1').

Proposition 1.31. *For the integrable irreducible pair* (13) *which satisfy the relation* (V_0) *there is no nonzero vector in* H *which is in the domain of any polynomial of* A*,* B *and which is analytic for both* A *and* B*.*

Proof. Indeed, any analytic vector for the derivation operator is an analytic function. From the other hand, any function on which all the operators of the form

$$A^n B^m = \frac{(-1)^n}{\lambda^n}\frac{d^m}{d\lambda^m}, \quad n, m \in \mathbb{N},$$

are defined, is zero together with all its derivatives on the point $\lambda = 0$. Analyticity then implies $f(\lambda) \equiv 0$. ∎

In the case (V_1), the situation is even more unusual.

Proposition 1.32. *For the integrable irreducible pair* (14) *which satisfies the relation* (V_1) *there exist no domain* Φ *which is dense in* H*, which is invariant with respect to* A *and* B*, and which is a core for the operators.*

Proof. Indeed, if $\Phi \subset D(A) \cap D(B)$ is invariant with respect to the derivation operator then $\Phi \subset C^\infty([-\frac{\pi}{2}, \frac{\pi}{2}])$. Since Φ is invariant under multiplying by $\operatorname{tg}\lambda$ it implies that any function $f(\cdot) \in \Phi$ is zero together with all its derivatives on the edges of an interval. But it is well-known that such domain is not a core for the derivation operator. ∎

Consider, at last, the relation (V_2). Since

$$[A, (A^2 + B)] = i(A^2 + B),$$

the operators A and $A^2 + B$ satisfy (formally) the relation (IV_2). If one set unbounded operators A and B to be integrable pair satisfying (V_2) if and only if the pair A and $A^2 + B$ is integrable one satisfying (IV_2), then the structure of pairs satisfying (V_2) is given by the structure statements for (IV_2).

1.5. Quantum relations

Pairs of self-adjoint operators that satisfy relations (VI_0) – (VII_1),

$$\begin{aligned} [A, B] &= i\alpha(A^2 \pm B^2) \qquad \alpha > 0 \\ [A, B] &= i\alpha(A^2 \pm B^2) + iI \qquad \alpha \in \mathbb{R}^1 \setminus \{0\} \end{aligned}$$

have been studied in physical and mathematical literature as the q-oscillator (relation (VI_1), see [6, 9, 35, 36] and references therein), real and complex q-plane (relations (VI_0) and (VII_0), see [11, 40, 41] etc.), q-hyperboloid (relation (VII_1), see [40] etc.). Here we follow[27, 29].

1. Consider the pairs of bounded self-adjoint operators satisfying the relation (VI_0)

$$[A,B] = i\alpha(A^2+B^2) \qquad \alpha > 0$$

Proposition 1.33. *If the pair of bounded self-adjoint operators A, B satisfies (VI_0) then $A = B = 0$.*

Proof. Introduce operators $X = A + iB$, $X^* = A - iB$. Then the operators X and X^* satisfy the relation of (complex) q-plane

$$XX^* = qX^*X, \qquad q = \frac{1+\alpha}{1-\alpha}.$$

For a polar decomposition of the operator $X = UC$ we have $UC^2 = qC^2U$. Then the spectrum of self-adjoint operator C^2 is invariant with respect to transformation $\lambda \mapsto q\lambda$ $(q \neq \pm 1)$, which is impossible for a bounded nonzero operator. ∎

For unbounded pairs we require, following [27] that for all $\Delta \in \mathfrak{B}(\mathbb{R}^1)$ the relations

$$E_{C^2}(\Delta)U = UE_{C^2}(q\Delta) \tag{15}$$

hold. Such pairs we call integrable.

Now we proceed to the study of irreducible integrable pairs, satisfying (VI_0). Let $H_0 = \ker C$. Then (15) implies that H_0 is invariant with respect to U, C and so, with respect to A, B. In H_0 operators are trivial. So, for the non-trivial irreducible pairs we have $\ker C = \{0\}$.

