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$\mathbf{L^p}$–$\mathbf{L^q}$–Estimates for functions of the Laplace–Beltrami operator on noncompact symmetric spaces, II*

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In this paper we continue the study of functional calculus for the Laplace–Beltrami operator on symmetric spaces of the noncompact type begun in [3]; this paper is dedicated to a study of the Poisson semigroup, which we define shortly.

Let G and K be a connected noncompact semisimple Lie group with finite center and a maximal compact subgroup thereof, and consider the symmetric space G/K, also denoted by X. We denote by n the dimension of X, by ℓ its real rank, and by ν the "pseudo–dimension" $2\left|\Sigma_0^+\right|+\ell$, where $\left|\Sigma_0^+\right|$ is the cardinality of the set of the positive indivisible (restricted) roots.

There is a canonical invariant Riemannian metric on X; denote by $-\mathcal{L}_0$ the associated Laplace–Beltrami operator. By general nonsense, $\mathcal{L}_0$ is positive and essentially self-adjoint on $C_c^\infty(X)$; let $\mathcal{L}$ be the unique self-adjoint extension of $\mathcal{L}_0$ and $\{P_\eta\}$ the spectral resolution of the identity for which

$$\mathcal{L}f = \int_{R_0^2}^\infty \eta \, dP_\eta f \qquad \forall f \in \mathrm{Dom}(\mathcal{L}),$$

where $R_0 = \langle \rho, \rho \rangle^{1/2}$, ρ being the usual half-sum of the positive roots.

For θ in $[0,1]$ and σ in $(0,1)$, the θ–heat and the (σ,θ)–Poisson semigroups $(\mathcal{H}_{t,\theta})_{t>0}$ and $(\mathcal{P}_{t,\theta}^\sigma)_{t>0}$ are defined thus:

$$\mathcal{H}_{t,\theta} f = \int_{R_0^2}^\infty \exp\bigl(-t(\eta - \theta R_0^2)\bigr)\, dP_\eta f \qquad \forall t \in (0,\infty) \quad \forall f \in L^2(X)$$

$$\mathcal{P}_{t,\theta}^\sigma f = \int_{R_0^2}^\infty \exp\bigl(-t(\eta - \theta R_0^2)^\sigma\bigr)\, dP_\eta f \qquad \forall t \in (0,\infty) \quad \forall f \in L^2(X).$$

The (σ,θ)–Poisson semigroup may be obtained from the θ–heat semigroup by subordination. However, while estimates from above can be proved using this

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fact, estimates from below cannot. If $1 \leq p, q \leq \infty$ and the operator $\mathcal{P}_{t,\theta}^{\sigma}$ satisfies a norm inequality of the form

$$\left\|\mathcal{P}_{t,\theta}^{\sigma} f\right\|_q \leq C \left\|f\right\|_p \qquad \forall f \in L^2(X) \cap L^p(X),$$

$\mathcal{P}_{t,\theta}^{\sigma}$ is said to be L^p–L^q-bounded.

In this paper we examine for which p and q the operator $\mathcal{P}_{t,\theta}^{\sigma}$ is L^p–L^q-bounded, and we study the behaviour of the L^p–L^q-operator norms $|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q}$ as t tends to 0 and to ∞ for all such p and q. As t tends to ∞, the expression describing the behaviour of $|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q}$ involves powers of t, in which the indices ν and ℓ play an important rôle. Two features of our study are noteworthy. First, while $\mathcal{H}_{t,\theta}$ is L^p–L^q-bounded whenever $1 \leq p \leq q \leq \infty$, $\mathcal{P}_{t,\theta}^{\sigma}$ is not L^p–L^q-bounded for many such p and q. Second, when p or q reach the critical index for L^p–L^q-boundedness, the exponent $\ell + 1$ appears; in our previous work, we saw only ℓ. We refer to [3] for an account of related work.

In order to state our main theorem, we introduce a little notation: if $0 \leq \theta \leq 1$,

$$\begin{aligned} p_\theta &= 2 \Big/ \left(1 + (1-\theta)^{1/2}\right), \\ R_\theta &= [(1-\theta)\, \langle \rho, \rho \rangle]^{1/2}, \\ R_{\theta,p} &= \big[(\frac{4}{pp'} - \theta)\, \langle \rho, \rho \rangle \big]^{1/2}; \end{aligned}$$

in the definition of $R_{\theta,p}$, $p_\theta \leq p \leq p_\theta{}'$. Observe that $1 \leq p_\theta \leq 2$, that R_θ defined here agrees with R_0 as defined previously when $\theta = 0$, that $R_{\theta,p} = R_\theta$ when $p = 2$, and that $R_{\theta,p} = 0$ when $p = p_\theta$ or $p = p_\theta{}'$.

Theorem 1. *Suppose that $0 \leq \theta \leq 1$, $0 < \sigma < 1$, and $1 \leq p, q \leq \infty$. The following conditions hold:*

(i) *if $t > 0$, then $\mathcal{P}_{t,\theta}^{\sigma}$ is L^p–L^q-bounded only if $p \leq q$, $p \leq p_\theta{}'$, and $q \geq p_\theta$;*

(ii) *if $p_\theta \leq p \leq p_\theta{}'$, then*
$$|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;p} = \exp\left(-R_{\theta,p}^{2\sigma}\, t\right) \qquad \forall t \in (0, \infty);$$

(iii) *if $p \leq q$, $p \leq p_\theta{}'$ and $q \geq p_\theta$, then*
$$|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q} \sim t^{-n(1/p-1/q)/2\sigma} \qquad \forall t \in (0, 1];$$

(iv) *if $p < q = 2$ or $2 = p < q$, then*
$$|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q} \sim t^{-\nu/4} \exp\left(-R_\theta^{2\sigma} t\right) \qquad \forall t \in [1, \infty);$$

(v) *if $p < 2 < q$, then*
$$|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q} \sim t^{-\nu/2} \exp\left(-R_\theta^{2\sigma} t\right) \qquad \forall t \in [1, \infty);$$

