Journal of Lie Theory
Volume **4** (1994) 17–29

Maximal Ol'shanskiĭ Semigroups

J. D. Lawson *

Communicated by K. H. Hofmann

Abstract. In this article we show for simple complex groups $G_{\mathbb{C}}$ that if a subsemigroup S contains a real form G such that G is not isolated in S, then S contains a minimal Ol'shanskiĭ semigroup and is contained in a maximal Ol'shanskiĭ semigroup. It follows in particular that maximal Ol'shanskiĭ semigroups are maximal semigroups.

Let G be a real form for a complex Lie group $G_{\mathbb{C}}$. A complex Ol'shanskiĭ semigroup in $G_{\mathbb{C}}$ is one of the form $\Gamma(\mathbf{C}) = \exp(i\mathbf{C})G$, where $\mathbf{C}$ is an $\mathrm{Ad}\, G$-invariant convex cone in the Lie algebra $\mathfrak{g}$ of G. These semigroups are noncommutative analogues of tube domains and play an important role in the representation theory of G ([13], [12]) and also arise in a variety of contexts of harmonic analysis on G ([5]). Their structure is presented in more detail in Section 1.

When one tries to establish in a concrete situation that a certain semigroup is an Ol'shanskiĭ semigroup, it is generally much easier to show that the appropriate Ol'shanskiĭ semigroup is contained in the given semigroup than to show the reverse containment. This paper provides an important theoretical tool for showing the reverse containment in the case of simple groups: if a subsemigroup S contains a real form G of $G_{\mathbb{C}}$ such that G is not isolated in S, then S contains a minimal Ol'shanskiĭ semigroup and is contained in a maximal Ol'shanskiĭ semigroup. The preliminary background is recalled in the first two sections and the proof is given in Sections 3 and 4. Section 5 discusses some important consequences and applications of this result.

It is a corollary of what is done here that in the context of simple groups maximal complex Ol'shanskiĭ semigroups are maximal semigroups, a fact first established by Hilgert and Neeb [7, Chapter 8]. The proof given here relies heavily on their earlier work, but there is also a significant simplication. In particular, one no longer needs the theory of highest weight representations nor San Martin's theory of invariant control sets [17]. This not only makes the proof shorter and more accessible, but also enables the generalization given in this paper. For related results in the real analytic case, see [8].

---

* The author gratefully acknowledges the support of NSF Grant DMS-9104582

ISSN 0940–2268 / $2.50

1. Complex Ol'shanskiĭ Semigroups

By a *wedge* or *cone* in a finite dimensional real or complex vector space, we mean a non-empty subset that (i) is closed under addition, (ii) is closed with respect to scalar multiplication by non-negative scalars, and (iii) is topologically closed. The terms "wedge" and "cone" are thus interchangeable; we shall, however, use the term "wedge" when we wish to emphasize the likelihood that the *edge* $\mathbf{W} \cap -\mathbf{W}$ of $\mathbf{W}$ is non-trivial. We shall tend to think of cones $\mathbf{C}$ as *pointed*, i.e., as satisfying $\mathbf{C} \cap -\mathbf{C} = \{0\}$. We shall generally be interested in cones or wedges $\mathbf{C}$ that are *generating*, i.e., that satisfy $\mathbf{C} - \mathbf{C}$ is the whole vector space. Note that $\mathbf{C}$ is generating if and only if it has non-empty interior, which we denote by $\mathbf{C}^\circ$. In particular, $\mathbf{C}$ always has interior in $\mathbf{C} - \mathbf{C}$; we call this interior the *algebraic interior* and denote it by $\operatorname{algint}(\mathbf{C})$.

In the Lie theory of semigroups, one assigns to a closed subsemigroup S of G an appropriate wedge $\mathcal{L}(S)$ (consisting of subtangent vectors to S at e) in the Lie algebra $\mathcal{L}(G)$ of G, a process that extends the the usual assigning to an analytic subgroup of the group G its subalgebra in $\mathcal{L}(G)$.

We recall basic facts from the Lie theory of semigroups concerning this assignment (see, for example, [6]). Let S be a closed subsemigroup of a Lie group G with Lie algebra $\mathcal{L}(G)$. We define the *subtangent set* of S at e by

$$\mathcal{L}(S) = \{X \in \mathcal{L}(G) \colon \exp(tX) \in S \ \text{ for all } \ t \geq 0\}.$$

It turns out that $\mathbf{W} := \mathcal{L}(S)$ is always a *Lie wedge*, i.e., $\mathbf{W}$ is a wedge which satisfies (i) the edge $\mathbf{W} \cap -\mathbf{W}$ (which is the largest vector subspace contained in $\mathbf{W}$) is a subalgebra, and (ii) the wedge $\mathbf{W}$ is invariant under the adjoint action of any connected analytic subgroup with Lie subalgebra the edge (equivalently $\mathbf{W}$ is invariant under $e^{\operatorname{ad} X}$ for all $X \in \mathbf{W} \cap -\mathbf{W}$). We say that S is *infinitesimally generated* if the subsemigroup generated by the exponential image of $\mathcal{L}(S)$ is dense in S and is *strictly infinitesimally generated* if the semigroup generated by the exponential image of $\mathcal{L}(S)$ is all of S.

Let $\mathfrak{g}$ be a Lie algebra, and let G be a corresponding Lie group. We denote by $\operatorname{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$. The group of inner automorphisms is the group $\operatorname{Inn}(\mathfrak{g}) := \langle e^{\operatorname{ad} \mathfrak{g}} \rangle$; if G is connected, it agrees with the group $\operatorname{Ad}(G)$.

A cone $\mathbf{C} \subseteq \mathfrak{g}$ is called an *invariant cone* if $\mathbf{C}$ is invariant under the adjoint action of G_0, the identity component of G. Note that this is equivalent to the invariance of the cone under all automorphisms of the form $\exp(\operatorname{ad} X)$ for $X \in \mathfrak{g}$, since these automorphisms generate the adjoint group. Thus invariance may be viewed as strictly a Lie algebra phenomenon.

These invariant cones give rise to one of the most standard and basic constructions for Ol'shanskiĭ semigroups (see [13], Section V.4 of [6], Chapter VII of [7] and [10]).

Definition 1.1. Let $\mathbf{C}$ be an invariant generating pointed cone in a finite dimensional real Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ be the complexification of $\mathfrak{g}$ and let $G_{\mathbb{C}}$ be a connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. A closed subsemigroup S of $G_{\mathbb{C}}$ is called the *complex Ol'shanskiĭ semigroup associated to* $\mathbf{C}$ *and* $G_{\mathbb{C}}$ if the following conditions are satisfied:

(i) The semigroup S is closed in $G_{\mathbb{C}}$ and satisfies $S = (\exp i\mathbf{C})G_0 = G_0(\exp i\mathbf{C})$, where G_0 is the analytic subgroup generated in $G_{\mathbb{C}}$ by $\exp \mathfrak{g}$ and is itself a closed subgroup of $G_{\mathbb{C}}$.