Since the operator U is unitary, it follows from (15) that the spectrum, $\sigma(C^2)$ is invariant under multiplication by q. Due to positivity of C^2 we also have that for $q < 0$ there are no nontrivial pairs satisfying (15). In what follows, we suppose $q > 1$ $(\alpha \in (0,1))$. Any invariant set having positive spectral measure determines an invariant subspace, so it is easy to show that in the irreducible case the spectrum of C^2 is discrete, $\sigma(C^2) = \{q^k\lambda \mid k \in \mathbb{Z}\}$, $\lambda \in [1,q)$ being a parameter. Also, the operator U is unitarily equivalent to a shift operator, and we have:

Proposition 1.34. *Any irreducible nontrivial pair, satisfying* (15) *is defined by*

$$Xe_k = \lambda q^k e_{k+1}, \qquad k \in \mathbb{Z}$$

for some $\lambda \in [1,q)$.

Remark 1.35. Note that closure of the operators A and B defined on finite linear combinations of $\{e_k\}$ are symmetric operators having defect indices $(1,1)$.

2. Similar to a previous case, relation (VI_1) can be rewritten as

$$XX^* = qX^*X + (q+1)I,$$

(q-oscillator) and as a definition of integrability one can require, following [27] that

$$UE_{C^2}(\Delta)P = E_{C^2}(q\Delta + (q+1))UP,$$

where $X = UC$ is a polar decomposition and P is a projectiononto $\ker C^{\perp}$.

Now we list all the irrerucible representations of (VI_1).

Proposition 1.36. (See [29]). *Let $q \in (0,1)$ $(\alpha \in (-1,0))$. The irreducible representations of (VI_1) occur in*

a) a one-dimensional family:

$$X_1, \phi = e^{i\phi}\sqrt{-1/\alpha} \qquad \phi \in [0, 2\pi);$$

b) a single one-dimensional (Fock representation): $H = l_2$,

$$X_{\infty,0} e_k = \sqrt{F^{\circ k}(0)}\, e_{k+1} \qquad k = 0, 1, \dots,$$

where $F(\lambda) = q\lambda + (q+1)$ and $F^{\circ k}(\cdot)$ is the k-th iteration of $F(\cdot)$;

c) an infinite-dimensional family: $H = l_2(\mathbb{Z})$,

$$X_{\infty,\lambda} e_k = \sqrt{F^{\circ(k+1)}(\lambda)}\, e_{k+1} \qquad k \in \mathbb{Z},$$

the parameter $\lambda \in (F(\lambda_0), \lambda_0]$; $\lambda_0 > -1/\alpha$ is fixed.

■

If $\alpha \geq 0$, there are no representations. If $\alpha < -1$, there is a family of one-dimensional representations and a unique infinite-dimensional of the form (b). For $\alpha \in (0,1)$, only the Fock representation (b) exists.

Note that for $\alpha < 0$ the Fock pair is bounded.

3. The relation (VII_0)

$$[A, B] = i\alpha(A^2 - B^2), \qquad \alpha > 0$$

can be rewritten in the form

$$XY = qYX, \qquad q = \frac{1 + i\alpha}{1 - i\alpha} \tag{16}$$

(real quantum plane $\mathbb{R}^2_q$, here $X = A + B$, $Y = A - B$). For bounded self-adjoint operators due to Fuglede-Putnam theorem we also have

$$XY = \overline{q}YX,$$

which, since $q \neq \overline{q}$ implies $XY = YX = 0$.

4. Now consider bounded pairs satisfying (VII_1),

$$\frac{1}{i}[A, B] = \alpha(A^2 - B^2) + I, \qquad \alpha \neq 0, \alpha \in \mathbb{R}.$$

For the operators $X = A + B$, $Y = A - B$ we have

$$XY = qYX + (q+1)I.$$

For bounded operators it implies

$$(XY + \frac{1}{2\alpha}I)X = qX(XY + \frac{1}{2\alpha}I),$$

and due to the Fuglede-Putnam theorem we also have

$$(XY + \frac{1}{2\alpha}I)X = \overline{q}X(XY + \frac{1}{2\alpha}I),$$

which is possible only if

$$XYX = X^2Y = -\frac{1}{2\alpha}X.$$

Then $H_0 = \ker X$ is invariant under A, B, so $H_0 = \{0\}$. On the subspace $H_0^{\perp}$ operator X is invertible and $XY = YX$. So we have $[A, B] = 0$ and $\alpha(A^2 - B^2) + I = 0$.