(vi) *if $p < q < 2$ and $q > p_\theta$, then*

$$\left\|\!\left|\mathcal{P}^{\sigma}_{t,\theta}\right|\!\right\|_{p;q} \sim t^{-\ell/2q'} \exp\left(-R^{2\sigma}_{\theta,q}\, t\right) \qquad \forall t \in [1,\infty);$$

(vii) *if* $p < q = p_\theta$, *then*

$$\left\|\!\left|\mathcal{P}^{\sigma}_{t,\theta}\right|\!\right\|_{p;q} \sim t^{-(\ell+1)/2\sigma q'} \qquad \forall t \in [1,\infty);$$

(viii) *if* $2 < p < q$ *and* $p < p_\theta{}'$, *then*

$$\left\|\!\left|\mathcal{P}^{\sigma}_{t,\theta}\right|\!\right\|_{p;q} \sim t^{-\ell/2p} \exp\left(-R^{2\sigma}_{\theta,p}\, t\right) \qquad \forall t \in [1,\infty);$$

(ix) *if* $p_\theta{}' = p < q$, *then*

$$\left\|\!\left|\mathcal{P}^{\sigma}_{t,\theta}\right|\!\right\|_{p;q} \sim t^{-(\ell+1)/2\sigma p} \qquad \forall t \in [1,\infty).$$

Section 1 of this paper is devoted to notation, and a summary of relevant material, such as the the spherical Fourier transformation. In Section 2 we prove our theorem.

1. Notation and Background Material

We use the standard notation of the theory of Lie groups and symmetric spaces, as in, for instance, S. Helgason [6]. Our notation here is also consistent with our paper [3], to which we refer several times.

For any x in G, we denote by $A(x)$ the element of $\mathfrak{a}$ such that $x \in N \exp A(x)\, K$ (the Iwasawa decomposition). For any (complex-valued) linear form Λ on $\mathfrak{a}$, the elementary spherical function ϕ_Λ is defined by the rule

$$\phi_\Lambda(x) = \int_K \exp\big((i\Lambda + \rho)\, A(kx)\big)\, dk \qquad \forall x \in G.$$

The spherical Fourier transform $\widetilde{f}$ of an $L^1(G)$–function f is defined by the formula

$$\widetilde{f}(\Lambda) = \int_G f(x)\, \phi_{-\Lambda}(x)\, dx \qquad \forall \Lambda \in \mathfrak{a}^*.$$

Harish–Chandra proved an inversion formula and a Plancherel formula for the spherical Fourier transformation, namely

$$f(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\Lambda)\, \phi_\Lambda(x)\, d\mu(\Lambda) \qquad \forall x \in G$$

for "nice" K–bi-invariant functions f on G, and

$$\|f\|_2 = \left[\int_{\mathfrak{a}^*} \left|\widetilde{f}(\Lambda)\right|^2\, d\mu(\Lambda)\right]^{1/2} \qquad \forall f \in L^2(K\backslash X),$$

where $d\mu(\Lambda) = c_G\, |\mathbf{c}(\Lambda)|^{-2}\, d\Lambda$, and $\mathbf{c}$ denotes the Harish–Chandra $\mathbf{c}$–function. For the details, see, for instance, Theorem IV.7.5 of Helgason [6]. We often deal with the inversion formula and the Plancherel formula with purely radial integrands. From the formula for the $\mathbf{c}$–function (Theorem IV.6.14 of [6]), it is clear that if $F : [0,\infty) \to \mathbf{C}$ and $f(\Lambda) = F(|\Lambda|)$ for all Λ in $\mathfrak{a}^*$, then

$$\text{(1)} \quad \int_{\mathfrak{a}^*} f(\Lambda)\, d\mu(\Lambda) = c_G \int_{\mathfrak{a}^*} F(|\Lambda|)\, |\mathbf{c}(\Lambda)|^{-2}\, d\Lambda \sim \int_0^\infty F(r)\, (1+r)^{n-\nu}\, r^{\nu-1}\, dr,$$

provided the integrals converge. We shall also use a modified version of the Plancherel measure, $\widetilde{\mu}$, defined by the rule

$$d\widetilde{\mu}(\Lambda)/d\Lambda = \prod_{\alpha\in\Sigma_0^+} (1+|\langle\alpha,\Lambda\rangle|)^{d_\alpha}$$

(see [3] for the notation). Clearly

$$C \leq d\widetilde{\mu}(\Lambda)/d\Lambda \leq C'(1+|\Lambda|^{n-\ell}),$$

where C and C' are constants. The modified Plancherel measure appears in several results of harmonic analysis on symmetric spaces; see [3].

Let $\mathbf{W}_1$ be the interior of the convex hull in $\mathfrak{a}^*$ of the images of ρ under the Weyl group of $(\mathfrak{g},\mathfrak{a})$. For δ in $(0,1)$, we denote by $\mathbf{W}_\delta$ and $\mathbf{T}_\delta$ the dilate of $\mathbf{W}_1$ by δ and the tube over the polygon $\mathbf{W}_\delta$, i.e., $\mathbf{T}_\delta = \mathfrak{a}^* + i\delta\mathbf{W}_1$. If $1 \leq p < 2$, the spherical Fourier transform of an $L^p(G)$-function extends to a bounded holomorphic function in the tube $\mathbf{T}_{\delta(p)}$, where $\delta(p)$ is defined by the rule

$$\delta(p) = 2/p - 1.$$

We define the quadratic function Q_θ on $\mathfrak{a}^*_{\mathbf{C}}$:

$$Q_\theta(\Lambda) = \langle\Lambda,\Lambda\rangle + (1-\theta)\langle\rho,\rho\rangle \qquad \forall\Lambda\in\mathfrak{a}^*_{\mathbf{C}},$$

and denote by $p^\sigma_{t,\theta}$ the K–bi-invariant function on G such that

$$\widetilde{p}^\sigma_{t,\theta}(\Lambda) = \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr) \qquad \forall\Lambda\in\mathbf{T}_{\delta(p_\theta)}.$$

Then

$$\mathcal{P}^\sigma_{t,\theta}f = f * p^\sigma_{t,\theta} \qquad \forall t\in(0,\infty) \quad \forall f\in L^2(X).$$

Note that Q^σ_θ and $\widetilde{p}^\sigma_{t,\theta}$ continue analytically to the tube $\mathbf{T}_{\delta(p_\theta)}$, but to no larger tube. We denote by h_t the kernel associated to the heat operator, i. e., $\widetilde{h}_t(\Lambda) = \exp\bigl(-tQ_0(\Lambda)\bigr)$.