(ii) $(X,h) \mapsto (\exp iX)h\colon \mathbf{C} \times G_0 \to S$ is a homeomorphism (and a diffeomorphism if that is appropriately defined for this context). Thus each $s \in S$ admits a unique *(Ol'shanskiĭ) polar decomposition* $s = (\exp iX)h$, $X \in \mathbf{C}$, $h \in G_0$. The left-right dual also holds.

(iii) The interior S° of S is a dense semigroup ideal, $S^\circ = (\exp i\mathbf{C}^\circ)G_0$, where $\mathbf{C}^\circ$ is the interior of $\mathbf{C}$ in $\mathfrak{g}$, S° is a complex manifold, and the multiplication on S° is holomorphic.

(iv) The mapping $s \mapsto s^*\colon S \to S$ defined by $s^* = g^{-1}\exp iX$ if $s = (\exp iX)g$ is a continuous antiautomorphism of order 2 on S, called the *adjoint involution.* It is antiholomorphic on S°. For $s \in S$, $s^* = s^{-1}$ if and only if $s \in G_0$ and $s^* = s$ if $s \in \exp i\mathbf{C}$.

(v) The polar decomposition is uniquely determined as $s = \exp(iX)h \in \exp(i\mathbf{C})G_0$, where X is the unique member of $\mathbf{C}$ such that $\exp(2iX) = ss^*$ and $h = \exp(-iX)s$.

(vi) The semigroup S is strictly infinitesimally generated and $\mathcal{L}(S) = \mathfrak{g} \oplus \mathbf{C}$.

The complex Ol'shanskiĭ semigroup associated to $\mathbf{C}$ and $G_{\mathbb{C}}$ is denoted by $\Gamma(\mathbf{C}, G_{\mathbb{C}})$, or simply $\Gamma(\mathbf{C})$ if $G_{\mathbb{C}}$ is understood. ■

The next theorem is an existence theorem in the semisimple setting for complex Ol'shanskiĭ semigroups which is very close to the original one of Ol'shanskiĭ [13]. More general results can be found in [10] and [7].

Theorem 1.2. *Let $\mathfrak{g}$ be finite dimensional semisimple Lie algebra which contains a pointed generating invariant cone $\mathbf{C}$, let $G_{\mathbb{C}}$ be a connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$, and let G_0 be the analytic subgroup of $G_{\mathbb{C}}$ generated by the exponential image of $\mathfrak{g}$ in $G_{\mathbb{C}}$. Then the complex Ol'shanskiĭ semigroup $\Gamma(\mathbf{C})$ always exists for the data $\mathbf{C}$ and $G_{\mathbb{C}}$. Furthermore, if complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ integrates to a conjugation involution on $G_{\mathbb{C}}$, then G_0 is the identity component of the fixed point set, and the adjoint involution on $\Gamma(\mathbf{C})$ is the restriction of the anti-isomorphism on $G_{\mathbb{C}}$ given by $g \mapsto g^* := \overline{g}^{-1}$.* ■

2. Invariant Cones in Semisimple and Hermitian Algebras

In the remainder of the paper let $\mathfrak{g}$ denote a finite dimensional semisimple Lie algebra which contains a compact Cartan algebra $\mathfrak{t}$ (i.e. $e^{\operatorname{ad}\mathfrak{t}}$ is a compact subgroup of $\operatorname{Inn}\mathfrak{g}$). We summarize the basic theory of invariant cones in such algebras as it appears in [15], [16], and Chapter 7 of [7] (see also [19], [14], [18], Chapter III of [6], and [11]). Associated to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ in the complexification $\mathfrak{g}_{\mathbb{C}}$ is a root decomposition as follows. For a linear functional $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ we set

$$\mathfrak{g}_{\mathbb{C}}^{\lambda} := \{X \in \mathfrak{g}_{\mathbb{C}} : (\forall Y \in \mathfrak{t}_{\mathbb{C}})[Y,X] = \lambda(Y)X\}$$

and

$$\Delta := \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) := \{\lambda \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} : \mathfrak{g}_{\mathbb{C}}^{\lambda} \neq \{0\}\}.$$

Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\lambda\in\Delta} \mathfrak{g}_{\mathbb{C}}^{\lambda},$$

$\lambda(\mathfrak{t}) \subseteq i\mathbb{R}$ for all $\lambda \in \Delta$ and $\sigma(\mathfrak{g}_{\mathbb{C}}^{\lambda}) = \mathfrak{g}_{\mathbb{C}}^{-\lambda}$, where σ denotes complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. Let $\mathfrak{k} \supseteq \mathfrak{t}$ denote a maximal compactly embedded subalgebra; $\mathfrak{k}$ always exists and is uniquely determined by $\mathfrak{t}$. Then a root is said to be *compact* if $\mathfrak{g}_{\mathbb{C}}^{\lambda} \subseteq \mathfrak{k}_{\mathbb{C}}$.

We write Δ_k for the set of compact roots and Δ_p for the set of non-compact roots. A subset $\Delta^+ \subseteq \Delta$ is called a *positive system of roots* if there exists an element $X \in i\mathfrak{t}$ such that $\Delta^+ = \{\alpha \in \Delta : \alpha(X) \geq 0\}$ and $\Delta^+ \cap -\Delta^+ = \emptyset$. The *Weyl group* associated to $\mathfrak{t}$ is the group

$$\mathcal{W}_{\mathfrak{k}} := N_G(\mathfrak{t})/Z_G(\mathfrak{t}) \cong N_K(\mathfrak{t})/Z_K(\mathfrak{t})$$

which coincides with the Weyl group of the compactly embedded Lie algbra $\mathfrak{k}$. A positive system Δ^+ is said to be $\mathfrak{k}$*-adapted* if Δ_p^+ is invariant under the Weyl group.

For a positive system Δ^+ of roots we define the cone

$$\mathfrak{c}_{\max} := \mathfrak{c}_{\max}(\Delta_p^+) := \{X \in \mathfrak{t} : (\forall \alpha \in \Delta_p^+) i\alpha(X) \geq 0\}$$

and

$$\mathfrak{c}_{\min} := \mathfrak{c}_{\min}(\Delta_p^+) := \operatorname{cone}\{i[\overline{X}_\alpha, X_\alpha] : \alpha \in \Delta_p^+, X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}\},$$

where for a subset A of a vector space $\operatorname{cone}(A)$ denotes the smallest closed convex cone containing A. For a cone $\mathbf{C}$ in a vector space V we write $\mathbf{C}^\star := \{\nu \in V^* : \nu(\mathbf{C}) \subseteq \mathbb{R}^+\}$ for the dual cone.

The next theorem is the major one on the existence of invariant cones in the semisimple case.