The possible definition of integrable pairs of unbounded pairs of selfadjoint operators satisfying (VII_0) and (VII_1) and the structure of such pairs were studied in [30, 40, 41] etc.

2. Pairs of self-adjoint operators satisfying polynomial relations

2.1. Dynamical systems

A study of the structure of pairs A, B that satisfy a polynomial (or functional) relation can sometimes be reduced to a study of the structure of unbounded operators C and X such that

$$CX = XF(C), \tag{17}$$

where C is a self-adjoint operator, $F(\cdot)$ is a fixed measurable real function on $\mathbb{R}^1$, X is a normal operator.

Example 2.1. Let $Z = A + iB$ and suppose that the relation for the self-adjoint operators A and B has the form

$$Z^*Z = F(ZZ^*). \tag{18}$$

Then (see [44]) by applying the polar decomposition to the operator $Z = \sqrt{Z^*Z}\,U$ and isolating isometric direct summands, we can reduce the study of pairs A, B that satisfy (18) to the unitary classification problem for pairs C, U such that C is self-adjoint, U is unitary, and $CU = UF(C)$.

a) The study of two self-adjoint operators A, B satisfying the polynomial relation

$$A^2 + B^2 + \frac{1}{i}[A, B] = P_n(A^2 + B^2 - \frac{1}{i}[A, B])$$

leads to the relation

$$CU = UP_n(C),$$

where C is positive, U is unitary.

b) Let the algebra of polynomials on the closed quantum unit disc (see [18]) be a unital $*$-algebra generated by two elements z and z^* with the following relation

$$[z, z^*] = \mu(I - zz^*)(I - z^*z) \quad (0 < \mu < 1),$$

or

$$z^*z = \frac{(1+\mu)zz^* - \mu}{\mu zz^* + (1-\mu)}.$$

Then a study of structure of these operators is equivalent to a study of pairs of positive C and unitary U such that

$$CU = UF(C)$$

where

$$F(\lambda) = \frac{(1+\mu)\lambda - \mu}{\mu\lambda + (1-\mu)} \quad (\lambda \in [0, \infty)).$$ ■

1. To make sense out of relation (17) if C and X are unbounded, it is natural to require that the operators C, $F(C)$, X be defined on a dense linear subset $\Phi \subset H$ invariant under these operators. But if we require, as in the case of a Lie algebra, that the resulting representation is integrable to a representation of the corresponding Lie group, it must be demanded that $\Phi \subset H^\omega(C, X)$ (Φ should consist of vectors which are analytic for the operators C and X).

Definition 2.2. We say that the operators C and X satisfy (17) if

$$CXu = XF(C)u$$

for all $u \in \Phi$, where

(a) Φ is invariant with respect to C, $F(C)$, X, X^*;
(b) Φ is a base for X, X^*;
(c) $\Phi \subset H^\omega(C, F(C))$.

Theorem 2.3. *Let conditions* (a), (b) *and* (c) *hold for* $\Phi \subset H$. *Then the following conditions are equivalent:*

(1) $(\forall u \in \Phi)\quad CXu = XF(C)u$;
(2) $(\forall u \in \Phi,\ \Delta \in \mathfrak{B}(\mathbb{R}^1))\quad E_C(\Delta)Xu = XE_C(F^{-1}\Delta))u$;
(3) $(\forall u \in \Phi,\ f(\cdot) \in L_\infty(\mathbb{R}^1, dE_C(\cdot)))\quad f(C)Xu = Xf(F(C))u$.