Throughout this paper, the following assumptions are made about the parameters:

$$\begin{gathered} 1 \leq p, q \leq \infty, \\ 1/p + 1/p' = 1, \\ 0 \leq \theta \leq 1, \\ 0 < \sigma < 1, \\ 0 < t < \infty. \end{gathered}$$

Recall that $p_\theta = 2\Big/\bigl[1+(1-\theta)^{1/2}\bigr]$, and note that $\delta(p_\theta) = (1-\theta)^{1/2}$. By C and C' we denote positive constants which may not be the same at different occurrences; C and C' may depend on anything quantified, implicitly or explicitly, before the formula in which they appear. The expression

$$A(t) \sim B(t) \qquad \forall t\in\mathbf{D},$$

where $\mathbf{D}$ is some subset of the domains of A and of B, means that there exist constants C and C' such that

$$C\,|A(t)| \leq |B(t)| \leq C'\,|A(t)| \qquad \forall t \in \mathbf{D}.$$

If the operator T on $L^2(X)$ satisfies a norm inequality of the form

$$\|Tf\|_q \leq C\,\|f\|_p \qquad \forall f \in L^2(X) \cap L^p(X),$$

then T extends uniquely to a bounded operator from $L^p(X)$ to $L^q(X)$ (where the weak-star topology should be used if $p = \infty$); conversely, if a continuous extension to a bounded operator from $L^p(X)$ to $L^q(X)$ exists, then such a norm inequality holds. We denote by $|\!|\!|T|\!|\!|_{p;q}$ the norm of the linear operator T from $L^p(X)$ to $L^q(X)$.

2. Estimates for the (σ,θ)–Poisson semigroup

First we prove a couple of technical results on integration, then we estimate the L^p–norms of $p^\sigma_{t,\theta}$ for various p. Finally we put the ingredients together to prove the main theorem.

Lemma 1. *Suppose that $0 \leq \alpha < \beta \leq \infty$, $\omega > 0$, and $\eta > 0$. Suppose also that ψ is a function on the interval $[\alpha,\beta)$, which is continuous and strictly positive throughout $[\alpha,\beta)$, such that, for constants C and k,*

$$|\psi(s)| \leq C\,(1+s)^k \qquad \forall s \in [\alpha,\beta).$$

Then

(i) *if* $\alpha = 0$,

$$\int_\alpha^\beta \exp(-ts^\omega)\,(s-\alpha)^{\eta-1}\,\psi(s)\,ds \sim t^{-\eta/\omega} \qquad \forall t \in [1,\infty);$$

(ii) *if* $\alpha > 0$,

$$\int_\alpha^\beta \exp(-ts^\omega)\,(s-\alpha)^{\eta-1}\,\psi(s)\,ds \sim t^{-\eta}\,\exp(-\alpha^\omega t) \qquad \forall t \in [1,\infty).$$

Proof. We assume initially that $\beta < \infty$.

We first prove (i). By changing variables ($v = ts^\omega$), we transform the integral to

$$\frac{t^{-\eta/\omega}}{\omega}\int_0^{t\beta^\omega} \exp(-v)\,v^{\eta/\omega-1}\,\psi\big((v/t)^{1/\omega}\big)\,dv;$$

the integrand is dominated by

$$\exp(-v)\,v^{\eta/\omega-1}\max\big\{|\psi(s)| : s \in [\alpha,\beta)\big\},$$

which is integrable on $\mathbf{R}^+$, and as t tends to ∞ the integral tends to

$$\int_0^\infty \exp(-v)\,v^{\eta/\omega-1}\,\psi(0)\,dv$$

by the Lebesgue dominated convergence theorem.

We now prove (ii). Let γ denote $\beta^\omega - \alpha^\omega$. Since ψ on $[\alpha,\beta)$ is continuous, bounded, and strictly positive, the function ϕ on $[0,\gamma)$, such that $\phi(0) = \omega^{1-\eta}\,\alpha^{\eta(1-\omega)}\,\psi(\alpha)$ and

$$\phi(v) = \Big[\frac{(v+\alpha^\omega)^{1/\omega}-\alpha}{v}\Big]^{\eta-1}\,\psi\big((v+\alpha^\omega)^{1/\omega}\big)\,(v+\alpha^\omega)^{1/\omega-1}\qquad \forall v\in(0,\gamma),$$

is too. By changing variables ($v = s^\omega - \alpha^\omega$), we transform the integral to

$$\frac{\exp(-t\alpha^\omega)}{\omega}\int_0^\gamma \exp(-tv)\,v^{\eta-1}\,\phi(v)\,dv.$$

We have therefore an integral of the form already treated in (i), and are done.