Theorem 2.1. *Let $\mathfrak{g}$ be a finite dimensional semisimple real Lie algebra, which contains a pointed generating invariant cone $\mathbf{C}$. Then compact Cartan algebras $\mathfrak{t}$ exist. For any compact Cartan subalgebra $\mathfrak{t}$ and the (unique) maximal compact algebra $\mathfrak{k}$ containing it, there exists $0 \neq Z \in \mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{t}$ and a $\mathfrak{k}$-adapted positive system of roots Δ^+ such that such that (i) Z is in the interior of $\mathbf{C}$, (ii) Z is in the interior of the elliptic (= compactly embedded) elements of $\mathfrak{g}$, (iii) $i\alpha(Z) > 0$ for all $\alpha \in \Delta_p^+$, and (iv) for Δ_p^+*

$$\mathfrak{c}_{\min} \subseteq \mathbf{C} \cap \mathfrak{t} \subseteq \mathfrak{c}_{\max}.$$

The invariant pointed generating cones containing $\mathfrak{c}_{\min}$ are in one-to-one correspondence via intersection with $\mathfrak{t}$ with those cones in $\mathfrak{t}$ containg $\mathfrak{c}_{\min}$, contained in $\mathfrak{c}_{\max}$, and invariant under the action of the Weyl group $\mathcal{W}_{\mathfrak{k}}$. In particular, there exist largest and smallest pointed generating invariant cones $\mathbf{C}_{\max}$ and $\mathbf{C}_{\min}$ containing $\mathfrak{c}_{\min}$ and these satisfy $\mathbf{C}_{\max} \cap \mathfrak{t} = \mathfrak{c}_{\max}$ and $\mathbf{C}_{\min} \cap \mathfrak{t} = \mathfrak{c}_{\min}$. ■

The simple Lie algebras which contain pointed generating invariant cones are precisely the hermitian simple ones, those for which the maximal compactly embedded subalgebras have a non-trivial (one-dimensional) center. Let $\mathfrak{g}$ be a hermitian simple Lie algebra, let $\mathfrak{k}$ be a maximal compact subalgebra, and let $B_\theta(\cdot,\cdot)$ be the positive-definite form on $\mathfrak{g}$ given by $B_\theta(X,Y) = -\kappa\big(X,\theta(Y)\big)$, where κ is the Cartan-Killing form and θ is the Cartan involution which fixes $\mathfrak{k}$. Let $Z \in \mathfrak{k}$ span the center of $\mathfrak{k}$, and let $\mathfrak{t}$ be a compact Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. Then $Z \in \mathfrak{t}$, by maximal commutativity of $\mathfrak{t}$, and there exists a $\mathfrak{k}$-adapted positive system of roots Δ^+ such that $i\alpha(Z) > 0$ for all $\alpha \in \Delta_p^+$. The following conditions hold (see [15]):

(I) $\mathfrak{c}_{\min}$ contains Z in its algebraic interior, and $\mathfrak{c}_{\max} = (\mathfrak{c}_{\min})^*$, where the dual is taken in $\mathfrak{t}$ with respect to the restriction of $B_\theta(\cdot,\cdot)$.

(II) Each pointed generating invariant cone in $\mathfrak{g}$ contains either Z or $-Z$ in its interior.

(III) For each pointed generating invariant cone $\mathbf{C}$, we have $(\mathbf{C}\cap\mathfrak{t})^* = \mathbf{C}^*\cap\mathfrak{t}$, where the first dual cone is computed in $\mathfrak{t}$ and the second in $\mathfrak{g}$, both with respect to $B_\theta(\cdot,\cdot)$.

3. Maximal Semigroups and Cartan Subalgebras

Let S be a subsemigroup of a Lie group G such that the interior S° of S in G is non-empty. We set

$$\mathcal{L}_\infty(S) := \{X \in G\colon \exp(tX) \in S^\circ \text{ for some } t > 0\}.$$

We list some elementary properties of $\mathcal{L}_\infty(S)$, which hold in an arbitrary Lie algebra.

Lemma 3.1. *Let $S \neq G$ be a subsemigroup with non-empty interior in a connected Lie group G with Lie algebra $\mathfrak{g}$. Then the following hold:*

(i) *The set $\mathcal{L}_\infty(S)$ is topologically open and is closed under multiplication by positive scalars.*

(ii) *If $X \in \mathcal{L}_\infty(S)$, then there exists $T > 0$ such that $t \geq T$ implies $\exp(tX) \in S$.*

(iii) *If $\exp(sX) \in \overline{S}$ for some $s > 0$ (in particular, if $X \in \mathcal{L}_\infty(S)$) and if $Y \in \mathcal{L}_\infty(S)$ and $[X,Y] = 0$, then $X + Y \in \mathcal{L}_\infty(S)$.*

(iv) *If $X \in \mathcal{L}_\infty(S)$, then $\exp(-tX) \notin \overline{S}$ for $t > 0$. In particular, $0 \notin \mathcal{L}_\infty(S)$.*

(v) *If $g, g^{-1} \in \overline{S}$ and $X \in \mathcal{L}_\infty(S)$, then $\mathrm{Ad}\, g(X) \in \mathcal{L}_\infty(S)$.*

Proof. (i) Consider the mapping $\Phi\colon (0,\infty) \times \mathfrak{g} \to G$ defined by $\Phi(t,X) = \exp(tX)$. Then Φ is continuous and $\mathcal{L}_\infty(S)$ is projection into the second coordinate of the open set $\Phi^{-1}(S^\circ)$, hence open. That $\mathcal{L}_\infty(S)$ is closed under multiplication by positive scalars is immediate.

(ii) Suppose that $X \in \mathcal{L}_\infty(S)$ and $\exp(tX) \in S^\circ$ for $t > 0$. Then there exists $\varepsilon > 0$ such that $\exp(sX) \in S^\circ$ for $t - \varepsilon < s < t + \varepsilon$. Note that for each

$n > 0$, $\exp(n \cdot (t - \varepsilon, t + \varepsilon)X) \subseteq S^\circ$, since the latter is a semigroup. As n is chosen larger, the intervals $n \cdot (t - \varepsilon, t + \varepsilon)$ expand in length. Hence if N is chosen large enough, then the intervals $n \cdot (t - \varepsilon, t + \varepsilon)$ overlap each other for consecutive integers greater than N, and thus their union over all $n > N$ contains the open ray (Nt, ∞). (Indeed one can choose N so that $1/N < (t + \varepsilon)/(t - \varepsilon) - 1$.) One now chooses $T = Nt + 1$.

(iii) Suppose that $\exp(sX) \in \overline{S}$ for $s > 0$. By (ii) there exists $T > 0$ such that $\exp(tY) \in S^\circ$ for $t \geq T$. Pick $n \in \mathbb{N}$ such that $ns > T$. Then for $t := ns$,

$$\exp\bigl(t(X + Y)\bigr) = \exp(tX)\exp(tY) \in \overline{S}S^\circ \subseteq S^\circ,$$

so $X + Y \in \mathcal{L}_\infty(S)$.