Proof. (1) $\Rightarrow$ (2). For all $n = 1, 2, \ldots$

$$C^nXu = C^{n-1}XF(C)u = \ldots = X(F(C))^n u,$$

and since the vectors of Φ are analytic,

$$e^{itC}Xu = Xe^{itF(C)}u, \qquad \forall t \in \mathbb{C}^1,\ |t| < t_0 = t_0(u).$$

For $u, v \in \Phi$, the function

$$k_1(t) = (e^{itC} Xu, v) = \int_{\mathbb{R}^1} e^{it\lambda}\, d(E_C(\lambda)Xu, v)$$

is analytic in the disk $|t| < t_0$. It is not difficult to show that it is analytic also in the strip $|\operatorname{Im} t| < t_0$. Similarly, the function

$$k_2(t) = (e^{itF(C)} u, X^* v) = \int_{\mathbb{R}^1} e^{it\lambda}\, d(E_C(F^{-1}(\lambda))u, X^* v)$$

is analytic for $|\operatorname{Im} t| < t_0$. It follows from the uniqueness property for analytic functions that $k_1(t) = k_2(t)$ for all $t \in \mathbb{R}^1$, or

$$(\forall t \in \mathbb{R}^1) \quad \int_{\mathbb{R}^1} e^{it\lambda}\, d(E_C(\lambda)Xu, v) = \int_{\mathbb{R}^1} e^{it\lambda}\, d(E_C(F^{-1}(\lambda))u, X^* v).$$

Due to uniqueness of Fourier transform for complex measures, we can deduce that

$$(E_C(\Delta)Xu, v) = (E_C(F^{-1}(\Delta))u, X^* v)$$

for all $u, v \in \Phi$, $\Delta \in \mathfrak{B}(\mathbb{R}^1)$. Since Φ is a base for X^*, we have

$$E_C(F^{-1}(\Delta))u \in D(X), \qquad E_C(\Delta)Xu = XE_C(F^{-1}(\Delta))u.$$

The implications (2) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (3) follow by considering the spectral decomposition for the operators C and $F(C)$. ■

Relations (17) could be also formulated in terms of bounded operators without using Φ. To do this, consider the polar decomposition of the operator $X = U|X|$ and the projection P into the initial space of the isometry U, $P = \operatorname{sign} |X|$.

Theorem 2.4. *For the operators C and X, the relation* (17) *is equivalent to the relations*

$$E_C(\Delta)UP = UE_C(F^{-1}(\Delta))P, \quad [E_{|X|}(\Delta), E_C(\Delta')] = 0, \qquad \Delta, \Delta' \in \mathfrak{B}(\mathbb{R}^1) \quad (19)$$

Proof. Let for all $u \in \Phi$,

$$E_C(\Delta)Xu = XE_C(F^{-1}(\Delta))u, \qquad \Delta \in \mathfrak{B}(\mathbb{R}^1).$$

For the adjoint operators, we get

$$X^* E_C(\Delta)u = E_C(F^{-1}(\Delta))X^* u,$$

and consequently,

$$X^* X E_C(\Delta)u = E_C(\Delta)X^* Xu.$$

Since the operator $|X|^2 = X^* X$ is essentially self-adjoint on Φ, it commutes with C in the sense of resolution of the identity. Consider the polar decomposition of the operator $X = U|X|$, where $|X| = \sqrt{X^*X}$, U is an isometry, and introduce $P = \operatorname{sign}|X| = U^*U$, a projection into the initial space of the isometry U. There, the operators P and $|X|$ commute with C. We have

$$E_C(\Delta)Xu = E_C(\Delta)U|X|u = U|X|E_C(F^{-1}(\Delta))u = UE_C(F^{-1}(\Delta))|X|u.$$

Since the operator $|X|$ is invertible on the image of the projection $\mathfrak{R}(P)$,

$$E_C(\Delta)UP = UE_C(F^{-1}(\Delta))P.$$

Conversely, from (19) we have for all $u \in \Phi$ that

$$E_C(\Delta)U|X|u = U|X|E_C(F^{-1}(\Delta))u,$$

or, equivalently, that

$$E_C(\Delta)Xu = XE_C(F^{-1}(\Delta))u$$

for all $u \in \Phi$. ■

2. If no additional information on the relation between the operators X and its adjoint X^* is given, the relation (17) is wild. If $X = X^*$, then it is possible to give a description of all the solutions of (17) by means of an integral of one- and two-dimensional irreducible ones (and to prove a structure theorem) [28]. If $F(\lambda) = \lambda$, then this is a classical spectral theorem for a pair of commuting self-adjoint operators, if $F(\lambda) = -\lambda$, it becomes a structure theorem for anticommuting operators [37].