The case where $\beta=\infty$ now follows: we write the integral as the sum of an integral from α to $\alpha+1$ and one from $\alpha+1$ to ∞; the first integral is treated by the result already established, and the second is easily shown to be $O(t^{k+\eta+1}\exp\big(-(\alpha+1)^\omega t)\big)$. $\square$

Lemma 2. *Suppose that $0<\tau<\infty$ and $R>0$. Then*

$$\int_0^1 \exp\big(-t[r^2+R^2]^\sigma\big)\,r^{\tau-1}\,dr \sim \begin{cases} 1 & \forall t\in(0,1]\\ t^{-\tau/2}\,\exp(-R^{2\sigma}t) & \forall t\in[1,\infty)\end{cases}$$

and

$$\int_1^\infty \exp\big(-t[r^2+R^2]^\sigma\big)\,r^{\tau-1}\,dr \sim \begin{cases} t^{-\tau/2\sigma} & \forall t\in(0,1]\\ t^{-1}\,\exp\big(-[1+R^2]^\sigma\,t\big) & \forall t\in[1,\infty).\end{cases}$$

Proof. It is trivial that the first integral behaves as claimed for t in $(0,1]$. To study its behaviour for t in $[1,\infty)$, we change variables $\big(s=[r^2+R^2]^{1/2}\big)$, and it becomes

$$\begin{aligned}&\int_R^{\sqrt{R^2+1}} \exp\big(-ts^{2\sigma}\big)\,(s^2-R^2)^{(\tau-2)/2}\,s\,ds\\ &= \int_R^{\sqrt{R^2+1}} \exp\big(-ts^{2\sigma}\big)\,(s-R)^{(\tau-2)/2}\,(s+R)^{(\tau-2)/2}\,s\,ds.\end{aligned}$$

The required behaviour for t in $[1,\infty)$ is a corollary of Lemma 1 (where $\omega=2\sigma$, $\eta=\tau/2$, $\alpha=R$, and $\psi(s)=(s+R)^{(\tau-2)/2}\,s$).

By performing the same change of variables, we transform the second integral to

$$\int_{\sqrt{R^2+1}}^\infty \exp\big(-ts^{2\sigma}\big)\,(s^2-R^2)^{(\tau-2)/2}\,s\,ds.$$

The result stated for t in $[1,\infty)$ follows from Lemma 1 (where $\omega=2\sigma$, $\eta=1$, $\alpha=[R^2+1]^{1/2}$, and $\psi(s)=(s^2-R^2)^{(\tau-2)/2}\,s$). By changing variables again, we transform the last integral to

$$t^{-\tau/2\sigma}\int_{\sqrt{R^2+1}\,t^{1/2\sigma}}^\infty \exp\big(-v^{2\sigma}\big)\,\big(v^2-(t^{1/2\sigma}R)^2\big)^{(\tau-2)/2}\,v\,dv$$

and the required behaviour for t in $(0,1]$ is an immediate consequence. $\square$

Note that the preceding lemma holds for any positive σ.
We now begin the harmonic analysis.

Lemma 3. *The following norm estimates for $p^\sigma_{t,\theta}$ hold:*

(i) *if* $p=2$, *then*
$$\left\|p^\sigma_{t,\theta}\right\|_p \sim \begin{cases} t^{-n/4\sigma} & \forall t\in(0,1] \\ t^{-\nu/4}\exp(-R_\theta^{2\sigma}t) & \forall t\in[1,\infty); \end{cases}$$
(ii) *if* $p=\infty$, *then*
$$\left\|p^\sigma_{t,\theta}\right\|_p \sim \begin{cases} t^{-n/2\sigma} & \forall t\in(0,1] \\ t^{-\nu/2}\exp(-R_\theta^{2\sigma}t) & \forall t\in[1,\infty); \end{cases}$$
(iii) *if* $p_\theta<p<2$, *then*
$$\left\|p^\sigma_{t,\theta}\right\|_p \sim \begin{cases} t^{-n/2\sigma p'} & \forall t\in(0,1] \\ t^{-\ell/2p'}\exp\big(-R_{\theta,p}^{2\sigma}\,t\big) & \forall t\in[1,\infty); \end{cases}$$
(iv) *if* $p=p_\theta<2$, *then*
$$\left\|p^\sigma_{t,\theta}\right\|_p \sim \begin{cases} t^{-n/2\sigma p'} & \forall t\in(0,1] \\ t^{-(\ell+1)/2\sigma p'} & \forall t\in[1,\infty). \end{cases}$$

Proof. To prove (i) we use the Plancherel formula and pass to polar coordinates, using formula (1):

$$\begin{aligned}
\left\|p^\sigma_{t,\theta}\right\|_2 &= \left[\int_{\mathfrak{a}^*} \exp\big(-2tQ_\theta(\Lambda)^\sigma\big)\,d\mu(\Lambda)\right]^{1/2} \\
&\sim \left[\int_0^\infty \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,(1+r)^{n-\nu}\,r^{\nu-1}\,dr\right]^{1/2} \\
&\sim \Bigg[\int_0^1 \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,r^{\nu-1}\,dr \\
&\qquad + \int_1^\infty \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,r^{n-1}\,dr\Bigg]^{1/2} \qquad \forall t\in(0,\infty).
\end{aligned}$$

Now, from Lemma 2,

$$\int_0^1 \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,r^{\nu-1}\,dr \sim \begin{cases} 1 & \forall t\in(0,1] \\ t^{-\nu/2}\,\exp(-2R_\theta^{2\sigma}t) & \forall t\in[1,\infty) \end{cases}$$

and

$$\int_1^\infty \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,r^{n-1}\,dr \sim \begin{cases} t^{-n/2\sigma} & \forall t\in(0,1] \\ t^{-1}\,\exp\big(-2[1+R_\theta^2]^\sigma\,t\big) & \forall t\in[1,\infty). \end{cases}$$

This proves (i).

To prove (ii), we proceed similarly, using the inversion formula. For any x in G,

$$\begin{aligned}
\left|p^{\sigma}_{t,\theta}(x)\right| &= \left|\int_{\mathfrak{a}^*} \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr)\, \phi_\Lambda(x)\, d\mu(\Lambda)\right| \\
&\le \int_{\mathfrak{a}^*} \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr)\, |\phi_\Lambda(x)|\, d\mu(\Lambda) \\
&\le \int_{\mathfrak{a}^*} \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr)\, \phi_\Lambda(e)\, d\mu(\Lambda) \\
&= p^{\sigma}_{t,\theta}(e),
\end{aligned}$$

so that $\left\|p^{\sigma}_{t,\theta}\right\|_\infty = p^{\sigma}_{t,\theta}(e)$. Now it is easy to see that

$$\left\|p^{\sigma}_{t,\theta}\right\|_\infty = \int_{\mathfrak{a}^*} \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr)\, d\mu(\Lambda) \sim \begin{cases} t^{-n/2\sigma} & \forall t \in (0,1] \\ t^{-\nu/2} \exp(-R_\theta^{2\sigma} t) & \forall t \in [1,\infty) \end{cases}$$

by a calculation like that of the $L^2(G)$–norm of $p^{\sigma}_{t,\theta}$.