(iv) Suppose $X \in \mathcal{L}_\infty(S)$ and $\exp(-tX) \in \overline{S}$ for some $t > 0$. By parts (i) and (iii), $0 \in \mathcal{L}_\infty(S)$. Thus S° contains some open neighborhood $\exp(B)$ of e, and hence is all of G, since G is connected, a contradiction.

(v) Suppose $g, g^{-1} \in \overline{S}$, and $s := \exp(tX) \in S^\circ$ for $t > 0$. Pick an open symmetric neighborhood B of the identity e so small that $BsB \subseteq S^\circ$. Then there exist $b, d \in B$ such that $gb, dg^{-1} \in S$. Then

$$gsg^{-1} = gbb^{-1}sd^{-1}dg^{-1} \in SBsBS \subseteq SS^\circ S \subseteq S^\circ.$$

Thus $\exp t\bigl(\operatorname{Ad} g(X)\bigr) = \exp\bigl(\operatorname{Ad} g(tX)\bigr) = g\exp(tX)g^{-1} \in S^\circ$. It follows that $\operatorname{Ad} g(X) \in \mathcal{L}_\infty(S)$. ∎

We record a lemma that will be useful for our later calculations.

Lemma 3.2. *Let B be a neighborhood of 0 in $\mathbf{sl}(2, \mathbb{C})$ and let $\mathfrak{a}$ be the one-dimensional subspace of diagonal matrices in $\mathbf{SL}(2, \mathbb{R})$. Then there exists $m \in \mathbb{N}$ such that $A^m = \mathbf{SL}(2, \mathbb{C})$, where $A := \exp \mathfrak{a} \cup \exp B$.*

Proof. Let U be an open set containing the identity e and contained in $\exp B$. Then the maximal compact subgroup $K := \mathbf{SU}(2)$ is contained in U^n for some n, since these increasing open sets cover K. Let $D = \exp \mathfrak{a}$ be the one-dimensional subgroup of diagonal matrices of determinant one with positive real entries on the diagonal. Then

$$\mathbf{SL}(2, \mathbb{C}) = KDK \subseteq U^n D U^n \subseteq A^{2n+1},$$

where the first equality follows from the Cartan decomposition of a semisimple Lie group (see e.g. Theorem IX.1.1 of [4]), or can be deduced directly by writing any matrix in its polar decomposition as a product of a special unitary and a positive definite hermitian matrix, and then converting the latter to a diagonal matrix by an orthonormal change of basis. ∎

In the next lemma we let $\mathfrak{g}$ be a semisimple Lie algebra containing a pointed generating invariant cone $\mathbf{C}$.

We assume the notation and setting of Theorem 2.1. We thus have in the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$ a positive system of roots Δ^+ such that $i\alpha(Z) > 0$ for all $\alpha \in \Delta^+$, where $Z \in \mathfrak{z}(\mathfrak{k})$. For each non-compact positive root $\alpha \in \Delta_p^+$, we set

$$\mathfrak{g}_\mathbb{C}(\alpha) := \mathfrak{g}_\mathbb{C}^\alpha + \mathfrak{g}_\mathbb{C}^{-\alpha} + [\mathfrak{g}_\mathbb{C}^\alpha, \mathfrak{g}_\mathbb{C}^{-\alpha}].$$

Note that $\mathfrak{g}_{\mathbb{C}}(\alpha)$ is isomorphic to $\mathbf{sl}(2,\mathbb{C})$.

Let $G_{\mathbb{C}}$ be a complex Lie group for $\mathfrak{g}_{\mathbb{C}}$ for which complex conjugation integrates to $G_{\mathbb{C}}$, and let G be the connected component of the fixed point set. Then G is closed and has Lie subalgebra $\mathfrak{g}$ in $\mathfrak{g}_{\mathbb{C}}$. *We henceforth consider only closed subsemigroups S of $G_{\mathbb{C}}$ with non-empty interior S° and with the property that $G \subseteq S \neq G_{\mathbb{C}}$.*

Lemma 3.3. *Suppose $iZ \in \mathcal{L}_\infty(S)$. If for $\alpha \in \Delta_p^+$, there exists $X \in \mathfrak{t}$ such that $iX \in \mathcal{L}_\infty(S)$ and $i\alpha(X) < 0$, then $\mathfrak{g}_{\mathbb{C}}(\alpha) \cap i\mathfrak{t} \subseteq \overline{\mathcal{L}_\infty(S) \cap i\mathfrak{t}}$.*

Proof. By hypothesis, we have that $iZ \in \mathcal{L}_\infty(S) \cap i\mathfrak{t}$, where Z is appropriately chosen as above in the center of $\mathfrak{k}$, and by choice of the positive system $\alpha(iZ) > 0$. Thus there exists a positive number r such that $i\alpha(rZ + X) = 0$. By Lemma 3.1(iv), $Y := i(rZ + X) \neq 0$, and by Lemma 3.1(iii), $Y \in \mathcal{L}_\infty(S)$. We can replace Y with a positive multiple such that $\exp(tY) \in S^\circ$; it remains true that $\alpha(tY) = 0$. We again call this new element Y. Consider the subalgebra $\mathfrak{g}_{\mathbb{C}}(\alpha) = \mathfrak{g}_{\mathbb{C}}^\alpha + \mathfrak{g}_{\mathbb{C}}^{-\alpha} + [\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^{-\alpha}]$. Then $\mathfrak{g}_{\mathbb{C}}(\alpha)$ is isomorphic to $\mathbf{sl}(2,\mathbb{C})$ and $\mathfrak{g}^{[\alpha]} := \mathfrak{g}_{\mathbb{C}}(\alpha) \cap \mathfrak{g}$ is a real form isomorphic to $\mathbf{sl}(2,\mathbb{R})$ (see e.g. Section 3.6 of [6] or Sections 7.1 and 7.2 of [7]). Since $\alpha(Y) = 0$, we see directly that $[\mathfrak{g}_{\mathbb{C}}(\alpha), Y] = 0$. Thus $\exp Y$ commutes with $\exp U$ for all $U \in \mathfrak{g}_{\mathbb{C}}(\alpha)$. Pick an open set B containing 0 in $\mathfrak{g}_{\mathbb{C}}(\alpha)$ such that $\exp(B + Y) \subseteq S^\circ$. Set $A := \exp B \cup \exp(\mathfrak{g}^{[\alpha]}$. Since the subgroup generated by $\exp(\mathfrak{g}_{\mathbb{C}}(\alpha))$ is a homomorphic image of the simply connected group $\mathbf{SL}(2,\mathbb{C})$, it follows directly from Lemma 3.2 that $H := \langle \exp \mathfrak{g}_{\mathbb{C}}(\alpha) \rangle \subseteq A^m$ for some positive m. Let $U \in \mathfrak{g}_{\mathbb{C}}(\alpha)$. Then $\exp U \in A^m$, so $\exp U = \exp X_1 \cdots \exp X_m$, where for each i, $X_i \in B$ or $X_i \in \mathfrak{g}^{[\alpha]}$. Then

$$\begin{aligned}\exp(mY + U) &= \exp(mY)\exp U = \exp Y \exp X_1 \exp Y \exp X_2 \cdots \exp Y \exp X_m \\ &= \exp(Y + X_1) \cdots \exp(Y + X_m).\end{aligned}$$