Theorem 2.5. *Let A, B be anticommuting self-adjoint operators on H. Then the decomposition $H = H_0 \oplus H_1 \oplus (\mathbb{C}^2 \otimes H_+)$ and the following orthogonal resolutions of the identity are uniquely defined: (1) $E_0(\cdot)$ defined on $\mathbb{R}^1$ with the values in the projectons onto the subspaces of H_0; (2) $E_1(\cdot,\cdot)$ defined on $M_1 = \{(\lambda, b) \in \mathbb{R}^2 \mid F(\lambda) = \lambda, b \neq 0\}$, with the values in the projections onto the subspaces of H_1; (3) $E_2(\cdot,\cdot)$ defined on $M_2 = \{(\lambda, b) \in \mathbb{R}^2 \mid F(\lambda) > \lambda, b > 0\}$, with the values in the projections onto the subspaces of H_2 such that*

$$\begin{aligned} A &= \int_{\mathbb{R}^1} \lambda\, dE_0(\lambda) + \int_{M_1} \lambda\, dE_1(\lambda, b) + \int_{M_2} \begin{pmatrix} \lambda & 0 \\ 0 & F(\lambda) \end{pmatrix} \otimes dE_2(\lambda, b), \\ B &= \int_{M_1} b\, dE_1(\lambda, b) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \int_{M_2} b\, dE_2(\lambda, b). \end{aligned}$$

■

3. If X is a normal operator (or X is f-normal, i.e. $XX^* = f(X^*X)$), then to study the pair C, X, we consider the pair C, U (C is self-adjoint, U is unitary) such that

$$CU = UF(C).$$

This relation will be tame or wild (see [27, 46]) depending upon the structure of the dynamical system $\lambda \mapsto F(\lambda)$ ($\lambda \in \mathbb{R}^1$). Relation (17) is tame if and only if for the dynamical system $\lambda \mapsto F(\lambda)$, there exists a measurable section (a Borel set that intersects every orbit of the dynamical system in a single point).

Example 2.6. The study of $*$-representations of an algebra with two self-adjoint generators satisfying the polynomial relation

$$A^2 + B^2 + \frac{1}{i}[A,B] = (A^2 + B^2 - \frac{1}{i}[A,B] - \alpha I)^2 \quad (\alpha \in \mathbb{R}^1)$$

leads to the relation $CU = UP(C)$, where C is positive, U unitary and $P(\cdot)$ is a polynomial of the second order. By using a real linear change of coordinates, one can reduce it to the form

$$CU = U(C - \alpha I)^2), \quad (\alpha \in \mathbb{R}^1).$$

(a) For $\alpha < -1/4$ there are no representations.

(b) $\alpha = -1/4$. The spectrum of the operator $\sigma(C) \subset [1/4, \infty)$. The mapping $P(\lambda) = (\lambda + 1/4)^2$ is one-to-one on $[1/4, \infty)$. Denote by $P^{\circ k}(\cdot)$ the k-th iteration of $P(\cdot)$. The irreducible representations could be:

(1) one-dimensional: $H = \mathbb{C}^1$, $C = 1/4$, $U = e^{i\mu}$ $(\mu \in [0, 2\pi))$;

(2) infinite-dimensional: $H = l_2(\mathbb{Z})$, the operators are

$$\begin{aligned} Ae_k &= P^{\circ k}(\lambda) e_k \qquad (\lambda \in [1, (1 + 1/4)^2), \\ Ue_k &= e_{k+1} \qquad (k \in \mathbb{Z}) \end{aligned}$$

(here C is unbounded);

(c) $-1/4 < \alpha \le 0$. The spectrum of the operator $\sigma(C) \subset [x_0, \infty)$ (we set $x_{0,1} = \frac{1}{2}(2\alpha + 1 \pm \sqrt{4\alpha + 1}))$. The mapping $P(\lambda) = (\lambda - \alpha)^2$ is bijective on $[x_0, \infty)$. The irreducible representations are:

(1) one-dimensional: $H = \mathbb{C}^1$, $C = x_0$, $U = e^{i\mu}$ $(\mu \in [0, 2\pi))$ and $C = x_1$, $U = e^{i\mu}$ $(\mu \in [0, 2\pi))$;