The hardest parts of this lemma are (iii) and (iv). We prove both cases by obtaining first a lower bound, then an upper bound.

We begin the proof of (iii). Suppose that $p_\theta < p < 2$. By Theorem 2.1 of [3], and the fact that $d\widetilde{\mu}(\Lambda)/d\Lambda \ge C$,

$$\begin{aligned}
(2)\quad \left\|p^{\sigma}_{t,\theta}\right\|_p &\ge C\left[\int_{\mathfrak{a}^*} \left|\widetilde{p}^{\sigma}_{t,\theta}(\Lambda + i\delta(p)\rho)\right|^{p'} d\widetilde{\mu}(\Lambda)\right]^{1/p'} \\
&\ge C\left[\int_{\mathfrak{b}} \exp\Bigl(-tp' \operatorname{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^\sigma\bigr)\Bigr)\, d\Lambda\right]^{1/p'} \qquad \forall t \in [1,\infty),
\end{aligned}$$

where $\mathfrak{b}$ denotes the unit ball in $\mathfrak{a}^*$. We denote the right hand side of the last inequality by $I(t)$. Now for all Λ in $\mathfrak{a}^*$,

$$\begin{aligned}
\operatorname{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^\sigma\bigr) &\le |Q_\theta(\Lambda + i\delta(p)\rho)|^\sigma \\
&= \bigl(\langle\Lambda,\Lambda\rangle + R^2_{\theta,p} + 2i\delta(p)\langle\rho,\Lambda\rangle\bigr)^{\sigma/2} \\
&\le \bigl([\langle\Lambda,\Lambda\rangle + R^2_{\theta,p}]^2 + 4\delta(p)^2 R_0^2 \langle\Lambda,\Lambda\rangle\bigr)^{\sigma/2} \\
&= \bigl(\langle\Lambda,\Lambda\rangle^2 + \gamma_1\langle\Lambda,\Lambda\rangle + \gamma_2\bigr)^{\sigma/2},
\end{aligned}$$

where $\gamma_1 = 2R^2_{\theta,p} + 4\delta(p)^2 R_0^2$ and $\gamma_2 = R^4_{\theta,p}$, so, by using polar co-ordinates, and then changing variables $\bigl(s = (r^4 + \gamma_1 r^2 + \gamma_2)^{1/4}\bigr)$, we see that

$$\begin{aligned}
I(t) &\ge C\left[\int_0^1 \exp\Bigl(-tp'\bigl(r^4 + \gamma_1 r^2 + \gamma_2\bigr)^{\sigma/2}\Bigr)\, r^{\ell-1}\, dr\right]^{1/p'} \\
&= C\left[\int_{R_{\theta,p}}^{\gamma} \exp\bigl(-tp' s^{2\sigma}\bigr)\, (s - R_{\theta,p})^{\ell/2-1}\, \psi(s)\, ds\right]^{1/p'} \qquad \forall t \in [1,\infty),
\end{aligned}$$

where $\gamma = (1+\gamma_1+\gamma_2)^{1/4}$ and ψ is a continuous strictly positive function on the interval $[R_{\theta,p},\gamma]$. Now, Lemma 1 applies (where $\omega = 2\sigma$, $\eta = \ell/2$ and $\alpha = R_{\theta,p}$) and we conclude that

$$I(t) \geq C\, t^{-\ell/2p'} \exp(-R_{\theta,p}^{2\sigma}\, t) \qquad \forall t \in [1,\infty); \tag{3}$$

combined with (2), this proves the lower bound of (iii).

We now prove the upper bound of (iii). From [3], Theorem 2.4, if $p_\theta \leq p < 2$, then

$$\left\|p_{t,\theta}^{\sigma}\right\|_p \leq \left\|p_{t,\theta}^{\sigma}\,\phi_{i\delta(p)\rho}\right\|_1^{\delta(p)} N^{1-\delta(p)},$$

where

$$\begin{aligned} N &= \left[\int_{\mathfrak{a}^*} \left|\widetilde{p}_{t,\theta}^{\sigma}(\Lambda + i\delta(p)\rho)\right|^2 d\widetilde{\mu}(\Lambda)\right]^{1/2} \\ &= \left[\int_{\mathfrak{a}^*} \left|\exp\bigl(-t[Q_\theta(\Lambda + i\delta(p)\rho)]^{\sigma}\bigr)\right|^2 d\widetilde{\mu}(\Lambda)\right]^{1/2} \\ &= \left[\int_{\mathfrak{a}^*} \exp\Bigl(-2t\,\mathrm{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^{\sigma}\bigr)\Bigr)\, d\widetilde{\mu}(\Lambda)\right]^{1/2}. \end{aligned}$$

Since $p_{t,\theta}^{\sigma}$ is a positive function,

$$\left\|p_{t,\theta}^{\sigma}\,\phi_{i\delta(p)\rho}\right\|_1 = \widetilde{p}_{t,\theta}^{\sigma}(-i\delta(p)\rho) = \exp\bigl(-R_{\theta,p}^{2\sigma}\, t\bigr).$$

Thus

$$\left\|p_{t,\theta}^{\sigma}\right\|_p \leq \exp\bigl(-\delta(p) R_{\theta,p}^{2\sigma}\, t\bigr)\, N^{1-\delta(p)}. \tag{4}$$