If $X_i \in B$, then $\exp(Y + X_i) \in S^\circ$ and if $X_i \in \mathfrak{g}^{[\alpha]}$, then $\exp(Y + X_i) = \exp Y \cdot \exp X_i \in S^\circ S \subseteq S^\circ$ since $X_i \in \mathfrak{g}$. Thus the product $\exp(mY + U) \in S^\circ$. Suppose further that $U \in \mathfrak{g}_{\mathbb{C}}(\alpha) \cap i\mathfrak{t}$. From the preceding paragraph, $mY + nU \in \mathcal{L}_\infty(S) \cap i\mathfrak{t}$ for each n, and $(1/n)(mY + nU)$ converges to U. Hence U is in the closure of $\mathcal{L}_\infty(S) \cap i\mathfrak{t}$. ∎

We specialize now to the case that $\mathfrak{g}$ is hermitian simple. As noted above, these are precisely the simple Lie algebras which contain an invariant cone.

Lemma 3.4. *Let $\mathfrak{t}$ be a compact Cartan subalgebra of $\mathfrak{g}$. If $\mathcal{L}_\infty(S) \cap i\mathfrak{t} \neq \emptyset$, then either $\pm iZ \in \mathcal{L}_\infty(S)$, where Z spans the center of the maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ containing $\mathfrak{t}$.*

Proof. Pick $X \in \mathfrak{t}$ such that $iX \in \mathcal{L}_\infty(S)$. Consider the element $Y := \sum_{\gamma \in \mathcal{W}_{\mathfrak{k}}} \gamma(iX)$. Then $Y \in \mathfrak{t}$ and $Y \in \mathcal{L}_\infty(S)$ by (iii) and (v) of Lemma 3.1 since $K \subseteq S$, where $\mathcal{L}(K) = \mathfrak{k}$. It then follows from 3.1(iv) that $Y \neq 0$. Since Y is fixed by the action of the Weyl group $\mathcal{W}_{\mathfrak{k}}$, it must be in the center of $\mathfrak{k}$. Since the center of $\mathfrak{k}$ is one-dimensional, the result follows from 3.1(i). ∎

Remark 3.5. By interchanging Z and $-Z$ and the positive and negative roots if necessary, we can in light of Lemma 3.4 assume that $iZ \in \mathcal{L}_\infty(S)$. We henceforth assume this to be the case. ∎

Theorem 3.6. *Let $\mathfrak{g}$ be a hermitian simple Lie algebra, let $G_\mathbb{C}$ be a complex Lie group for $\mathfrak{g}_\mathbb{C}$ for which complex conjugation integrates to $G_\mathbb{C}$, and let G be the connected component of the fixed point set.*

Furthermore assume that $\mathfrak{t}$ is a compact Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{k}$ is the unique maximal compact subalgebra containg $\mathfrak{t}$, $Z \in \mathfrak{t}$ spans the one-dimensional center of $\mathfrak{k}$, and Δ^+ is a $\mathfrak{k}$-adapted system of positive roots satisfying $\alpha(iZ) > 0$ for each $\alpha \in \Delta_p^+$.

If S is a subsemigroup of $G_\mathbb{C}$ containing G and having non-empty interior, then either $\mathcal{L}_\infty(S) \cap i\mathfrak{t} \subseteq \mathfrak{c}_{\max}$ or $\mathcal{L}_\infty(S) \cap i\mathfrak{t} \subseteq -\mathfrak{c}_{\max}$.

Proof. We assume that $\mathcal{L}_\infty(S) \cap i\mathfrak{t} \neq \emptyset$, otherwise there is nothing to prove. By Lemma 3.4 and Remark 3.5, we may assume without loss of generality that $iZ \in \mathcal{L}_\infty(S)$. In this case we show $\mathcal{L}_\infty(S) \cap i\mathfrak{t} \subseteq \mathfrak{c}_{\max}$. Indeed, if this is not the case, then there exists $X \in \mathfrak{t}$ such that $iX \in \mathcal{L}_\infty(S)$ and $i\alpha(X) < 0$ for some $\alpha \in \Delta_p^+$, by the definition of $\mathfrak{c}_{\max}$.

Let $0 \neq X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha$. Then it is standard that $\overline{X}_\alpha \in \mathfrak{g}_\mathbb{C}^{-\alpha}$ (see e.g., Theorem 7.4 of [7]), and thus $-i[\overline{X}_\alpha, X_\alpha] \in \mathfrak{g}_\mathbb{C}(\alpha) \cap -\mathfrak{c}_{\min}$, by the definitions of $\mathfrak{g}_\mathbb{C}(\alpha)$ and $\mathfrak{c}_{\min}$. Since $i\mathfrak{g}_\mathbb{C}(\alpha) = \mathfrak{g}_\mathbb{C}(\alpha)$, We have

$$Y := [\overline{X}_\alpha, X_\alpha] = i(-i)[\overline{X}_\alpha, X_\alpha] \in \mathfrak{g}_\mathbb{C}(\alpha) \cap -i\mathfrak{c}_{\min} \subseteq \mathfrak{g}_\mathbb{C}(\alpha) \cap i\mathfrak{t}.$$

It follows from Lemma 3.3 that $Y \in \overline{\mathcal{L}_\infty(S) \cap i\mathfrak{t}}$.

Consider $U := \sum_{\gamma \in \mathcal{W}_\mathfrak{k}} \gamma(Y)$. Since $Y \in -i\mathfrak{c}_{\min}$, which is invariant under the action of the Weyl group $\mathcal{W}_\mathfrak{k}$, we conclude that $0 \neq U \in -i\mathfrak{c}_{\min}$. Since U is invariant under the action of $\mathcal{W}_\mathfrak{k}$ and is in $-i\mathfrak{c}_{\min}$, we conclude that U is some negative multiple of iZ. If $Y_n \to Y$, where $Y_n \in i\mathfrak{t} \cap \mathcal{L}_\infty(S)$, then $U_n := \sum_{\gamma \in \mathcal{W}_\mathfrak{k}} \gamma(Y_n)$ is in $\mathcal{L}_\infty(S)$ from parts (iii) and (v) of Lemma 3.1 and the sequence converges to U. Then $-U_n$ converges to $-U \in (\mathbb{R}^+ \setminus \{0\})(iZ) \subseteq \mathcal{L}_\infty(S)$. It follows (Lemma 3.1(i)) that $-U_n \in \mathcal{L}_\infty(S)$ for large n, and hence $0 = -U_n + U_n \in \mathcal{L}_\infty(S)$ by Lemma 3.1(iii), but a contradiction to Lemma 3.1(iv). ∎

4. Maximal Subsemigroups

In this section we work in the following setting. Let $\mathfrak{g}$ be a hermitian simple Lie algebra, let $\mathfrak{g}_\mathbb{C}$ be its complexification, let $G_\mathbb{C}$ be a corresponding complex Lie group for which complex conjugation integrates to $\sigma: G_\mathbb{C} \to G_\mathbb{C}$, and let G be the identity component of the fixed point set. Then G is a closed connected Lie subgroup with Lie algebra $\mathfrak{g}$. Let $g^* := \sigma(g)^{-1}$ denote the adjoint involution, the involutive antiautomorphism arising from σ.