(2) infinite-dimensional with bounded C: $H = l_2(\mathbb{Z})$,

$$\begin{aligned} Ce_k &= P^{\circ k}(\lambda) e_k, \\ Ue_k &= e_{k+1} \qquad (\lambda \in ((\lambda_0 - \alpha)^2, \lambda_0],\ k \in \mathbb{Z}) \end{aligned}$$

(here $\lambda_0 \in (x_0, x_1)$ is fixed), and with unbounded C:

$$\begin{aligned} Ce_k &= P^{\circ k}(\lambda) e_k, \\ Ue_k &= e_{k+1} \qquad (\lambda \in [\lambda_1, (\lambda_1 - \alpha)^2),\ k \in \mathbb{Z}) \end{aligned}$$

(here $\lambda_1 > x_1$ is fixed).

For $1/4 \le \alpha \le 0$, any representation of the relation can be represented as an integral of irreducible representations.

d) Consider the relation for $\alpha = \alpha^*$ ($\alpha^* = 1.4\ldots$ is a certain number such that the mapping $P(\lambda) = (\lambda - \alpha^*)^2 \colon [0, x_1] \to [0, x_1]$ has cycles of all the periods equal to 2^k, $k = 1, 2\ldots$ and does not have cycles of other periods). The spectrum of the operator $\sigma(C) \subset [0, \infty)$. Then

(1) The mapping $P(\cdot)$ is bijective on $[x_1, \infty)$ and so $H'_\infty = E_C((x_1, \infty))H$ is a subspace invariant with respect to C and U. The operators C and U restricted to H'_∞ have a very simple structure, they are "glued together" from irreducible

ones: unbounded $Ce_k = P^{\circ k}(\lambda)e_k$ and the unitary shift operator $Ue_k = e_{k+1}$ are both defined on $l_2(\mathbb{Z})$ ($\lambda \in (x_1, P(x_1))$, $k \in \mathbb{Z}$).

(2) The structure of representations with bounded operators on $H \ominus H'_\infty$ is more complicated. The mapping $P(\cdot)\colon [0, x_1] \to [0, x_1]$ is not bijective, however, $P(\cdot)\colon K \to K$ where $K = \overline{\{P^{\circ n}(\alpha^*) \mid n = 0, 1, \ldots\}}$ is homeomorphic to the Cantor set, is one-to-one (see, e.g. [42]). The dynamical system $(K, P(\cdot))$ has a unique ergodic invariant probability measure μ_0 [22]. Following [24], we can use the measure $\mu_0(\cdot)$ to construct factor representation of type II_1. This shows that the problem of describing an infinite-dimensional representation of the relation with bounded operators is wild for $\alpha = \alpha^*$.

e) For $\alpha > \alpha^*$ the corresponding dynamical system and, consequently, the relation are wild.

2.2. Semi-linear relations

Consider a relation that is more general than (17),

$$\sum_{k=1}^{N} f_k(C) X g_k(C) = h(C) \quad (k = 1, \ldots, N). \tag{20}$$

Here, the pair $C = C^*, X = X^* \in L(H)$ is a pair of bounded selfadjoint operators and $f_k(\cdot)$, $g_k(\cdot)$, $h(\cdot)$ ($k = 1, \ldots, N$) are complex functions bounded on the spectrum of C.

If the function

$$\phi(t) = \frac{h(t)}{\sum_{k=1}^{N} f_k(t) g_k(t)}$$

is assumed to be bounded on the spectrum of C, then a particular solution of non-homogeneous equation (20) can be taken to be $X = \phi(C)$. Then there is a correspondence between the pairs (C, X) that satisfy the homogeneous equation

$$\sum_{k=1}^{N} f_k(C) X g_k(C) = 0 \tag{21}$$

and the pairs $(C, X + \phi(C))$ that satisfy (20). So, we restrict ourselves by considering only homogeneous relations (21).

To a semi-linear relation (21), there correspond

a) the characteristic function

$$\Phi(t, s) = \sum_{k=1}^{N} f_k(t) g_k(s) \qquad (t, s \in \mathbb{R}^1);$$

b) the characteristic binary relation

$$\Gamma = \{(t, s) \mid \Phi(t, s) = 0\} \subset \mathbb{R}^2$$

c) an oriented graph (D, Γ), where an edge $\underset{t}{\bullet}\!\longrightarrow\!\underset{s}{\bullet}$ belongs to the graph Γ if and only if $\Phi(t, s) = 0$.