To estimate N, we observe that

$$\mathrm{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^{\sigma}\bigr) \geq \bigl(\mathrm{Re}[Q_\theta(\Lambda + i\delta(p)\rho)]\bigr)^{\sigma} = [\langle\Lambda,\Lambda\rangle + R_{\theta,p}^2]^{\sigma} \qquad \forall \Lambda \in \mathfrak{a}^*,$$

because, for any complex number z with nonnegative real part and any σ in $(0,1)$, $\mathrm{Re}\bigl(z^{\sigma}\bigr) \geq \bigl(\mathrm{Re}(z)\bigr)^{\sigma}$. We recall that $d\widetilde{\mu}(\Lambda)/d\Lambda \leq C(1+|\Lambda|^{n-\ell})$, and pass to polar co-ordinates, to deduce that

$$\begin{aligned} N &\leq C\left[\int_0^{\infty} \exp\bigl(-2t[r^2 + R_{\theta,p}^2]^{\sigma}\bigr)\bigl(1 + r^{n-\ell}\bigr)\, r^{\ell-1}\, dr\right]^{1/2} \\ &\sim \left[\int_0^1 \exp\bigl(-2t[r^2 + R_{\theta,p}^2]^{\sigma}\bigr)\, r^{\ell-1}\, dr \right. \\ &\qquad\qquad \left. + \int_1^{\infty} \exp\bigl(-2t[r^2 + R_{\theta,p}^2]^{\sigma}\bigr)\, r^{n-1}\, dr\right]^{1/2} \\ &\leq C\, t^{-\ell/4} \exp\bigl(-R_{\theta,p}^{2\sigma}\, t\bigr) \qquad \forall t \in [1,\infty), \end{aligned}$$

by Lemma 2, so that

$$\left\|p_{t,\theta}^{\sigma}\right\|_p \leq \exp\bigl(-\delta(p) R_{\theta,p}^{2\sigma}\, t\bigr)\, N^{1-\delta(p)} \leq C\, t^{-\ell/2p'} \exp\bigl(-R_{\theta,p}^{2\sigma}\, t\bigr) \qquad \forall t \in [1,\infty),$$

as required to prove (iii).

We now consider (iv). If $p > 1$ and $\ell > 1$, then, much as argued to prove (2),

$$\begin{aligned}\left\| p^\sigma_{t,\theta} \right\|_p &\geq C \left[\int_{\mathfrak{a}^*} \left| \widetilde{p}^\sigma_{t,\theta}(\Lambda + i\delta(p)\rho) \right|^{p'} d\widetilde{\mu}(\Lambda) \right]^{1/p'} \\ &\geq C \left[\int_{\mathfrak{c}} \exp\bigl(-tp' \left| Q_\theta(\Lambda + i\delta(p)\rho) \right|^\sigma \bigr) \, d\Lambda \right]^{1/p'} \qquad \forall t \in [1, \infty),\end{aligned}$$

where $\mathfrak{c}$ denotes the subset of $\mathfrak{a}^*$ of all elements of the form $\Lambda_0 + \lambda_1 \rho$, such that $\langle \Lambda_0, \rho \rangle = 0$, $\langle \Lambda_0, \Lambda_0 \rangle \leq 1$, and $0 \leq \lambda_1 \leq 1$. We denote the right hand side of the last inequality by $J(t)$. Now if Λ may be written in this way, then

$$\begin{aligned}\left| Q_\theta(\Lambda + i\delta(p)\rho) \right| &= \left| \langle \Lambda_0, \Lambda_0 \rangle + \lambda_1^2 R_0^2 + 2i\delta(p)\lambda_1 R_0^2 \right| \\ &\leq \langle \Lambda_0, \Lambda_0 \rangle + \lambda_1^2 R_0^2 + 2\lambda_1 R_0^2 \\ &\leq \langle \Lambda_0, \Lambda_0 \rangle + 3\lambda_1 R_0^2,\end{aligned}$$

so, passing to polar co-ordinates in $\rho^\perp$, and then changing variables, we deduce that

$$\begin{aligned}J(t) &\geq C \left[\int_0^1 \int_0^1 \exp\bigl(-tp' \left| \lambda_0^2 + 3\lambda_1 R_0^2 \right|^\sigma \bigr) \, \lambda_0^{\ell-2} \, d\lambda_0 \, d\lambda_1 \right]^{1/p'} \\ &\geq C \, t^{-(\ell+1)/2\sigma p'} \left[\int_0^{t^{1/2\sigma}} \int_0^{t^{1/\sigma}} \exp\bigl(-p' \left| \lambda_0^2 + 3\lambda_1 R_0^2 \right|^\sigma \bigr) \, \lambda_0^{\ell-2} \, d\lambda_0 \, d\lambda_1 \right]^{1/p'} \\ &\geq C \, t^{-(\ell+1)/2\sigma p'} \qquad \forall t \in [1, \infty),\end{aligned}$$

as required to prove the lower bound. If $p = 1$ or $\ell = 1$, the argument simplifies but the conclusion is the same.

To prove the upper bound, we note first that (4) continues to hold, and that $R^{2\sigma}_{\theta,p} = 0$, so that

$$\left\| p^\sigma_{t,\theta} \right\|_p \leq N^{1-\delta(p)}.$$

To estimate N, we write $\Lambda = \Lambda_0 + \lambda_1 \rho$, where $\langle \Lambda_0, \rho \rangle = 0$, as above, and note that

$$\begin{aligned}\operatorname{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^\sigma \bigr) &= \operatorname{Re}\bigl([\langle \Lambda_0, \Lambda_0 \rangle + \lambda_1^2 R_0^2 + 2i\delta(p)\lambda_1 R_0^2]^\sigma \bigr) \\ &\geq \operatorname{Re}\bigl([\langle \Lambda_0, \Lambda_0 \rangle + 2i\delta(p)\lambda_1 R_0^2]^\sigma \bigr).\end{aligned}$$