We further assume that S is a subsemigroup of $G_\mathbb{C}$ which contains G such that G is not *isolated* in S, i.e., $(S \cap GU) \setminus G \neq \emptyset$ for all open neighborhoods U of the identity. Our goal in this section and the main goal of the paper to establish the following:

Theorem 4.1. *A subsemigroup $S \neq G_{\mathbb{C}}$ of $G_{\mathbb{C}}$ containing G such that G is not isolated in S contains a minimal Ol'shanskiĭ semigroup and is contained in the corresponding maximal Ol'shanskiĭ semigroup. In particular, a maximal Ol'shanskiĭ semigroup is maximal in the class of all proper subsemigroups.*

After the work of Section 3, the proof requires only minor modification of the proof of Hilgert and Neeb of the maximality of the maximal Ol'shanskiĭ semigroup. We recall the following proposition from their work which, although it is non-trivial, can be derived rather quickly using fairly standard Lie theory (see Proposition 8.48 of [7] and the preceding Proposition 8.32 and Lemma 8.47).

Proposition 4.2. *Let $\mathfrak{g}$ be a simple hermitian Lie algebra, let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compact Cartan subalgebra contained in the maximal compact subalgebra $\mathfrak{k}$, let Δ^+ be a $\mathfrak{k}$-adapted system of positive roots, and let $\mathfrak{c}_{\max}$ be the corresponding maximal cone in $\mathfrak{t}$. If $S \subseteq G_{\mathbb{C}}$ is a closed subsemigroup which contains G and has dense interior and if*

$$S \cap \exp(i\mathfrak{t}) \subseteq \exp(i\mathfrak{c}_{\max} \cup -i\mathfrak{c}_{\max}),$$

then

$$S \subseteq \overline{GN_{G_{\mathbb{C}}}(\mathfrak{t})G},$$

where $N_{G_{\mathbb{C}}}(\mathfrak{t})$ is the normalizer of $\mathfrak{t}$ in $G_{\mathbb{C}}$. ∎

Proof of Theorem 4.1. Let S be a subsemigroup of $G_{\mathbb{C}}$ containing G such that G is not isolated in S. For a countable base U_n at e, pick $t_n = g_n u_n \in S \setminus G$, where $g_n \in G$ and $u_n \in U_n$. Then the sequence $s_n := g_n^{-1} t_n$ is a sequence in $S \setminus G$ converging to e.

Step 1: *S contains a minimal Ol'shanskiĭ semigroup.*
For large n, we may write $s_n = (\exp X_n)(\exp iY_n)$, where $X_n, Y_n \in \mathfrak{g}$ converge to 0 (see, e.g., Lemma IV.7.10 of [6]). Then $t_n := \exp(-X_n)s_n = \exp(iY_n)$ is also a sequence in S converging to e, since $G \subseteq S$. So the tangent Lie wedge $\mathcal{L}(S)$ must meet $i\mathfrak{g}$ non-trivially, since any limit point of $iY_n/\|Y_n\|$ will be in $\mathcal{L}(S)$. Since $i\mathfrak{g}$ and $\mathcal{L}(S)$ are invariant under $\operatorname{Ad} G$, their intersection is an invariant cone in $i\mathfrak{g}$, and hence must contain a minimal invariant cone $i\mathbf{C}$. Since S is closed, it is standard that S contains $\exp\big(\mathcal{L}(S)\big)$, and hence contains the minimal Ol'shanskiĭ semigroup $\exp(i\mathbf{C})G$.

Step 2: *S is contained in the closure of $GN_{G_{\mathbb{C}}}(\mathfrak{t})G$.*
Since the minimal Ol'shanskiĭ semigroup has the identity e in the closure of its interior, so does S, and hence S has dense interior. We choose a compact Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, let $\mathfrak{k}$ be the corresponding maximal compactly embedded subalgebra, and choose a $\mathfrak{k}$-adapted positive root system Δ^+ so that $i\mathbf{C} \cap i\mathfrak{t} = i\mathfrak{c}_{\min} \subseteq \exp^{-1}(S) \cap i\mathfrak{t}$. We deduce from Theorem 3.6 that the hypotheses of Proposition 4.2 are satisfied, and we thus conclude

$$S \subseteq \overline{GN_{G_{\mathbb{C}}}(\mathfrak{t})G}.$$

Before we proceed to Step 3, we need some additional notation. Let $\mathfrak{c}_{\max}$ be the corresponding maximal cone in $\mathfrak{t}$ for the positive root system Δ^+ just introduced. Let $\mathcal{W}$ denote the big Weyl group of $i\mathfrak{t}$ with respect to $\mathfrak{g}_{\mathbb{C}}$ (i.e., the normalizer of $i\mathfrak{t}$ in $\mathrm{Ad}(G_{\mathbb{C}})$ modulo the centralizer of $i\mathfrak{t}$ in $\mathrm{Ad}(G_{\mathbb{C}})$), and let $\mathcal{W}_{\mathfrak{k}}$ denote those members of $\mathcal{W}$ which have representatives of the form $\mathrm{Ad}(g)$, where $g \in K = \exp \mathfrak{k}$. Note this agrees with our previous definition of $\mathcal{W}_{\mathfrak{k}}$, except now we consider the action on $i\mathfrak{t}$ instead of $\mathfrak{t}$.

Step 3: *If $\gamma \in \mathcal{W}$ and $\gamma{\cdot}i\mathfrak{c}_{\min} \subseteq i\mathfrak{c}_{\max}$, then $\gamma \in \mathcal{W}_{\mathfrak{k}}$.*