In the case when $\Phi(t,s)=0$ if and only if $\overline{\Phi(s,t)}=0$, the graph Γ together with the edge $\underset{t}{\bullet}\!\longrightarrow\!\underset{s}{\bullet}$ also contains the edge $\underset{t}{\bullet}\!\longleftarrow\!\underset{s}{\bullet}$, i.e., the graph can be considered as non-oriented.

In this article we will be limited only to the case when the *bounded self-adjoint* operators $A=A^*$, $B=B^*$ are such that

$$\sum_{k=1}^{N} f_k(A)Bg_k(A)=0, \tag{22}$$

where $\Phi(t,s)=\sum f_k(t)g_k(s)=\overline{\Phi(s,t)}$. We will discuss the unbounded operators only in the very end of the article.

1. We will start with the simplest case when the operator A has a finite discrete spectrum. Relation (22) can be expressed in a different way. Let $\sigma(A)=\{\lambda_1,\dots,\lambda_m\}$ and $H_j=H_{\lambda_j}$ be the eigenspaces of the operator A. Relatively to the decomposition $H=H_1\oplus\cdots\oplus H_m$, every operator can be written in a block matrix form,

$$X=(X_{ij})_{i,j=1}^{m}$$

Proposition 2.7. *For the pair A, B to satisfy* (22) *it is necessary and sufficient that the block matrix $B=(B_{ij})_{i,j=1}^{m}$ have its support on $\Gamma|_{\sigma(A)}$, i.e., that $B_{ij}=0$ for $(\lambda_i,\lambda_j)\notin\Gamma$.*

Proof. The proof follows from the equality

$$\Big(\sum_{k=1}^{N} f_k(A)Bg_k(A)x,y\Big)=\Phi(\lambda_i,\lambda_j)(Bx,y)$$

for $x\in H_i$, $y\in H_j$. ∎

Proposition 2.8. *Let A, B be a pair that satisfies* (22). *If the pair A, B is irreducible then the graph $\Gamma|_{\sigma(A)}$ is connected. For every finite connected subgraph $(D,\Gamma|_D)$, there exists an irreducible pair A, B that satisfies* (22) such that $D=\sigma(A)$.

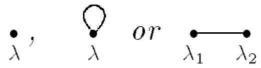
Proof. The pair A, B such that $\sigma(A)=D$,

$$\begin{aligned} A &= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}, \qquad \lambda_k\in D,\ \lambda_i\neq\lambda_j \text{ for } i\neq j, \\ B &= (b_{ij})_{i,j=1}^{m} \qquad b_{ij}=\begin{cases} 0, & (\lambda_i,\lambda_j)\notin\Gamma|_D \\ 1, & (\lambda_i,\lambda_j)\in\Gamma|_D \end{cases} \end{aligned}$$

is irreducible because $\Gamma|_D$ is connected. ∎

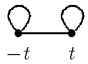
It depends on the structure of the graph Γ whether or not it is possible to describe up to unitary equivalence all irreducible pairs A, B that satisfy relation (22) such that $\sigma(A)=\{\lambda_1,\dots,\lambda_n\}$.

Proposition 2.9. *If $\sigma(A) = \{\lambda_1, \dots, \lambda_m\}$ then relation (22) is tame if and only if $m \le 2$ and $\Gamma|_{\sigma(A)}$ has the form*

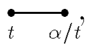
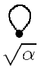


Proof. For the graphs of the form ,  or , irreducible pairs are either one-dimensional or two-dimensional. If $\Gamma|_{\sigma(A)}$ contains a subgraph of the form or , then the unitary classification problem for these pairs contain a wild problem of classifying pairs of self-adjoint operators without relations. ■

Example 2.10. a) the relation $[A^2, B] = 0$ is wild since $\Phi(t, s) = t^2 - s^2$ and the connected components of the corresponding graph are

 $(t \neq 0)$ and ;

b) The relation $ABA = \alpha B$ ($\alpha \in \mathbb{R}^1$) is tame for $\alpha \neq 0$ because $\Phi(t, s) = ts - \alpha$ and the corresponding graphs are

 $(t \neq 0 \text{ or } t \neq \sqrt{\alpha})$,  or .

c) The relation $ABA = 0$ is wild because $\Phi(t, s) = ts$ and any vertex of Γ is connected to zero.