We let Φ and $\Psi : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ be the functions given by the formulae

$$\begin{aligned}\Phi(\lambda_0, \lambda_1) &= \operatorname{Re}\bigl([\lambda_0^2 + 2i\delta(p) R_0^2 \lambda_1]^\sigma \bigr) \qquad \forall \lambda_0 \in (0, \infty), \quad \forall \lambda_1 \in (-\infty, \infty) \\ \Psi(\lambda_0, \lambda_1) &= \bigl(1 + \lambda_0 + |\lambda_1| \bigr)^{n-\ell} \qquad \forall \lambda_0 \in (0, \infty), \quad \forall \lambda_1 \in (-\infty, \infty),\end{aligned}$$

and recall that $d\widetilde{\mu}(\Lambda)/d\Lambda \leq C(1+|\Lambda|^{n-\ell})$. By passing to polar co-ordinates in $\rho^\perp$ and changing variables, we deduce that

$$\begin{aligned} N &= \left[\int_{\mathfrak{a}^*} \exp\Bigl(-2t\,\mathrm{Re}\bigl([Q_\theta(\Lambda+i\delta(p)\rho)]^\sigma\bigr)\Bigr)\,d\widetilde{\mu}(\Lambda)\right]^{1/2} \\ &\leq C\left[\int_{-\infty}^{\infty}\int_0^\infty \exp\bigl(-2t\,\Phi(\lambda_0,\lambda_1)\bigr)\,\Psi(\lambda_0,\lambda_1)\,\lambda_0^{\ell-2}\,d\lambda_0\,d\lambda_1\right]^{1/2} \\ &= C\,t^{-(\ell+1)/4\sigma}\left[\int_{-\infty}^{\infty}\int_0^\infty \exp\bigl(-2\,\Phi(\lambda_0,\lambda_1)\bigr)\,\Psi\Bigl(\frac{\lambda_0}{t^{1/2\sigma}},\frac{\lambda_1}{t^{1/\sigma}}\Bigr)\,\lambda_0^{\ell-2}\,d\lambda_0\,d\lambda_1\right]^{1/2} \\ &\leq C\,t^{-(\ell+1)/4\sigma} \qquad \forall t\in[1,\infty). \end{aligned}$$

Now we can conclude that

$$\bigl\|p_{t,\theta}^\sigma\bigr\|_p \leq C\,t^{-(1-\delta(p))(\ell+1)/4\sigma} = C\,t^{-(\ell+1)/2\sigma p'} \qquad \forall t\in[1,\infty),$$

as required. This finishes the proof of (iv), and of Lemma 3. $\square$

For convenience, we list the results of our main theorem.

(i) if $t>0$, then $\mathcal{P}_{t,\theta}^\sigma$ is L^p – L^q-bounded only if $p\leq q$, $p\leq p_\theta{}'$, and $q\geq p_\theta$;

(ii) if $p_\theta\leq p\leq p_\theta{}'$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;p}=\exp\bigl(-R_{\theta,p}^{2\sigma}\,t\bigr)$ for all t in $(0,\infty)$;

(iii) if $p\leq p_\theta{}'$ and $q\geq p_\theta$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q}\sim t^{-n(1/p-1/q)/2\sigma}$ for all t in $(0,1]$;

(iv) if $p<q=2$ or $2=p<q$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q}\sim t^{-\nu/4}\exp(-R_\theta^{2\sigma}t)$ for all t in $[1,\infty)$;

(v) if $p<2<q$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q}\sim t^{-\nu/2}\exp(-R_\theta^{2\sigma}t)$ for all t in $[1,\infty)$;

(vi) if $p<q<2$ and $q>p_\theta$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q}\sim t^{-\ell/2q'}\exp\bigl(-R_{\theta,q}^{2\sigma}\,t\bigr)$ for all t in $[1,\infty)$;

(vii) if $p<q=p_\theta$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q}\sim t^{-(\ell+1)/2\sigma q'}$ for all t in $[1,\infty)$;

(viii) if $2<p<q$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q}\sim t^{-\ell/2p}\exp\bigl(-R_{\theta,p}^{2\sigma}\,t\bigr)$ for all t in $[1,\infty)$;

(ix) if $p_\theta{}'=p<q$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q}\sim t^{-(\ell+1)/2\sigma p}$ for all t in $[1,\infty)$.

Proof of the theorem. First, a result of L. Hörmander [8] shows that $\mathcal{P}_{t,\theta}^\sigma$ cannot be L^p–L^q-bounded unless $p\leq q$. Next, $\mathcal{P}_{t,\theta}^\sigma$ cannot be L^p–L^q-bounded when $q<p_\theta$, by the sufficient condition of J. L. Clerc and E. M. Stein [1] and the fact that $\widetilde{p}_{t,\theta}^\sigma$ continues analytically to $\mathbf{T}_{\delta(p_\theta)}$ but to no larger tube. By duality, $\mathcal{P}_{t,\theta}^\sigma$ cannot be L^p–L^q-bounded if $p>p_\theta{}'$. This proves (i). Observe that parts (ii) to (ix) imply that $\mathcal{P}_{t,\theta}^\sigma$ is L^p–L^q-bounded if $p\leq q$, $p\leq p_\theta{}'$, and $q\geq p_\theta$.

We now prove (ii), for p in $[p_\theta,2]$. Define Λ_p to be $i\delta(p)\rho$. By C. S. Herz' *principe de majoration* [7] and spherical Fourier analysis, we have that

$$|||\mathcal{P}_{t,\theta}^\sigma|||_{p;p}=\int_G p_{t,\theta}^\sigma(x)\,\phi_{\Lambda_p}(x)\,dx=\widetilde{p}_{t,\theta}^\sigma(-\Lambda_p)=\exp\bigl(-R_{\theta,p}^{2\sigma}\,t\bigr) \qquad \forall t\in(0,\infty).$$

By duality, this result also holds for p in $[2, p_\theta']$.

Estimate (iii) is a consequence of the theory of ultracontractive semigroups, combined with the fact that $|||\mathcal{P}^\sigma_{t,\theta}|||_{1;\infty} = \|p^\sigma_{t,\theta}\|_\infty$ and the estimate for $\|p^\sigma_{t,\theta}\|_\infty$ in Lemma 3 above. See, e.g., Cowling and Meda [4], E. B. Davies [5], or N. Th. Varopoulos *et al.* [9].