Pick $iX \in i\mathfrak{c}_{\min}$ such that the stabilizer of iX in $\mathcal{W}$ is trivial (this is possible since $i\mathfrak{c}_{\min}$ has interior in $i\mathfrak{t}$). Then $\gamma{\cdot}iX \in i\mathfrak{c}_{\max}$, and from the definition of $\mathfrak{c}_{\max}$, $\alpha(\gamma{\cdot}iX) = i\alpha(\gamma{\cdot}X) \geq 0$ for all $\alpha \in \Delta_p^+$. We can rotate $\gamma{\cdot}iX$ to a Δ_k^+-positive chamber by an element of $\mathcal{W}_{\mathfrak{k}}$, i.e., there exists $\gamma' \in \mathcal{W}_{\mathfrak{k}}$ such that $\alpha(\gamma'\gamma{\cdot}iX) \geq 0$ for each $\alpha \in \Delta_k^+$. Now since Δ^+ is $\mathfrak{k}$-adapted, it is still the case that $\alpha(\gamma'\gamma{\cdot}iX) \geq 0$ for each $\alpha \in \Delta_p^+$. Similarly since $iX \in i\mathfrak{c}_{\min} \subseteq i\mathfrak{c}_{\max}$, we can find $\gamma'' \in \mathcal{W}_{\mathfrak{k}}$ such that $\alpha(\gamma''{\cdot}iX) \geq 0$ for all $\alpha \in \Delta^+$. Since each closed Weyl chamber is a fundamental domain for $\mathcal{W}$ (e.g., [1, Ch. V, §3, no. 3.3, Thm. 2]), it follows that $\gamma'\gamma{\cdot}iX = \gamma''{\cdot}iX$, and hence that $\gamma'\gamma = \gamma''$ from the triviality of the stabilizer of iX. Thus $\gamma = (\gamma')^{-1}\gamma'' \in \mathcal{W}_{\mathfrak{k}}$.

Step 4: *$S^\circ \cap N_{G_{\mathbb{C}}}(\mathfrak{t}) \subseteq S_{\max}$, where $S_{\max}$ is the unique maximal Ol'shanskiĭ semigroup containing the minimal one appearing in step 1.*

Let $s \in S^\circ \cap N_{G_{\mathbb{C}}}(\mathfrak{t})$. Let $\gamma := \mathrm{Ad}(s)|_{i\mathfrak{t}}$ be the member of the Weyl group corresponding to s. It follows from the finiteness of the Weyl group $\mathcal{W}$ that there exists $n \in \mathbb{N}$ such that $\gamma^n = \mathrm{Ad}(s^n)|_{i\mathfrak{t}} = \mathrm{id}_{i\mathfrak{t}}$. Then

$$\exp(\gamma{\cdot}i\mathfrak{c}_{\min})s^n = \mathrm{Ad}(s)\big(\exp(i\mathfrak{c}_{\min})\big)s^n = s\exp(i\mathfrak{c}_{\min})s^{n-1} \subseteq S^\circ S S^\circ \subseteq S^\circ,$$

since $\exp(i\mathfrak{c}_{\min}) \subseteq S$ from step 1.

On the other hand, it is standard that $s^n \in Z_{G_{\mathbb{C}}}(\mathfrak{t}) = \exp \mathfrak{t}_{\mathbb{C}}$, so that there exists $t \in T := \exp \mathfrak{t}$ and $Y \in i\mathfrak{t}$ such that $s^n = t\exp(Y)$. Let $A := \exp(i\mathfrak{t})$. It follows that

$$\exp(\gamma{\cdot}i\mathfrak{c}_{\min})t\exp(Y) = t\exp(\gamma{\cdot}i\mathfrak{c}_{\min} + Y) \subseteq S^\circ \cap (TA) = T(S^\circ \cap A) \subseteq T\exp(i\mathfrak{c}_{\max}).$$

The last equality follows from $T \subseteq G \subseteq S$ and the last containment follows from Theorem 3.6.

It now follows from the uniqueness of the polar decomposition $G_{\mathbb{C}} = K'\exp(\mathfrak{p} + i\mathfrak{k})$, where $K' = \exp(\mathfrak{k} + i\mathfrak{p})$, that $\gamma{\cdot}i\mathfrak{c}_{\min} + Y \subseteq i\mathfrak{c}_{\max}$. For each $X \in \mathfrak{c}_{\min}$, $\gamma \cdot iX + (1/n)Y = \frac{1}{n}(n\gamma \cdot iX + Y) \in i\mathfrak{c}_{\max}$. Thus $\gamma \cdot iX \in i\mathfrak{c}_{\max}$. Hence $\gamma \cdot i\mathfrak{c}_{\min} \subseteq i\mathfrak{c}_{\max}$, and thus from step 3, $\gamma \in \mathcal{W}_{\mathfrak{k}}$. Consequently $\gamma = \mathrm{Ad}(k)|_{i\mathfrak{t}}$ for some $k \in N_K(\mathfrak{t})$. Now $\mathrm{Ad}(k^{-1}s)$ restricted to $i\mathfrak{t}$ is the identity, i.e., $k^{-1}s \in Z_{G_{\mathbb{C}}}(i\mathfrak{t}) = \exp \mathfrak{t}_{\mathbb{C}}$. Thus $s \in N_K(i\mathfrak{t})(\exp \mathfrak{t}_{\mathbb{C}}) \cap S$. Since $N_K(i\mathfrak{t}) \subseteq K \subseteq G \subseteq S$ and $\exp \mathfrak{t}_{\mathbb{C}} = TA$, we have

$$\begin{aligned} N_K(i\mathfrak{t})(TA) \cap S &= N_K(i\mathfrak{t})\big((TA) \cap S\big) = N_K(i\mathfrak{t})T(S \cap A) \\ &\subseteq N_K(i\mathfrak{t})T\exp(i\mathfrak{c}_{\max}) \subseteq G(i\mathbf{C}_{\max}) = S_{\max}, \end{aligned}$$

and thus $s \in S_{\max}$.

Step 5: *$S \subseteq S_{\max}$, the appropriate maximal Ol'shanskiĭ semigroup.*

Since S° is dense in S and $S_{\max}$ is closed, it suffices to show $S^\circ \subseteq S_{\max}$. Let $s \in S^\circ$, and let B be any open set containing s such that $B \subseteq S^\circ$. We have already seen that S is contained in the closure of $GN_{G_{\mathbb{C}}}(\mathfrak{t})G$, so there exists $x \in B \cap GN_{G_{\mathbb{C}}}(\mathfrak{t})G$. Pick $g_1, g_2 \in G$ and $y \in N_{G_{\mathbb{C}}}(\mathfrak{t})$ such that $x = g_1 y g_2$. Then $y = g_1^{-1} x g_2^{-1} \in GS^\circ G \subseteq S^\circ$. It follows from step 4 that $y \in S_{\max}$. Since also $G \subseteq S_{\max}$, we conclude that $x = g_1 y g_2 \in S_{\max}$. Since B was arbitrary containing s, we conclude $s \in \overline{S}_{\max} = S_{\max}$. This concludes the proof.