2) In the same way as it was done in Section 1.1, it is possible to classify all cubic relations linear with respect to B,

$$P_3(A, B) = \alpha\{A^2, B\} + \beta\{A, B\} + \gamma ABA + \delta B + \frac{\hbar_1}{i}[A^2, B] + \frac{\hbar_2}{i}[A, B] = 0$$
$$(\alpha, \beta, \gamma, \delta, \hbar_1, \hbar_2 \in \mathbb{R}^1)$$

and to determine for each of them whether it is tame or wild by using the zeros of the characteristic function

$$\Phi(t, s) = \alpha(t^2 + s^2) + \beta(t + s) + \gamma ts + \delta + \frac{\hbar_1}{i}(t^2 - s^2) + \frac{\hbar_2}{i}(t - s) = 0. \qquad ■$$

Remark 2.11. The problems of graph representation theory arise here for non-selfadjoint B. In particular, tame oriented graphs which correspond to *indecomposable* representations are described using extended Dynkin diagrams (Gabriel theorem, see [12]).

Proposition 2.12. *Under the assumption that the spectrum of A is finite, the pairs of bounded self-adjoint operators A, B satisfy semi-linear relation* (22) *if and only if the support of B is contained in the graph of this relation, constructed with respect to $\sigma(A)$.*

We give one of the analogues of this statement for a pair of bounded self-adjoint operators A, B that satisfy relation (22) (the spectrum of the operator A is not assumed to be finite).

Let $E_A(\cdot)$ be a resolution of the identity of a self-adjoint operator A, $K \subset \mathbb{R}^2$ be a Borel set. We will say that the operator B has the support in K with respect to A if $E_A(\Delta) B E_A(\Delta') = 0$ for any Borel sets $\Delta, \Delta' \subset \mathbb{R}^1$ such that $\Delta \times \Delta'$ do not intersect K.

Theorem 2.13. (See [5]). *Let $f_k(\cdot)$, $g_k(\cdot)$ $(k = 1, \dots, N)$ be polynomials. For A, B to satisfy semi-linear relation* (22) *it is necessary and sufficient that B have the support in the binary relation $\Gamma \subset \mathbb{R}^2$ constructed relatively to A.*

Note that this statement implies both a generalization of Fuglede-Putnam theorem and the following statement which is a generalization of the corollary of Kleineke-Shirokov theorem for self-adjoint operators.

Corollary 2.14. *If two homogeneous polynomial relations have the characteristic functions $\Phi_1(t,s)$ and $\Phi_2(t,s)$ such that $\Phi_1(t,s) = 0$ if and only if $\Phi_2(t,s) = 0$ then a pair of bounded self-adjoint operators A, B either satisfy both of these relations or neither of them.*

Example 2.15. The relation $\mathrm{ad}^n_{q,A}(B) = 0$ $(|q| = 1)$ for bounded self-adjoint operators A, B implies that $\mathrm{ad}_{q,A}(B) = AB - qBA = 0$ since these relations have the characteristic functions $\Phi_1(t,s) = (t - qs)^n$ and $\Phi_2(t,s) = (t - qs)$.

A procedure similar to one above can be applied to a pair A, B that satisfies one semi-linear relation (22) (without any restriction on $\sigma(A)$) to determine whether the relation is tame or wild. For theorems on the structure of pairs A, B that satisfy a number of relations, one of which is semi-linear, see [38].

3. Integrable semi-linear relations for unbounded operators were investigated in the particular case

$$F(A)B = BG(A)$$

for $A = A^*$, $B = B^*$ and $F(\cdot)$, $G(\cdot)\colon \mathbb{R}^1 \to \mathbb{R}^1$, see [31].

Problem 2.16. What is an "integrable" pair of unbounded self-adjoint operators satisfying $[A,[A,B]] = 0$?

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