Next we prove (iv). By duality, it suffices to treat the case where $p < 2$ and $q = 2$. On the one hand, by the Kunze–Stein phenomenon [2], $|||\mathcal{P}^\sigma_{t,\theta}|||_{p;2} \leq C\,\|p^\sigma_{t,\theta}\|_2$, so that

$$|||\mathcal{P}^\sigma_{t,\theta}|||_{p;2} \leq C\, t^{-\nu/4} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty),$$

from Lemma 3. On the other hand,

$$\|h_1 * p^\sigma_{t,\theta}\|_2 \leq |||\mathcal{P}^\sigma_{t,\theta}|||_{p;2}\, \|h_1\|_p = C\, |||\mathcal{P}^\sigma_{t,\theta}|||_{p;2} \qquad \forall t \in [1,\infty);$$

by the Plancherel formula, formula (1), and Lemma 2,

$$\begin{aligned}
\|h_1 * p^\sigma_{t,\theta}\|_2 &= \left[\int_{\mathfrak{a}^*} \exp\bigl(-2tQ_\theta(\Lambda)^\sigma - 2Q_0(\Lambda)\bigr)\, d\mu(\Lambda)\right]^{1/2} \\
&\sim \left[\int_0^\infty \exp\bigl(-2t[r^2 + R_\theta^2]^\sigma - 2[r^2 + R_0^2]\bigr)\,(1+r)^{n-\nu}\, r^{\nu-1}\, dr\right]^{1/2} \\
&\geq C\left[\int_0^1 \exp\bigl(-2t[r^2 + R_\theta^2]^\sigma\bigr)\, r^{\nu-1}\, dr\right]^{1/2} \\
&\sim t^{-\nu/4} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty),
\end{aligned}$$

so that

$$|||\mathcal{P}^\sigma_{t,\theta}|||_{p;2} \geq C\, t^{-\nu/4} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty).$$

It follows that $|||\mathcal{P}^\sigma_{t,\theta}|||_{p;2} \sim t^{-\nu/4} \exp(-R_\theta^{2\sigma} t)$ for all t in $[1,\infty)$, and (iv) is proved.

To prove (v), we proceed similarly. On the one hand, from (iv),

$$|||\mathcal{P}^\sigma_{t,\theta}|||_{p;q} \leq |||\mathcal{P}^\sigma_{t/2,\theta}|||_{p;2}\, |||\mathcal{P}^\sigma_{t/2,\theta}|||_{2;q} \sim t^{-\nu/2} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty).$$

On the other hand,

$$|||\mathcal{H}_{1,0}|||_{1;p}\, |||\mathcal{P}^\sigma_{t,\theta}|||_{p;q}\, |||\mathcal{H}_{1,0}|||_{q;\infty} \geq \|h_2 * p^\sigma_{t,\theta}\|_\infty \sim t^{-\nu/2} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty),$$

by an argument similar to that used in the proof of (iv) above. This completes the proof of (v).

We now prove (vi). On the one hand, Theorem 2.2 in [3] and Lemma 3 may be invoked to show that

$$|||\mathcal{P}^\sigma_{t,\theta}|||_{p;q} \leq C\,\|p^\sigma_{t,\theta}\|_q \leq C\, t^{-\ell/2q'} \exp\bigl(-R_{\theta,q}^{2\sigma}\, t\bigr) \qquad \forall t \in [1,\infty).$$

On the other hand,

$$\left\| h_1 * p^\sigma_{t,\theta} \right\|_q \leq \left\vert\!\left\vert\!\left\vert \mathcal{P}^\sigma_{t,\theta} \right\vert\!\right\vert\!\right\vert_{p;q} \left\| h_1 \right\|_p = C \left\vert\!\left\vert\!\left\vert \mathcal{P}^\sigma_{t,\theta} \right\vert\!\right\vert\!\right\vert_{p;q} \qquad \forall t \in [1,\infty).$$

As argued to prove (2), if $\mathfrak{b}$ again denotes the unit ball in $\mathfrak{a}^*$,

$$\begin{aligned} \left\| h_1 * p^\sigma_{t,\theta} \right\|_q &\geq C \left[\int_{\mathfrak{a}^*} \left| \widetilde{h}_1 (\Lambda + i\delta(q)\rho) \, \widetilde{p}^\sigma_{t,\theta} (\Lambda + i\delta(q)\rho) \right|^{q'} d\widetilde{\mu}(\Lambda) \right]^{1/q'} \\ &\geq C \left[\int_{\mathfrak{b}} \left| \widetilde{h}_1 (\Lambda + i\delta(q)\rho) \, \widetilde{p}^\sigma_{t,\theta} (\Lambda + i\delta(q)\rho) \right|^{q'} d\Lambda \right]^{1/q'} \\ &\geq C \left[\int_{\mathfrak{b}} \left| \widetilde{p}^\sigma_{t,\theta} (\Lambda + i\delta(q)\rho) \right|^{q'} d\Lambda \right]^{1/q'} \qquad \forall t \in [1,\infty), \end{aligned}$$

since for Λ in $\mathfrak{b}$, $\left| \widetilde{h}_1 (\Lambda + i\delta(q)\rho) \right|$ is bounded away from 0. This integral was treated in the proof of Lemma 3 (see (3)), and we conclude that

$$\left\vert\!\left\vert\!\left\vert \mathcal{P}^\sigma_{t,\theta} \right\vert\!\right\vert\!\right\vert_{p;q} \geq C \, t^{-\ell/2q'} \exp\bigl(-R^{2\sigma}_{\theta,q} \, t\bigr) \qquad \forall t \in [1,\infty),$$

completing the proof of (vi).

Finally, (vii) is proved in the same way as (vi), but part (iv) of Lemma 3 is used instead of part (iii), and (viii) and (ix) follow from (vi) and (vii) by duality. □

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