The last assertion follows by beginning with $S_{\max}$, supposing it is contained in some proper subsemigroup S, and applying the first part of the theorem to conclude that $S \subseteq S_{\max}$. ∎

Corollary 4.3. *Let $G_{\mathbb{C}}$ satisfy the hypotheses at the beginning of this section and let $S \neq G_{\mathbb{C}}$ be a closed subsemigroup which contains G and $\exp(i\mathfrak{c}_{\max})$. Then S is the maximal Ol'shanskiĭ semigroup corresponding to $\mathfrak{c}_{\max}$.*

Proof. Since S contains $\exp(i\mathfrak{c}_{\max})$, the hypotheses of Theorem 4.1 are satisfied. Hence $S \subseteq S_{\max}$, the maximal Ol'shanskiĭ semigroup corresponding to $\mathfrak{c}_{\max}$. On the other hand, $\mathcal{L}(S) \cap i\mathfrak{g}$ is a G-invariant wedge in $i\mathfrak{g}$ containing $i\mathfrak{c}_{\max}$, and hence must be the maximal invariant cone $i\mathbf{C}_{\max}$ (see Section 2). Thus $S_{\max} = \exp(i\mathbf{C}_{\max})G \subseteq \big(\exp \mathcal{L}(S)\big)G \subseteq SS = S$. ∎

5. Applications and Reflections

We state first of all what is essentially a reformulation of the major result, Theorem 4.1.

Theorem 5.1. *Let G be a complex Lie group with complex Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be a simple algebra which is a real form of $\mathfrak{g}$. If $S \neq G$ is a closed subsemigroup of G such that S contains the connected subgroup G_0 with Lie algebra $\mathfrak{g}_0$ and G_0 is not isolated in S, then $\mathfrak{g}_0$ is a hermitian Lie algebra, and there exist unique minimal and maximal Ol'shanskiĭ semigroups $S_{\min}$ and $S_{\max}$ each with group of units G_0 such that $S_{\min} \subseteq S \subseteq S_{\max}$.*

Proof. We first consider the G_0-invariant Lie wedge $\mathcal{L}(S)$. It follows directly that $i\mathbf{C} := \mathcal{L}(S) \cap i\mathfrak{g}_0$ is invariant and proper in $i\mathfrak{g}$ (since $S \neq G$). Thus $\mathbf{C}$ is an invariant wedge in $\mathfrak{g}_0$, and hence is a pointed generating cone since $\mathfrak{g}_0$ is simple. Thus $\mathfrak{g}_0$ is hermitian. One is now in a position to apply Theorem 4.1. ∎

Remark 5.2. The Theorems 4.1 and 5.1 give general conditions for concluding that a semigroup is a subsemigroup of an Ol'shanskiĭ semigroup. Once one obtains this conclusion, then one has the Ol'shanskiĭ polar factorization in S, and each of the factors will also belong to S since S contains G_0. The availabilty of this factorization can be a useful theoretical tool. For example, one easily derives the polar decompositions in contraction semigroups found in the work of Brunet and Kramer ([3] and [2]) by this means. ∎

Remark 5.3. It is a consistent feature of Lie theory to try to reduce the study of various problems and objects to the study of Cartan subalgebras. We remark that this is the case for those semigroups arising in Theorems 4.1 and 5.1. Indeed if S is such a closed semigroup, then $S = \overline{G(S \cap i\mathfrak{t})G}$, where $\mathfrak{t}$ is any compact Cartan subalgebra of the Lie algebra of the group of units of S. This follows directly from the facts that S has dense interior, that the interior of the maximal Ol'shanskiĭ semigroup is of the form $G\exp(i\mathbf{C}^\circ)G$, and that $G \subseteq S \subseteq S_{\max}$. Hence determining S reduces to determining $S \cap i\mathfrak{t}$. The latter is generally a much more manageable problem. It might be an interesting problem to try to determine precisely which sets between $i\mathfrak{c}_{\min}$ and $i\mathfrak{c}_{\max}$ are traces of such semigroups. ∎

Remark 5.4. Ol'shanskiĭ [13] has shown that that the compression semigroups of simple hermitian symmetric spaces are maximal Ol'shanskiĭ semigroups. The difficult direction in his proof is showing that the compression semigroup is contained in the maximal Ol'shanskiĭ semigroup, but this can be easily deduced using Corollary 4.3. Similar observations apply to the compression semigroup of complex linear transformations of determinant 1 preserving the set of positive (or negative) elements of a pseudohermitian form. ∎

References

[1] Bourbaki, N., "Groupes et algèbres de Lie," Chap. IV–VI, Hermann, Paris, 1968.

[2] Brunet, M., *The metaplectic semigroup and related topics*, Reports on Mathematical Physics **22** (1985), 149–170.

[3] Brunet, M., and P. Kramer, *Semigroups of length increasing transformations*, Reports on Mathematical Physics **15** (1979), 287–304.

[4] Helgason, S., "Differential Geometry, Lie Groups, and Symmetric Spaces," Academic Press, Orlando, 1978.

[5] Hilgert, J., *Applications of Lie semigroups in analysis*, in: Hofmann,K. H., Lawson, J. D., and J. S. Pym (eds.), "The Analytical and Topological Theory of Semigroups," de Gruyter, Berlin 1990, 27–50.

[6] Hilgert, J., Hofmann, K. H., and J. D. Lawson, "Lie Groups, Convex Cones, and Semigroups," Oxford Press, Oxford 1989.

[7] Hilgert, J., and K.-H. Neeb, "Basic Theory of Lie Semigroups and Applications," Springer Lecture Notes in Mathematics **1552**, 1993.

[8] —, *Maximality of compression semigroups*, Semigroup Forum Vol **50** (1994), to appear.

[9] Howe, R., *The oscillator semigroup*, in: "The Mathematical Heritage of Hermann Weyl," Proceedings of Symposia in Pure Mathematics **48,** American Mathematical Society, Providence, 1988, 61–132.

[10] Lawson, J. D., *Polar and Ol'shanskiĭ Decompositions*, J.reine u. angewandte Math. **448** (1994), 191–219.

[11] Neeb, K.-H., "Invariant subsemigroups of Lie groups," Memoirs of the AMS **499**, 1993.

[12] —, "Holomorphic representation theory and coadjoint orbits of convexity type," Habilitationsschrift, Technische Hochschule, Darmstadt, 1993.

[13] Ol'shanskiĭ, G. I., *Invariant cones in Lie algebras, Lie semigroups, and the holomorphic discrete series*, Funct. Anal. Appl. **15** (1982), 275–285.

[14] —, *Convex cones in symmetric Lie algebras, Lie semigroups and invariant causal (order) structures on pseudo-Riemannian symmetric spaces*, Soviet Math Dokl., Amer. Math. Soc. translation **26** (1982), 97–101.

[15] Paneitz, S. M., *Invariant convex cones and causality in semisimple Lie algebras and groups*, J. of Funct. Anal. **43** (1981), 313–359.

[16] —, *Determination of invariant convex cones in simple Lie algebras*, Arkiv för Mat. **21** (1984), 217–228.

[17] San Martin, L., *Invariant control sets on flag manifolds*, Math. of Control, Signals and Systems, to appear.

[18] Spindler, K., "Invariante Kegel in Liealgebren," Mitteilungen Math. Seminar Giessen (**188**), 1988.

[19] Vinberg, E. B., *Invariant cones and orderings in Lie groups*, Funct. Anal. and Appl. **14** (1980), 1–13.

J. D. Lawson
Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana 70803
USA
LAWSON@MARAIS.MATH.LSU.EDU

Received June 30, 1994